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Time-Consistent Conditional Expectation Under Probability Distortion

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Abstract. We introduce a new notion of conditional nonlinear expectation under probability distortion. Such a distorted nonlinear expectation is not subadditive in general, so it is beyond the scope of Peng's framework of nonlinear expectations. A more fundamental problem when extending the distorted expectation to a dynamic setting is *time inconsistency*, that is, the usual "tower property" fails. By localizing the probability distortion and restricting to a smaller class of random variables, we introduce a so-called distorted probability and construct a conditional expectation in such a way that it coincides with the original nonlinear expectation at time zero, but has a time-consistent dynamics in the sense that the tower property remains valid. Furthermore, we show that in the continuous time model this conditional expectation corresponds to a parabolic differential equation whose coefficient involves the law of the underlying diffusion. This work is the first step toward a new understanding of nonlinear expectations under probability distortion and will potentially be a helpful tool for solving time-inconsistent stochastic optimization problems.

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1. Introduction

In this paper we propose a new notion of nonlinear conditional expectation under *probability distortion*. Such a nonlinear expectation is by nature not subadditive; thus, it is different from Peng's well-studied nonlinear expectations (see, e.g., [18], [19]). Our goal is to find an appropriate definition of conditional nonlinear expectations such that it is *time consistent* in the sense that the usual "tower property" holds.

Probability distortion has been largely motivated by empirical findings in behavioral economics and finance (see, e.g., Kahneman-Tversky [13], [23], Zhou [26], and the references therein). It describes the natural human tendency to exaggerate small probabilities for certain events, contradicting the classical axiom of rationality. Mathematically, this can be characterized by a nonlinear expectation where the underlying probability scale is modified by a *distortion function*. More precisely, let ξ be a nonnegative random variable representing the outcome of an uncertain event. The usual (linear) expectation of ξ can be written in the form

$$\mathbb{E}[\xi] = \int_0^\infty \mathbb{P}(\xi \ge x)\, dx. \tag{1}$$

Probability distortion, on the other hand, considers a "distorted" version of the expectation

$$\mathscr{E}[\xi] := \int_0^\infty \varphi(\mathbb{P}(\xi \ge x))\, dx, \tag{2}$$

where the distortion function $\varphi : [0,1] \to [0,1]$ is continuous, is strictly increasing, and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. Economically the most interesting case is that φ is reverse S-shaped, that is, φ is concave when $p \approx 0$ and is convex when $p \approx 1$. In the special case $\varphi(p) \equiv p$, (2) reduces to (1). In general, the distorted expectation $\mathscr{E}[\cdot]$ is nonlinear, that is, neither subadditive nor superadditive.

Although (2) is useful in many contexts, a major difficulty occurs when one tries to define the "conditional," or "dynamic," version of the distorted expectation. Consider, for example, a "naively" defined distorted conditional expectation given the information $\mathcal{F}_t$ at time t:

$$\mathscr{E}_t[\xi] = \int_0^\infty \varphi(\mathbb{P}(\xi > x|\mathcal{F}_t))\, dx. \tag{3}$$

Then it is easy to check that in general $\mathscr{E}_s[\mathscr{E}_t[\xi]] \neq \mathscr{E}_s[\xi]$ for $s < t$, that is, the "tower property" or the flow property fails. This is often referred to as a type of "time inconsistency" and is studied extensively in stochastic optimal control (see Section 1.1 for more discussion).

Motivated by the work of Karnam et al. [15], which provides a new perspective for time-inconsistent optimization problems, in this paper we find a different way to define the distorted conditional expectation so that it remains *time consistent* in terms of preserving the tower property. To be specific, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$. We look for a family of operators $\{\mathscr{E}_t\}_{0\le t\le T}$ such that, for a given $\mathcal{F}_T$-measurable random variable ξ, it holds that $\mathscr{E}_0[\xi] = \mathscr{E}[\xi]$ as in (2), and for $0 \le s < t \le T$, the tower property holds: $\mathscr{E}_s[\mathscr{E}_t[\xi]] = \mathscr{E}_s[\xi]$. More generally, we shall construct operators $\mathscr{E}_{s,t}$ for $0 \le s \le t \le T$ such that $\mathscr{E}_{r,s}[\mathscr{E}_{s,t}[\xi]] = \mathscr{E}_{r,t}[\xi]$ for $\mathcal{F}_t$-measurable ξ and $r \le s \le t$. We shall argue that this is possible at least for a large class of random variables: $\xi = g(X_t)$, where $g : \mathbb{R} \to [0,\infty)$ is increasing, and X is either a binomial tree or a one-dimensional diffusion

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \ge 0. \tag{4}$$

It is worth noting that, although the aforementioned class of random variables are somewhat restricted, especially the monotonicity of g, which plays a crucial role in our approach, it contains a large class of practically useful random variables considered in most publications about probability distortion, where X is the state process and g is a utility function, whence monotone.

The main idea of our approach is based on the following belief: in a dynamic distorted expectation the form of the distortion function should depend on the prospective time horizon. Simply put, the distortion function over $[0,T]$ such as that in (2) is very likely to be different from that in (3), which is applied only to subintervals of $[0,T]$. We believe this is why (3) becomes time inconsistent. Similar to the idea of "dynamic utility" in [15], we propose to *localize* the distortion function as follows: given a collection of initial distortion functions φ_t corresponding to intervals of the form $[0,t]$, we look for a dynamic distortion function $\Phi(s,t,x;p)$ such that $\Phi(0,t,X_0;\cdot) = \varphi_t$ (e.g., $\varphi_t \equiv \varphi$) and that the resulting distorted conditional expectation

$$\mathscr{E}_{s,t}[\xi] = \int_0^\infty \Phi\big(s,t,X_s;\mathbb{P}\big(\xi > y|\mathcal{F}_s\big)\big)dy, \quad 0 \le s < t \le T, \tag{5}$$

is time consistent for all $\xi = g(X_t)$ with g increasing. Intuitively, the dependence of the distortion function Φ on (s,t,x) could be thought of as the agent's (distorted) view toward the prospective random events at future time t at current time s and state x.

We shall first illustrate this idea in discrete time using a binomial tree model to present all the main elements. The diffusion case is conceptually similar, but the analysis is much more involved. In both cases, however, the dynamic distortion function has an interesting interpretation: there exists a probability $\mathbb{Q}$ (equivalent to $\mathbb{P}$ and independent of the increasing function g) such that

$$\Phi\big(s,t,x;\mathbb{P}\big(X_t \ge y|X_s = x\big)\big) = \mathbb{Q}\big(X_t \ge y|X_s = x\big)$$

(see Theorems 1 and 2 as well as Remark 2). We shall refer to $\mathbb{Q}$ as the *distorted probability*, so that (5) renders the distorted conditional expectation as a usual linear conditional expectation under $\mathbb{Q}$. We should note that such a hidden linear structure, because of the restriction $\xi = g(X_t)$, has not been explored in previous works. In particular, in the continuous time setting, this enables us to show that the conditional expectation $\mathscr{E}_{s,t}[\xi]$ in (5) can be written as $\mathscr{E}_{s,t}[\xi] = u(s, X_s)$, where the function u satisfies a linear parabolic partial differential equation (PDE) whose coefficients depend on the distortion function φ and the density of the underlying diffusion X defined by (4).

We would like to emphasize that although this paper considers only the conditional expectations, it is the first step toward a long-term goal of investigating stochastic optimization problems under probability distortion, as well as other time-inconsistent problems. In fact, in a very recent paper He et al. [10] studied an optimal investment problem under probability distortion and showed that a time-consistent dynamic distortion function of the form $\Phi = \Phi(s,t;p)$ exists if and only if it belongs to the family introduced in Wang [24] or the agent does not invest in the risky assets. This result in part validates our general framework, which aims at a large class of optimization problems of similar type in a general setting, by allowing Φ to depend on the state X_s, and even its law.

The rest of the paper is organized as follows. In Section 1.1 we review some approaches in the literature for time-inconsistent stochastic optimization problems, which will put this paper in the proper perspective.

In Section 2 we recall the notion of probability distortion and introduce our dynamic distortion function. In Section 3 we construct a time-consistent dynamic distortion function in a discrete time binomial tree framework. In Section 4 we consider the diffusion case (4) with constant σ, and the results are extended to the case with general σ in Section 5. Finally, in Section 6 we study the density of the underlying state process X, which is crucial for constructing our dynamic distortion function Φ.

1.1. Discussion: Time-Inconsistency in Stochastic Control

We begin by recalling the usual meaning of "time inconsistency" in a stochastic optimization problem. Consider a stochastic control problem over time horizon $[0, T]$, denote it by $P_{[0,T]}$, and assume $u^*_{0,T}$ is an optimal control. Now for any $t < T$ we consider the same problem over time horizon $[t, T]$ and denote it by $P_{[t,T]}$. The dynamic problems $\{P_{[t,T]}\}_{t\in[0,T]}$ is said to be *time consistent* if $u^*_{0,T}|_{[t,T]}$ remains optimal for each $P_{[t,T]}$ and *time inconsistent* if it is not.

Following Strotz [21], there are two main approaches for dealing with time-inconsistent problems: *pre-commitment strategy* and *consistent planning*. The former approach essentially ignores the inconsistency issue and studies only the problem $P_{[0,T]}$, so it can be viewed as a static problem. The consistent planning approach, also known as the *game approach*, assumes that the agent plays with future selves and tries to find an equilibrium. This approach is by nature dynamic, backward in time, and time consistent; and the solution is subgame optimal. Since Ekeland andLazrak [7], the game approach has gained strong traction in the math finance community (see, e.g., Bjork andMurgoci [3], Bjork et al. [4], Hu et al. [11], and Yong [25], to mention a few). We remark, however, that mathematically the two approaches actually produce different values.

In Karnam et al. [15] the authors suggested a different perspective. Instead of using a predetermined "utility" function for all problems $P_{[t,T]}$ as in the game approach (in the context of probability distortion this means using the same φ in (3) for all $0 \le s < t \le T$), in [15] a *dynamic utility* is introduced, in the spirit of the *predictable forward utility* in Musiela and Zariphopoulou [16], [17] and Angoshtar et al. [1], to formulate a new dynamic problem $\tilde{P}_{[t,T]}$, $t \in [0, T]$. This new dynamic problem is time consistent, and in the meantime, $\tilde{P}_{[0,T]}$ coincides with the precommitment $P_{[0,T]}$. We should note that similar idea also appeared in the works Cui et al. [6] and Feinstein andRudloff [8], [9]. In [15] it is also proposed to use the dynamic programing principle (DPP) to characterize the time consistency, rather than the aforementioned original definition using optimal control. Such a modification is particularly important in situations where the optimal control does not exist. Noting that the DPP is nothing but the "tower property" in the absence of control; we thus consider this paper the first step toward a more general goal.

2. Static and Dynamic Probability Distortions

In this section we define probability distortion and introduce the notion of a time-consistent dynamic distortion function.

2.1. Nonlinear Expectation Under Probability Distortion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathbb{L}^0_+(\mathcal{F})$ be the set of $\mathcal{F}$-measurable random variables $\xi \ge 0$. The notion of probability distortion (see, e.g., Zhou [26]) consists of two elements: (i) a "distortion function" and (ii) a Choquet-type integral that defines the "distorted expectation." More precisely, we have the following definition.

Definition 1.

(i) A mapping $\varphi : [0, 1] \to [0, 1]$ is called a distortion function if it is continuous, is strictly increasing, and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

(ii) For any random variable $\xi \in \mathbb{L}^0_+(\mathcal{F})$, the distorted expectation operator (with respect to the distortion function φ) is defined by (2). We denote $\mathbb{L}^1_\varphi(\mathcal{F}) := \{\xi \in \mathbb{L}^0_+(\mathcal{F}) : \mathscr{E}[\xi] < \infty\}$.

Remark 1.

(i) The requirement $\xi \ge 0$ is imposed mainly for convenience.

(ii) If $\varphi(p) = p$, then $\mathscr{E}[\xi] = \mathbb{E}^{\mathbb{P}}[\xi]$ is the standard expectation under $\mathbb{P}$.

(iii) The operator $\mathscr{E}[\cdot]$ is law invariant, namely, $\mathscr{E}[\xi]$ depends only on the law of ξ.

The following example shows that $\mathscr{E}$ is in general neither subadditive nor superadditive. In particular, it is beyond the scope of Peng [19], which studies subadditive nonlinear expectations.

Example 1. Assume ξ_1 is a Bernoulli random variable: $\mathbb{P}(\xi_1 = 0) = p$, $\mathbb{P}(\xi_1 = 1) = 1 - p$, and $\xi_2 := 1 - \xi_1$. Then clearly $\mathscr{E}[\xi_1 + \xi_2] = \mathscr{E}[[1]] = 1$. However, by (8), we have

$$\mathscr{E}[\xi_1] = \varphi(1-p), \quad \mathscr{E}[\xi_2] = \varphi(p), \quad \text{and thus} \quad \mathscr{E}[\xi_1] + \mathscr{E}[\xi_2] = \varphi(p) + \varphi(1-p).$$

Depending on φ and p, $\mathscr{E}[\xi_1] + \mathscr{E}[\xi_2]$ can be greater than or less than 1.

Proposition 1. *Assume all the random variables below are in $\mathbb{L}^0_+(\mathcal{F})$. Let $c, c_i \ge 0$ be constants.*
(i) $\mathscr{E}[c] = c$ and $\mathscr{E}[c\xi] = c\mathscr{E}[\xi]$.
(ii) If $\xi_1 \le \xi_2$, then $\mathscr{E}[\xi_1] \le \mathscr{E}[\xi_2]$. In particular, if $c_1 \le \xi \le c_2$, then $c_1 \le \mathscr{E}[\xi] \le c_2$.
(iii) Assume ξ_k converges to ξ in distribution, and $\xi^* := \sup_k \xi_k \in \mathbb{L}^1_\varphi(\mathcal{F})$. Then $\mathscr{E}[\xi_k] \to \mathscr{E}[\xi]$.

Proof. Since φ is increasing, (i) and (ii) can be verified straightforwardly. To see (iii), note that $\lim_{k\to\infty} \mathbb{P}(\xi_k \ge x) = \mathbb{P}(\xi \ge x)$ for all but countably many values of $x \in (0,\infty)$. By the continuity of φ, we have $\lim_{k\to\infty} \varphi(\mathbb{P}(\xi_k \ge x)) = \varphi(\mathbb{P}(\xi \ge x))$ for Lebesgue-a.e. $x \in [0,\infty)$. Moreover, since φ is increasing, $\varphi(\mathbb{P}(\xi_k \ge x)) \le \varphi(\mathbb{P}(\xi^* \ge x))$ for all k. By (2) and the dominated convergence theorem we have $\mathscr{E}[\xi_k] \to \mathscr{E}[\xi]$. □

We now present two special cases that will play a crucial role in our analysis. In particular, they will lead naturally to the concept of *distorted probability*. Let

$$\mathcal{I} := \{g : \mathbb{R} \to [0,\infty) : g \text{ is bounded, continuous, and increasing}\}. \tag{6}$$

Proposition 2.
(i) Assume $\eta \in \mathbb{L}^1_\varphi(\mathcal{F})$ takes only finitely many values $x_1, \cdots, x_n$. Then

$$\mathscr{E}[\eta] = \sum_{k=1}^{n} x_{(k)}\big[\varphi\big(\mathbb{P}\big(\eta \ge x_{(k)}\big)\big) - \varphi\big(\mathbb{P}\big(\eta \ge x_{(k+1)}\big)\big)\big], \tag{7}$$

where $x_{(1)} \le \cdots \le x_{(n)}$ are the ordered values of $x_1, \cdots, x_n$, and $x_{(n+1)} := \infty$.
In particular, if $x_1 < \cdots < x_n$ and $g \in \mathcal{I}$, then

$$\mathscr{E}[g(\eta)] = \sum_{k=1}^{n} g(x_k)\big[\varphi\big(\mathbb{P}\big(\eta \ge x_k\big)\big) - \varphi\big(\mathbb{P}\big(\eta \ge x_{k+1}\big)\big)\big]. \tag{8}$$

(ii) Assume $\eta \in \mathbb{L}^0(\mathcal{F})$ has density ρ, and $g \in \mathcal{I}$, $\varphi \in C^1([0,1])$. Then

$$\mathscr{E}[g(\eta)] = \int_{-\infty}^{\infty} g(x)\rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx. \tag{9}$$

Proof.
(i) Denote $x_{(0)} := 0$. It is clear that $\mathbb{P}(\eta \ge x) = \mathbb{P}(\eta \ge x_{(k)})$ for $x \in (x_{(k-1)}, x_{(k)}]$. Then

$$\mathscr{E}[\eta] = \int_0^\infty \varphi\big(\mathbb{P}(\eta \ge x)\big)dx = \sum_{k=1}^{n} \big[x_{(k)} - x_{(k-1)}\big]\varphi\big(\mathbb{P}\big(\eta \ge x_{(k)}\big)\big).,$$

which implies (8) by using a simple Abel rearrangement as well as the fact that $\varphi(\mathbb{P}(\eta \ge x_{(n+1)})) = 0$.
(ii) We proceed in four steps.
Step 1. Assume g is bounded, strictly increasing, and differentiable. Let $a := g(-\infty), b := g(\infty)$. Then, $\varphi(\mathbb{P}(g(\eta) \ge x)) = 1$, $x \le a$; $\varphi(\mathbb{P}(g(\eta) \ge x)) = 0$, $x \ge b$; and integration by parts yields

$$\begin{aligned}\mathscr{E}[g(\eta)] &= a + \int_a^b \varphi\big(\mathbb{P}(g(\eta) \ge x)\big)dx = a + \int_{-\infty}^{\infty} \varphi\big(\mathbb{P}(\eta \ge x)\big)g'(x)dx \\ &= a + \varphi\big(\mathbb{P}(\eta \ge x)\big)g(x)\big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} g(x)\frac{d}{dx}\big(\varphi\big(\mathbb{P}(\eta \ge x)\big)\big)dx \\ &= \int_{-\infty}^{\infty} g(x)\rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx. \end{aligned} \tag{10}$$

Step 2. Assume g is bounded, increasing, and continuous. One can easily construct g_n such that each g_n satisfies the requirements in Step 1 and g_n converges to g uniformly. By Step 1, (9) holds for each g_n. Sending $n \to \infty$ and applying Proposition 1(iii), we prove (9) for g.

Step 3. Assume g is increasing and bounded by a constant C. For any $\varepsilon > 0$, one can construct a continuous and increasing function g_ε and an open set O_ε such that $|g_\varepsilon| \le C$, $|g_\varepsilon(x) - g(x)| \le \varepsilon$ for $x \notin O_\varepsilon$, and the Lebesgue measure $|O_\varepsilon| \le \varepsilon$. Then (9) holds for each g_ε. Note that

$$\mathbb{E}\big[|g_\varepsilon(\eta) - g(\eta)|\big] \le \varepsilon + 2C\mathbb{P}(\eta \in O_\varepsilon) = \varepsilon + 2C\int_{O_\varepsilon} \rho(x)dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Then $g_\varepsilon(\eta) \to g(\eta)$ in distribution, and thus, $\mathscr{E}[g_\varepsilon(\eta)] \to \mathscr{E}[g(\eta)]$ by Proposition 1(iii). Similarly,

$$\int_{-\infty}^{\infty} |g_\varepsilon(x) - g(x)|\rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx \le \varepsilon + 2C\int_{O_\varepsilon} \rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx \to 0.$$

Then we obtain (9) for g.

Step 4. In the general case, denote $g_n := g \wedge n$. Then (9) holds for each g_n and $g_n \uparrow g$. By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx = \int_{-\infty}^{\infty} g(x)\rho(x)\varphi'\big(\mathbb{P}(\eta \ge x)\big)dx.$$

If $g(\eta) \in \mathbb{L}^1_\varphi(\mathcal{F})$, then by Proposition 1(iii) we obtain (9) for g. Now assume $\mathscr{E}[g(\eta)] = \infty$. Following the arguments in Proposition 1(iii), note that $\mathbb{P}(g_n(\eta) \ge x) \uparrow \mathbb{P}(g(\eta) \ge x)$ for Lebesgue-almost everywhere (a.e.) $x \in [0,\infty)$, as $n \to \infty$. Then by the monotone convergence theorem one can verify that $\mathscr{E}[g_n(\eta)] = \int_0^\infty \varphi(\mathbb{P}(g_n(\eta) \ge x))dx \uparrow \int_0^\infty \varphi(\mathbb{P}(g(\eta) \ge x))dx = \mathscr{E}[g(\eta)]$, proving (9) again. □

Remark 2.

(i) In the discrete case, Equation (8) can be interpreted as follows. For each k, define the *distorted probability* q_k by

$$q_k := \varphi\big(\mathbb{P}(\eta \ge x_k)\big) - \varphi\big(\mathbb{P}(\eta \ge x_{k+1})\big), \quad k = 1, 2, \ldots, n. \tag{11}$$

Then $q_k \ge 0$, $\sum_{k=1}^n q_k = 1$, and $\mathscr{E}[g(\eta)] = \sum_{k=1}^n g(x_k)q_k$. So $\{q_k\}$ plays the role of a "probability distribution," and $\mathscr{E}$ is the usual linear expectation under the (distorted) probability $\{q_k\}$. This observation will be the foundation of our analysis below.

(ii) In the continuous case, the situation is similar. Indeed, denote $\widetilde{\rho}(x) := \rho(x)\varphi'(\mathbb{P}(\eta \ge x))$. Then $\widetilde{\rho}$ is also a density function, and by (9), $\mathscr{E}[g(\eta)] = \int_{-\infty}^\infty g(x)\widetilde{\rho}(x)dx$ is the usual expectation under the distorted density $\widetilde{\rho}$ of η.

(iii) Although the operator $\mathscr{E} : \mathbb{L}^1_\varphi(\mathcal{F}) \to [0,\infty)$ is nonlinear in general, for fixed η, the restricted mapping $g \in \mathcal{I} \mapsto \mathscr{E}[g(\eta)]$ is linear under nonnegative linear combinations.

(iv) Actually, for any $\xi \in \mathbb{L}^1_\varphi(\mathcal{F})$, note that $F_\xi(x) := 1 - \varphi(\mathbb{P}(\xi \ge x))$, $x \ge 0$, is a cumulative distribution function (cdf) and thus defines a distorted probability measure $\mathbb{Q}^\xi$ such that $\mathscr{E}[\xi] = \mathbb{E}^{\mathbb{Q}^\xi}[\xi]$. However, this $\mathbb{Q}^\xi$ depends on ξ. The main feature in (8) and (9) is that, for a given η, we find a common distorted probability measure for all $\xi \in \{g(\eta) : g \in \mathcal{I}\}$.

2.2. Time Inconsistency

Let $0 \in \mathcal{T} \subset [0,\infty)$ be the set of possible times, and let $X = \{X_t\}_{t\in\mathcal{T}}$ be a Markov process with deterministic X_0. Denoting $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}} = \mathbb{F}^X$ as the filtration generated by X, we want to define an $\mathcal{F}_t$-measurable conditional expectation $\mathscr{E}_t[\xi]$ such that each $\mathscr{E}_t[\xi]$ is $\mathcal{F}_t$-measurable, and the following "tower property" (or "flow property") holds (we will consider $\mathscr{E}_{s,t}$ later on):

$$\mathscr{E}_s[\mathscr{E}_t[\xi]] = \mathscr{E}_s[\xi], \quad \text{for all } s,t \in \mathcal{T} \text{ such that } s < t. \tag{12}$$

We note that the tower property (12) is standard for the usual (linear) expectation as well as the sublinear G-expectation of Peng [19]. It is also a basic requirement of the so-called *dynamic risk measures* (see, e.g., BieleckI et al.[2]). However, under probability distortion, the simple-minded definition of the conditional expectation given by (3) could very well be time inconsistent. Here is a simple explicit example.

Example 2. Consider a two-period binomial tree model: $X_t = \sum_{i=1}^t \zeta_i$, $t \in \mathcal{T} := \{0,1,2\}$, where ζ_1 and ζ_2 are independent Rademacher random variables with $\mathbb{P}(\zeta_i = \pm 1) = \frac{1}{2}$, $i = 1/2$. Let $\varphi(p) := p^2$, let $\xi := g(X_2)$ for some strictly increasing function g, and let $\mathcal{E}_1[\xi]$ be defined by (3). Then

$$\mathcal{E}[\mathcal{E}_1[\xi]] \neq \mathcal{E}[\xi]. \tag{13}$$

Proof. By (8), we have

$$\mathcal{E}_1[\xi]|_{X_1=-1} = g(-2)\left[1-\varphi\left(\frac{1}{2}\right)\right] + g(0)\varphi\left(\frac{1}{2}\right), \quad \mathcal{E}_1[\xi]|_{X_1=1} = g(0)\left[1-\varphi\left(\frac{1}{2}\right)\right] + g(2)\varphi\left(\frac{1}{2}\right).$$

Note that $\mathcal{E}_1[\xi]|_{X_1=-1} < \mathcal{E}_1[\xi]|_{X_1=1}$ since g is strictly increasing. Then, by (8) again, we have

$$\begin{aligned}\mathcal{E}[\mathcal{E}_1[\xi]] &= \mathcal{E}_1[\xi]|_{X_1=-1}\left[1-\varphi\left(\frac{1}{2}\right)\right] + \mathcal{E}_1[\xi]|_{X_1=1}\varphi\left(\frac{1}{2}\right)\\ &= g(-2)\left[1-\varphi\left(\frac{1}{2}\right)\right]^2 + 2g(0)\varphi\left(\frac{1}{2}\right)\left[1-\varphi\left(\frac{1}{2}\right)\right] + g(2)\left[\varphi\left(\frac{1}{2}\right)\right]^2 = \frac{9}{16}g(-2) + \frac{3}{8}g(0) + \frac{1}{16}g(2). \end{aligned} \tag{14}$$

On the other hand, by (8) we also have

$$\mathcal{E}[\xi] = g(-2)\left[1-\varphi\left(\frac{3}{4}\right)\right] + g(0)\left[\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{4}\right)\right] + g(2)\varphi\left(\frac{1}{4}\right) = \frac{7}{16}g(-2) + \frac{1}{2}g(0) + \frac{1}{16}g(2). \tag{15}$$

Comparing (14) and (15) and noting that $g(-2) < g(0)$, we obtain $\mathcal{E}[\mathcal{E}_1[\xi]] < \mathcal{E}[\xi]$. □

2.3. Time-Consistent Dynamic Distortion Function

As mentioned in the Introduction, an apparent reason for the time inconsistency of the "naive" distorted conditional expectation (3) is that the distortion function φ is time invariant. Motivated by the idea of *dynamic utility* in Karnam et al. [15], we introduce the notion of a *time-consistent dynamic distortion function* which forms the framework of this paper. Denote

$$\mathcal{T}_2 := \{(s,t) \in \mathcal{T} \times \mathcal{T} : s < t\}.$$

Definition 2.

(i) A mapping $\Phi : \mathcal{T}_2 \times \mathbb{R} \times [0,1] \to [0,1]$ is called a dynamic distortion function if it is jointly Lebesgue measurable in (x,p) for any $(s,t) \in \mathcal{T}_2$ and, for each $(s,t,x) \in \mathcal{T}_2 \times \mathbb{R}$, the mapping $p \in [0,1] \mapsto \Phi(s,t,x;p)$ is a distortion function in the sense of Definition 1.

(ii) Given a dynamic distortion function Φ, for any $(s,t) \in \mathcal{T}_2$ we define $\mathcal{E}_{s,t}$ as follows:

$$\mathcal{E}_{s,t}[\xi] := \int_0^\infty \Phi(s,t,X_s;\mathbb{P}(\xi \geq x|\mathcal{F}_s))dx, \quad \xi \in \mathbb{L}^0_+(\sigma(X_t)). \tag{16}$$

(iii) We say a dynamic distortion function Φ is time consistent if the tower property holds:

$$\mathcal{E}_{r,t}[g(X_t)] = \mathcal{E}_{r,s}[\mathcal{E}_{s,t}[g(X_t)]], \quad r,s,t \in \mathcal{T},\ 0 \leq r < s < t \leq T,\ g \in \mathcal{I}. \tag{17}$$

Remark 3.

(i) Compared with the naive definition (3), the dynamic distortion function in (16) depends also on the current time s, the "terminal" time t, and the current state x. This enables us to describe different (distorted) perceptions of future events at different times and states. For example, people may feel very differently toward a catastrophic event that might happen tomorrow as opposed to ten years later with the same probability.

(ii) In this paper we apply $\mathcal{E}_{s,t}$ only on $\xi = g(X_t)$ for some $g \in \mathcal{I}$. As we saw in Remark 2, in this case the operator $\mathcal{E}_{s,t}$ will be linear in g. The general case with nonmonotone g (or even path-dependent ξ) seems to be very challenging and will be left to future research (see Remark 4). It is worth noting, however, that in many applications g is a utility function, which is indeed increasing.

(iii) Given $g \in \mathcal{I}$, one can easily show that $\mathcal{E}_{s,t}[g(X_t)] = u(s,X_s)$ for some function $u(s,\cdot) \in \mathcal{I}$. This justifies the right-hand side of (17).

Now, for each $0 < t \in \mathcal{T}$, we assume that an initial distortion function $\varphi_t(\cdot)$ is given (a possible choice is $\varphi_t \equiv \varphi$) as the perspective at time 0 toward the future events at $t > 0$. Our goal is to construct a time-consistent dynamic distortion function Φ such that $\Phi(0, t, X_0; \cdot) = \varphi_t(\cdot)$ for all $0 < t \in \mathcal{T}$. We shall consider models both in discrete time and in continuous time.

3. The Binomial Tree Case

In this section we consider a binomial tree model which contains all the main ideas of our approach. Let $\{\varphi_t\}_{t\in\mathcal{T}\setminus\{0\}}$ be a given family of initial distortion functions.

3.1. The Two-Period Binomial Tree Case

To illustrate our main idea, let us first consider the simplest case when X follows a two-period binomial tree as in Example 2 (see the left graph in Figure 1). Let $\xi = g(X_2)$, where $g \in \mathcal{I}$. We shall construct $\Phi(1, 2, x; p)$ and $\mathscr{E}_{1,2}[\xi]$.

Note that $\Phi(0, t, 0; \cdot) = \varphi_t(\cdot)$ for $t = 1, 2$. By (8) we have

$$\mathscr{E}_{0,2}[\xi] = g(-2)\left[\varphi_2(1) - \varphi_2\left(\frac{3}{4}\right)\right] + g(0)\left[\varphi_2\left(\frac{3}{4}\right) - \varphi_2\left(\frac{1}{4}\right)\right] + g(2)\left[\varphi_2\left(\frac{1}{4}\right) - \varphi_2(0)\right]. \tag{18}$$

Here we write $\varphi_2(0)$ and $\varphi_2(1)$, although their values are 0 and 1, so that Equation (22) will be more informative when extending it to multiperiod models. Assume $\mathscr{E}_{1,2}[\xi] = u(1, X_1)$. Then by definition we should have

$$u(1, -1) = g(-2)\left[1 - \Phi\left(1, 2, -1; \frac{1}{2}\right)\right] + g(0)\Phi\left(1, 2, -1; \frac{1}{2}\right), \tag{19}$$

$$u(1, 1) = g(0)\left[1 - \Phi\left(1, 2, 1; \frac{1}{2}\right)\right] + g(2)\Phi\left(1, 2, 1; \frac{1}{2}\right). \tag{20}$$

Assume now that $u(1, \cdot)$ is also increasing. Then by (8) again we have

$$\mathscr{E}_{0,1}\left[\mathscr{E}_{1,2}[\xi]\right] = \mathscr{E}_{0,1}[u(1, X_1)] = u(1, -1)\left[\varphi_1(1) - \varphi_1\left(\frac{1}{2}\right)\right] + u(1, 1)\left[\varphi_1\left(\frac{1}{2}\right) - \varphi_1(0)\right]. \tag{21}$$

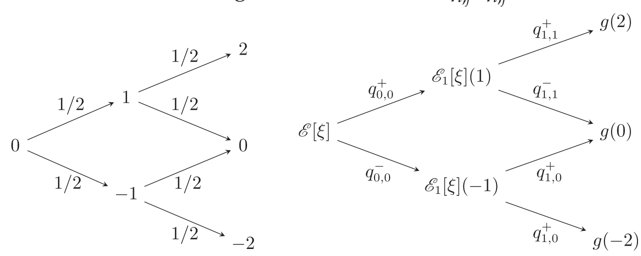
Plugging (19) into (21) gives

$$\begin{aligned}\mathscr{E}_{0,1}\left[\mathscr{E}_{1,2}[\xi]\right] = {}& g(-2)\left[1 - \Phi\left(1, 2, -1; \frac{1}{2}\right)\right]\left[\varphi_1(1) - \varphi_1\left(\frac{1}{2}\right)\right] + g(2)\Phi\left(1, 2, 1; \frac{1}{2}\right)\left[\varphi_1\left(\frac{1}{2}\right) - \varphi_1(0)\right] \\ & + g(0)\left[\Phi\left(1, 2, -1; \frac{1}{2}\right)\left[\varphi_1(1) - \varphi_1\left(\frac{1}{2}\right)\right] + \left[1 - \Phi\left(1, 2, 1; \frac{1}{2}\right)\right]\left[\varphi_1\left(\frac{1}{2}\right) - \varphi_1(0)\right]\right].\end{aligned}$$

Recall from (17) that we want the above to be equal to (18) for all $g \in \mathcal{I}$. This leads to a natural and unique choice:

$$\Phi\left(1, 2, -1; \frac{1}{2}\right) := \frac{\varphi_2\left(\frac{3}{4}\right) - \varphi_1\left(\frac{1}{2}\right)}{\varphi_1(1) - \varphi_1\left(\frac{1}{2}\right)}, \quad \Phi\left(1, 2, 1; \frac{1}{2}\right) := \frac{\varphi_2\left(\frac{1}{4}\right) - \varphi_1(0)}{\varphi_1\left(\frac{1}{2}\right) - \varphi_1(0)}. \tag{22}$$

Figure 1. Two-period binomial tree: left for X and right for $\mathscr{E}_t[\xi]$, with $(q^+_{i,j}, q^-_{i,j})$ in (26).



Consequently, (19) now reads

$$u(1,-1) = g(-2)\left[1 - \Phi\left(1,2,-1;\frac{1}{2}\right)\right] + g(0)\Phi\left(1,2,-1;\frac{1}{2}\right), \tag{23}$$

$$u(1,1) = g(0)\left[1 - \Phi\left(1,2,1;\frac{1}{2}\right)\right] + g(2)\Phi\left(1,2,1;\frac{1}{2}\right). \tag{24}$$

Note that $\varphi_2(\cdot)$ is strictly increasing. Assuming further $\varphi_2(1/4) < \varphi_1(1/2) < \varphi_2(3/4)$ and using (22), we have

$$0 < \Phi\left(1,2,-1;\frac{1}{2}\right) < 1, \quad 0 < \Phi\left(1,2,1;\frac{1}{2}\right) < 1. \tag{25}$$

Note that (23) and (25) imply that $u(1,-1) \le g(0) \le u(1,1)$; thus, $u(1,\cdot)$ is indeed increasing.

Finally, we note that the distorted expectations $\mathscr{E}_{0,1}[u(1,X_1)]$ and $\mathscr{E}_{0,2}[g(X_2)]$ and the distorted conditional expectation $\mathscr{E}_{1,2}[g(X_2)]$ can be viewed as a standard expectation and conditional expectation, but under a new *distorted probability measure* described in the right graph in Figure 1, where

$$q^+_{0,0} := \varphi_1\left(\frac{1}{2}\right), \quad q^+_{1,1} := \frac{\varphi_2(\frac{1}{4}) - \varphi_1(0)}{\varphi_1(\frac{1}{2}) - \varphi_1(0)}, \quad q^+_{1,0} := \frac{\varphi_2(\frac{3}{4}) - \varphi_1(\frac{1}{2})}{\varphi_1(1) - \varphi_1(\frac{1}{2})}, \quad q^-_{i,j} := 1 - q^+_{i,j}. \tag{26}$$

This procedure resembles finding the risk-neutral measure in option pricing theory, whereas the arguments of φ_t in (26) represent the quantiles of the simple random walk.

Remark 4.

(i) We now explain why it is crucial to assume $g \in \mathcal{I}$. Indeed, assume instead that g is decreasing. Then by (7) and following similar arguments we can see that

$$\Phi\left(1,2,1;\frac{1}{2}\right) = \frac{\varphi_2(\frac{3}{4}) - \varphi_1(\frac{1}{2})}{\varphi_1(1) - \varphi_1(\frac{1}{2})}, \quad \Phi\left(1,2,-1;\frac{1}{2}\right) = \frac{\varphi_2(\frac{1}{4}) - \varphi_1(0)}{\varphi_1(\frac{1}{2}) - \varphi_1(0)}.$$

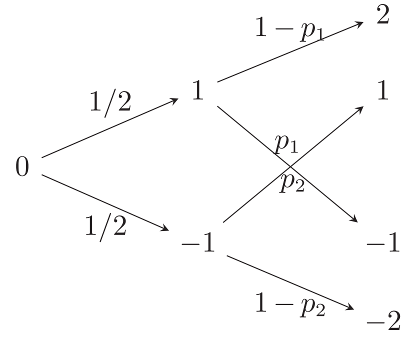
This is in general different from (22). That is, we cannot find a common time-consistent dynamic distortion function which works for both increasing and decreasing functions g.

(ii) For a fixed (possibly nonmonotone) function $g : \mathbb{R} \to [0,\infty)$, it is possible to construct Φ such that $\mathscr{E}_{0,2}[g(X_2)] = \mathscr{E}_{0,1}[\mathscr{E}_{1,2}[g(X_2)]]$. However, this Φ may depend on g. It seems to us that this is too specific and, thus, is not desirable.

(iii) Another challenging case is when X has crossing edges. This destroys the crucial monotonicity in a different way and Φ may not exist, as we shall see in Example 3. There are two ways to understand the main difficulty here: for the binary tree in Figure 2 and for $g \in \mathcal{I}$.

- $u(1,-1)$ is the weighted average of $g(-2)$ and $g(1)$, and $u(1,1)$ is the weighted average of $g(-1)$ and $g(2)$. Since $g(-1) < g(1)$, for any given Φ, there exists some $g \in \mathcal{I}$ such that $u(1,-1) > u(1,1)$, namely, $u(1,\cdot)$ is not increasing in x.
- In $\mathscr{E}_{1,2}[g(X_2)]$ the conditional probability $p_2 = \mathbb{P}(X_2 = 1 | X_1 = -1)$ would contribute to the weight of $g(1)$, but not to that of $g(-1)$. However, since $g(-1) < g(1)$, in $\mathscr{E}_{0,2}[g(X_2)]$ the p_2 will contribute to the weight of $g(1)$

Figure 2. A two-period binary tree with crossing edges.



as well. This discrepancy destroys the tower property. The same issue also arises in continuous time when the diffusion coefficient is nonconstant.

The following example shows that it is essential to require that the tree is recombining.

Example 3. Assume X follows the binary tree in Figure 2 and $g \in \mathcal{I}$ is strictly increasing. Then in general there is no time-consistent dynamic distortion function Φ.

Proof. By (7) we have

$$\mathscr{E}_{0,2}[g(X_2)] = g(-2)\left[1-\varphi_2\left(\frac{1+p_2}{2}\right)\right] + g(-1)\left[\varphi_2\left(\frac{1+p_2}{2}\right) - \varphi_2\left(\frac{1-p_1+p_2}{2}\right)\right]$$
$$+ g(1)\left[\varphi_2\left(\frac{1-p_1+p_2}{2}\right) - \varphi_2\left(\frac{1-p_1}{2}\right)\right] + g(2)\varphi_2\left(\frac{1-p_1}{2}\right).$$

Assume $\mathscr{E}_{1,2}[g(X_2)] = u(1, X_1)$. Then by definition we should have

$$u(1,-1) = g(-2)[1-\Phi(1,2,-1;p_2)] + g(1)\Phi(1,2,-1;p_2),$$
$$u(1,1) = g(-1)[1-\Phi(1,2,1;1-p_1)] + g(2)\Phi(1,2,1;1-p_1).$$

Assume without loss of generality that $u(1,-1) < u(1,1)$, and the case $u(1,1) < u(1,-1)$ can be analyzed similarly. Then

$$\mathscr{E}_{0,1}[\mathscr{E}_{1,2}[g(X_2)]] = g(-2)[1-\Phi(1,2,-1;p_2)]\left[1-\varphi_1\left(\frac{1}{2}\right)\right] + g(1)\Phi(1,2,-1;p_2)\left[1-\varphi_1\left(\frac{1}{2}\right)\right]$$
$$+ g(-1)[1-\Phi(1,2,1;1-p_1)]\varphi_1\left(\frac{1}{2}\right) + g(2)\Phi(1,2,1;1-p_1)\varphi_1\left(\frac{1}{2}\right).$$

If the tower property $\mathscr{E}_{0,2}[g(X_2)] = \mathscr{E}_{0,1}[\mathscr{E}_{1,2}[g(X_2)]]$ holds for all $g \in \mathcal{I}$, then comparing the weights of $g(-1)$ and $g(2)$ we have

$$[1-\Phi(1,2,1;1-p_1)]\varphi_1\left(\frac{1}{2}\right) = \varphi_2\left(\frac{1+p_2}{2}\right) - \varphi_2\left(\frac{1-p_1+p_2}{2}\right),$$
$$\Phi(1,2,1;1-p_1)\varphi_1\left(\frac{1}{2}\right) = \varphi_2\left(\frac{1-p_1}{2}\right).$$

Adding the two terms above, we have

$$\varphi_1\left(\frac{1}{2}\right) = \varphi_2\left(\frac{1+p_2}{2}\right) - \varphi_2\left(\frac{1-p_1+p_2}{2}\right) + \varphi_2\left(\frac{1-p_1}{2}\right). \tag{27}$$

This equality does not always hold. In other words, unless φ satisfies (27), there is no time-consistent Φ for the model in Figure 2. □

3.2. The General Binomial Tree Case

We now extend our idea to a general binomial tree model. Let $\mathcal{T}$ consist of the points $0 = t_0 < \cdots < t_N$, and let $X = \{X_{t_i}\}_{0 \le i \le N}$ be a finite-state Markov process such that, for each $i = 0, \ldots, N$, X_{t_i} takes values $x_{i,0} < \cdots < x_{i,i}$ and has the following transition probabilities:

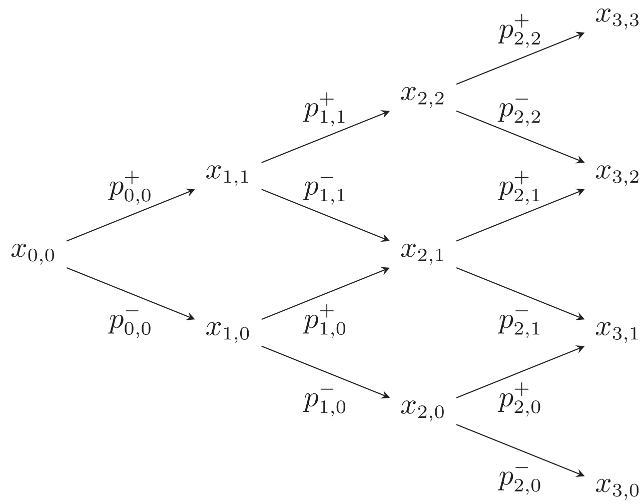
$$\mathbb{P}(X_{t_{i+1}} = x_{i+1,j+1} | X_{t_i} = x_{i,j}) = p^+_{i,j}, \quad \mathbb{P}(X_{t_{i+1}} = x_{i+1,j} | X_{t_i} = x_{i,j}) = p^-_{i,j} := 1 - p^+_{i,j}, \tag{28}$$

where $p^\pm_{ij} > 0$ (see Figure 3 for the case $N = 3$). We also assume that for each $t_i \in \mathcal{T} \setminus \{0\}$ we are given a distortion function φ_{t_i}.

Motivated by the analysis in Section 3.1, we shall find a distorted probability measure $\mathbb{Q}$ so that

$$\mathscr{E}_{s,t}[g(X_t)] = \mathbb{E}^{\mathbb{Q}}[g(X_t) | \sigma(X_s)] \quad \text{for all } g \in \mathcal{I}. \tag{29}$$

Figure 3. Three-period binomial tree for X.



This implies that the tower property of $\mathscr{E}_{s,t}$ immediately and naturally leads to a time-consistent dynamic distortion function. Keeping (26) in mind, we define the following distorted probabilities for the binomial tree model: for $0 \le j \le i \le N$,

$$q^+_{i,j} := \frac{\varphi_{t_{i+1}}(G_{i+1,j+1}) - \varphi_{t_i}(G_{i,j+1})}{\varphi_{t_i}(G_{i,j}) - \varphi_{t_i}(G_{i,j+1})}, \quad q^-_{i,j} := 1 - q^+_{i,j}, \quad \text{where} \quad G_{i,j} := \mathbb{P}(X_{t_i} \ge x_{i,j}). \tag{30}$$

We assume further that $G_{i,i+1} := 0$ and $\varphi_0(p) := p$. From (30), in order to have $0 < q^+_{i,j} < 1$, it suffices to (and we will) assume that

$$\varphi_{t_i}(G_{i,j+1}) < \varphi_{t_{i+1}}(G_{i+1,j+1}) < \varphi_{t_i}(G_{i,j}), \quad \text{for all } (i,j). \tag{31}$$

Intuitively, (31) is a technical condition which states that $\varphi_\cdot$ cannot change too quickly in time. Clearly this condition is satisfied when $\varphi_t \equiv \varphi$. Now let $\mathbb{Q}$ be the (equivalent) probability measure under which X is Markov with transition probabilities given by

$$\mathbb{Q}(X_{t_{i+1}} = x_{i+1,j+1} | X_{t_i} = x_{i,j}) = q^+_{i,j}, \quad \mathbb{Q}(X_{t_{i+1}} = x_{i+1,j} | X_{t_i} = x_{i,j}) = q^-_{i,j}. \tag{32}$$

We first have the following simple lemma.

Lemma 1. *Assume* (31) *holds and* $g \in \mathcal{I}$. *For* $0 < n \le N$, *define* $u_n(x) := g(x)$, *and for* $i = n-1, \dots, 0$,

$$u_i(x_{i,j}) := q^+_{i,j} u_{i+1}(x_{i+1,j+1}) + q^-_{i,j} u_{i+1}(x_{i+1,j}), \quad j = 0, \dots, i. \tag{33}$$

Then u_i *is increasing and* $\mathbb{E}^{\mathbb{Q}}[g(X_{t_n}) | \mathcal{F}_{t_i}] = u_i(X_{t_i})$.

Proof. It is obvious from the binomial tree structure that $\mathbb{E}^{\mathbb{Q}}[g(X_{t_n}) | \mathcal{F}_{t_i}] = u_i(X_{t_i})$. We prove the monotonicity of u_i by backward induction. First, $u_n = g$ is increasing. Assume u_{i+1} is increasing. Then, noting that $x_{i,j}$'s are increasing in j and $q^+_{i,j} + q^-_{i,j} = 1$ for all i, j, by (33) we have

$$\begin{aligned} u_i(x_{i,j}) &\le q^+_{i,j} u_{i+1}(x_{i+1,j+1}) + q^-_{i,j} u_{i+1}(x_{i+1,j+1}) = u_{i+1}(x_{i+1,j+1}) \\ &\le q^+_{i,j+1} u_{i+1}(x_{i+1,j+2}) + q^-_{i,j+1} u_{i+1}(x_{i+1,j+1}) = u_i(x_{i,j+1}). \end{aligned}$$

Thus, u_i is also increasing. □

We remark that (33) can be viewed as a "discrete PDE." This idea motivates our treatment of the continuous time model in the next section.

The following is our main result of this section.

Theorem 1. *Assume* (31). *Then there exists a unique time-consistent dynamic distortion function* Φ *such that* $\Phi(t_0, t_n, x_{0,0}; p) = \varphi_{t_n}(p)$ *for* $n = 1, \dots, N$, *and for all* $0 \le i < n \le N$, $0 \le j \le i$, *and* $0 \le k \le n$, *we have*

$$\Phi(t_i, t_n, x_{i,j}; \mathbb{P}\{X_{t_n} \ge x_{n,k} | X_{t_i} = x_{i,j}\}) = \mathbb{Q}\{X_{t_n} \ge x_{n,k} | X_{t_i} = x_{i,j}\}. \tag{34}$$

Here uniqueness is only at the conditional survival probabilities for all k in the left-hand side of (34). Moreover, the corresponding conditional nonlinear expectation satisfies (29).

Proof. We first show that (34) has a solution satisfying the desired initial conditon. Note that both $\mathbb{P}\{X_{t_n} \ge x_{n,k}|X_{t_i} = x_{i,j}\}$ and $\mathbb{Q}\{X_{t_n} \ge x_{n,k}|X_{t_i} = x_{i,j}\}$ are strictly decreasing in k, for fixed $0 \le i < n \le N$ and $x_{i,j}$. Then one can easily define a function Φ, depending on $t_i, t_n, x_{i,j}$, so that (34) holds for all $x_{n,k}$, $0 \le k \le n$. Moreover, the initial condition $\Phi(t_0, t_n, x_{0,0}; p) := \varphi_{t_n}(p)$ is equivalent to

$$\varphi_{t_n}\big(\mathbb{P}\{X_{t_n} \ge x_{n,k}\}\big) = \mathbb{Q}\{X_{t_n} \ge x_{n,k}\}, \quad 0 \le n \le N, \quad 0 \le k \le n. \tag{35}$$

We shall prove (35) by induction on n. First recall $\varphi_0(p) = p$ and that $\mathbb{P}\{X_{t_0} = x_{0,0}\} = \mathbb{Q}\{X_{t_0} = x_{0,0}\} = 1$; thus, (35) obviously holds for $n = 0$. Assume now it holds for $n < N$. Then

$$\begin{aligned}
\mathbb{Q}\{X_{t_{n+1}} = x_{n+1,k}\} &= \mathbb{Q}\{X_{t_n} = x_{n,k-1}\}q^+_{n,k-1} + \mathbb{Q}\{X_{t_n} = x_{n,k}\}q^-_{n,k}\\
&= \big[\mathbb{Q}\{X_{t_n} \ge x_{n,k-1}\} - \mathbb{Q}\{X_{t_n} \ge x_{n,k}\}\big]q^+_{n,k-1} + \big[\mathbb{Q}\{X_{t_n} \ge x_{n,k}\} - \mathbb{Q}\{X_{t_n} \ge x_{n,k+1}\}\big]q^-_{n,k}\\
&= \Big[\varphi_{t_n}(G_{n,k-1}) - \varphi_{t_n}(G_{n,k})\Big]q^+_{n,k-1} + \Big[\varphi_{t_n}(G_{n,k}) - \varphi_{t_n}(G_{n,k+1})\Big]\big[1 - q^+_{n,k}\big]\\
&= \Big[\varphi_{t_{n+1}}(G_{n+1,k}) - \varphi_{t_n}(G_{n,k})\Big] + \Big[\varphi_{t_n}(G_{n,k}) - \varphi_{t_{n+1}}(G_{n+1,k+1})\Big]\\
&= \varphi_{t_{n+1}}(G_{n+1,k}) - \varphi_{t_{n+1}}(G_{n+1,k+1}).
\end{aligned}$$

This leads to (35) for $n+1$ immediately and, thus, completes the induction step.

We next show that the above-constructed Φ is indeed a time-consistent dynamic distortion function. We first remark that, for this discrete model, only the values of Φ on the left-hand side of (34) are relevant, and one may extend Φ to all $p \in [0,1]$ by linear interpolation. Then by (34) it is straightforward to show that $\Phi(t_i, t_n, x_{i,j}; \cdot)$ satisfies Definition 1(i). Moreover, by (16), (8), and (34), for any $g \in \mathcal{I}$ we have

$$\begin{aligned}
\mathscr{E}_{t_i,t_n}[g(X_{t_n})]\big|_{X_{t_i}=x_{i,j}} &= \sum_{k=0}^{n} g(x_{n,k})\big[\Phi\big(t_i, t_n, x_{i,j}; \mathbb{P}\{X_{t_n} \ge x_{n,k}|X_{t_i} = x_{i,j}\}\big)\\
&\qquad -\Phi\big(t_i, t_n, x_{i,j}; \mathbb{P}\{X_{t_n} \ge x_{n,k+1}|X_{t_i} = x_{i,j}\}\big)\big]\\
&= \sum_{k=0}^{n} g(x_{n,k})\big[\mathbb{Q}\{X_{t_n} \ge x_{n,k}|X_{t_i} = x_{i,j}\} - \mathbb{Q}\{X_{t_n} \ge x_{n,k+1}|X_{t_i} = x_{i,j}\}\big]\\
&= \mathbb{E}^{\mathbb{Q}}\big[g(X_{t_n})|X_{t_i} = x_{i,j}\big].
\end{aligned}$$

That is, (29) holds. Moreover, fix n and g, and let u_i be as in Lemma 1. Since u_m is increasing, we have

$$\mathscr{E}_{t_i,t_m}\big[\mathscr{E}_{t_m,t_n}[g(X_{t_n})]\big] = \mathscr{E}_{t_i,t_m}[u_m(X_{t_m})] = u_i(X_{t_i}) = \mathscr{E}_{t_i,t_n}[g(X_{t_n})], \quad 0 \le i < m < n.$$

This verifies (17). Thus, Φ is a time-consistent dynamic distortion function.

It remains to prove the uniqueness of Φ. Assume Φ is an arbitrary time-consistent dynamic distortion function. For any appropriate i, j, and $g \in \mathcal{I}$, following the arguments of Lemma 1 we see that

$$\begin{aligned}
u(t_i, x_{i,j}) &:= \mathscr{E}_{t_i,t_{i+1}}[g(X_{t_{i+1}})]\big|_{X_{t_i}=x_{i,j}}\\
&= g(x_{i+1,j})\Big[1 - \Phi\Big(t_i, t_{i+1}, x_{i,j}; p^+_{i,j}\Big)\Big] + g(x_{i+1,j+1})\Phi\Big(t_i, t_{i+1}, x_{i,j}; p^+_{i,j}\Big)
\end{aligned}$$

is increasing in $x_{i,j}$. Then by (8) and the tower property we have

$$\begin{aligned}
&\sum_k g(x_{i+1,k})\Big[\varphi_{t_{i+1}}(G_{i+1,k}) - \varphi_{t_{i+1}}(G_{i+1,k+1})\Big] = \mathscr{E}_{0,t_{i+1}}[g(X_{t_{i+1}})] = \mathscr{E}_{0,t_i}\big[\mathscr{E}_{t_i,t_{i+1}}[g(X_{t_{i+1}})]\big].\\
&\quad = \mathscr{E}_{0,t_i}[u(t_i, X_{t_i})] = \sum_j u(t_i, x_{i,j})\Big[\varphi_{t_i}(G_{i,j}) - \varphi_{t_i}(G_{i,j+1})\Big]\\
&\quad = \sum_j \Big[g(x_{i+1,j})\Big[1 - \Phi\Big(t_i, t_{i+1}, x_{i,j}; p^+_{i,j}\Big)\Big] + g(x_{i+1,j+1})\Phi\Big(t_i, t_{i+1}, x_{i,j}; p^+_{i,j}\Big)\Big]\\
&\quad\quad \times \Big[\varphi_{t_i}(G_{i,j}) - \varphi_{t_i}(G_{i,j+1})\Big].
\end{aligned}$$

By the arbitrariness of $g \in \mathcal{I}$, this implies that

$$
\begin{aligned}
1-\varphi_{t_{i+1}}(G_{i+1,1}) &= \Big[1-\Phi\big(t_i,t_{i+1},x_{i,0};p^+_{i,0}\big)\Big]\Big[1-\varphi_{t_i}(G_{i,1})\Big];\\
\varphi_{t_{i+1}}\big(G_{i+1,k}\big)-\varphi_s t_{i+1}\big(G_{i+1,k+1}\big) &= \Big[1-\Phi\big(t_i,t_{i+1},x_{i,k};p^+_{i,k}\big)\Big]\Big[\varphi_{t_i}\big(G_{i,k}\big)-\varphi_{t_i}\big(G_{i,k+1}\big)\Big]\\
&\quad+\Phi\big(t_i,t_{i+1},x_{i,k-1};p^+_{i,k-1}\big)\Big[\varphi_{t_i}\big(G_{i,k-1}\big)-\varphi_{t_i}\big(G_{i,k}\big)\Big],\quad k=1,\dots,i+1.
\end{aligned}
$$

This is equivalent to, denoting $a_k := \Phi(t_i,t_{i+1},x_{i,k};p^+_{i,k})[\varphi_{t_i}(G_{i,k})-\varphi_{t_i}(G_{i,k+1})]$,

$$
\begin{aligned}
a_0 &= \varphi_{t_{i+1}}\big(G_{i+1,1}\big)-\varphi_{t_i}\big(G_{i,1}\big);\\
a_{k-1}-a_k &= \Big[\varphi_{t_{i+1}}\big(G_{i+1,k}\big)-\varphi_{t_{i+1}}\big(G_{i+1,k+1}\big)\Big]-\Big[\varphi_{t_i}\big(G_{i,k}\big)-\varphi_{t_i}\big(G_{i,k+1}\big)\Big].
\end{aligned}
$$

Clearly the above equations have a unique solution, so we must have $\Phi(t_i,t_{i+1},x_{i,k};p^+_{i,k}) = q^+_{i,k}$. This implies further that $\mathscr{E}_{t_i,t_{i+1}}[g(X_{t_{i+1}})] = \mathbb{E}^{\mathbb{Q}}[g(X_{t_{i+1}})|\mathcal{F}_{t_i}.]$. Now both $\mathscr{E}_{t_i,t_n}$ and $\mathbb{E}^{\mathbb{Q}}[\cdot|\cdot]$ satisfy the tower property. Then $\mathscr{E}_{t_i,t_n}[g(X_{t_n})] = \mathbb{E}^{\mathbb{Q}}[g(X_{t_n})|\mathcal{F}_{t_i}.]$ for all $t_i < t_n$ and all $g \in \mathcal{I}$. So $\mathscr{E}_{t_i,t_n}$ is unique, which implies immediately the uniqueness of Φ. □

Remark 5.

(i) We should note that the dynamic distortion function Φ that we constructed actually depends on the survival function of X under both $\mathbb{P}$ and $\mathbb{Q}$ (see also (57)).

(ii) Our construction of Φ is local in time. In particular, all the results can be easily extended to the case with infinite times: $0 = t_0 < t_1 < \cdots$.

(iii) Our construction of Φ is also local in state, in the sense that $\Phi(t_i,t_n,x_{i,j};\cdot)$ involves only the subtree rooted at $(t_i,x_{i,j})$.

4. The Constant-Diffusion Case

In this section we set $\mathcal{T} = [0,T]$ and consider the case where the underlying state process X is a one-dimensional Markov process satisfying the following SDE with constant-diffusion coefficient:

$$X_t = x_0 + \int_0^t b(s,X_s)ds + B_t, \tag{36}$$

where B is a one-dimensional standard Brownian motion on a given filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},\mathbb{P})$. Again we are given initial distortion functions $\{\varphi_t = \varphi(t,\cdot)\}_{0<t\le T}$ and $\varphi_0(p) \equiv p$. Our goal is to construct a time-consistent dynamic distortion function Φ and the corresponding time-consistent distorted conditional expectations $\mathscr{E}_{s,t}$ for $(s,t) \in \mathcal{T}_2$. We shall impose the following technical conditions.

Assumption 1. *The function b is sufficiently smooth, and both b and the required derivatives are bounded.*

Clearly, under the assumption the SDE (36) is well posed. The further regularity of b is used to derive some tail estimates for the density of X_t, which are required for our construction of the time-consistent dynamic distortion function Φ and the distorted probability measure $\mathbb{Q}$. By investigating our arguments more carefully, we can figure out the precise technical conditions we will need. However, since our main focus is the dynamic distortion function Φ, we prefer not to carry out these details for the sake of the readability of the paper.

4.1. Binomial Tree Approximation

Our idea is to approximate X by a sequence of binomial trees and then apply the results from the previous section. To this end, for fixed N, denote $h := T/N$ and $t_i := ih$, $i = 0,\cdots,N$. Then (36) may be discretized as follows:

$$X_{t_{i+1}} \approx X_{t_i} + b\big(t_i,X_{t_i}\big)h + B_{t_{i+1}} - B_{t_i}. \tag{37}$$

We first construct the binomial tree on $\mathcal{T}_N := \{t_i, i = 0,\dots,N\}$ as in Section 3.2 with

$$x_{0,0} = x_0,\quad x_{i,j} = x_0 + \big(2j-i\big)\sqrt{h},\quad b_{i,j} := b\big(t_i,x_{i,j}\big),\quad p^+_{i,j} := \frac{1}{2}+\frac{1}{2}b_{i,j}\sqrt{h}. \tag{38}$$

Since b is bounded, we shall assume h is small enough so that $0 < p^+_{i,j} < 1$. Let X^N denote the Markov chain corresponding to this binomial tree under the probability $\mathbb{P}_N$ specified by (38). Then our choice of $p^+_{i,j}$ ensures that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_N}\Big[X^N_{t_{i+1}} - X^N_{t_i}\Big|X^N_{t_i} = x_{i,j}\Big] = p^+_{i,j}\sqrt{h} - p^-_{i,j}\sqrt{h} = b_{i,j}h;\\&\mathbb{E}^{\mathbb{P}_N}\Big[\Big(X^N_{t_{i+1}} - X^N_{t_i} - b_{i,j}h\Big)^2\Big|X^N_{t_i} = x_{i,j}\Big] = p^+_{i,j}\Big(\sqrt{h} - b_{i,j}h\Big)^2 + p^-_{i,j}\Big(\sqrt{h} + b_{i,j}h\Big)^2 = h - b^2_{i,j}h^2.\end{aligned} \tag{39}$$

Clearly, as a standard Euler approximation, X^N matches the conditional expectation and conditional variance of X in (37), up to terms of order $o(h)$.

Next we define the other terms in Section 3.2:

$$\begin{cases} G^N_{i,j} := \mathbb{P}_N\Big\{X^N_{t_i} \geq x_{i,j}\Big\}, \quad q^{N,+}_{i,j} := \dfrac{\varphi_{t_{i+1}}\Big(G^N_{i+1,j+1}\Big) - \varphi_{t_i}\Big(G^N_{i,j+1}\Big)}{\varphi_{t_i}\Big(G^N_{i,j}\Big) - \varphi_{t_i}\Big(G^N_{i,j+1}\Big)}, \quad q^{N,-}_{i,j} := 1 - q^{N,+}_{i,j};\\ \mathbb{Q}_N\Big\{X^N_{t_{i+1}} = x_{i+1,j+1}\Big|X^N_{t_i} = x_{i,j}\Big\} = q^{N,+}_{i,j}, \quad \mathbb{Q}_N\Big\{X^N_{t_{i+1}} = x_{i+1,j}\Big|X^N_{t_i} = x_{i,j}\Big\} = q^{N,-}_{i,j};\\ \Phi_N\Big(t_i, t_n, x_{i,j}; \mathbb{P}_N\Big\{X^N_{t_n} \geq x_{n,k}\Big|X^N_{t_i} = x_{i,j}\Big\}\Big) := \mathbb{Q}_N\Big\{X^N_{t_n} \geq x_{n,k}\Big|X^N_{t_i} = x_{i,j}\Big\}.\end{cases} \tag{40}$$

We shall send $N \to \infty$ and analyze the limits of the above terms. In this section we evaluate the limits heuristically, by assuming all functions involved exist and are smooth.

Define the survival probability function and density function of the X in (36), respectively:

$$G(t,x) := \mathbb{P}(X_t \geq x), \quad \rho(t,x) := -\partial_x G(t,x), \quad 0 < t \leq T. \tag{41}$$

Note that, as the survival function of the diffusion process (36), G satisfies the following PDE:

$$\partial_t G = \frac{1}{2}\partial_{xx}G - b\partial_x G = -\frac{1}{2}\partial_x\rho + b\rho. \tag{42}$$

It is reasonable to assume $G^N_{i,j} \approx G(t_i, x_{i,j})$. Note that $t_{i+1} = t_i + h$, $x_{i,j+1} = x_{i,j} + 2\sqrt{h}$, $x_{i+1,j+1} = x_{i,j} + \sqrt{h}$. Rewrite $\varphi(t,p) := \varphi_t(p)$. Then, for $(t,x) = (t_i, x_{i,j})$, by (42) and applying Taylor expansion we have (suppressing variables when the context is clear):

$$\begin{aligned}&\varphi\Big(t+h, G\Big(t+h, x+\sqrt{h}\Big)\Big) - \varphi(t, G(t,x))\\&\quad= \partial_t\varphi h + \partial_p\varphi\Big[\partial_t G h + \partial_x G\sqrt{h} + \frac{1}{2}\partial_{xx}G h\Big] + \frac{1}{2}\partial_{pp}\varphi[\partial_x G]^2 h + o(h)\\&\quad= -\partial_p\varphi\rho\sqrt{h} + \Big[\partial_t\varphi + \partial_p\varphi b\rho - \partial_p\varphi\partial_x\rho + \frac{1}{2}\partial_{pp}\varphi\rho^2\Big]h + o(h);\\&\varphi\Big(t, G\Big(t, x+2\sqrt{h}\Big)\Big) - \varphi(t, G(t,x)) = \partial_p\varphi\Big[\partial_x G 2\sqrt{h} + \frac{1}{2}\partial_{xx}G 4h\Big] + \frac{1}{2}\partial_{pp}[\partial_x G]^2 4h + o(h)\\&\quad= -2\partial_p\varphi\rho\sqrt{h} - 2[\partial_p\varphi\partial_x\rho - \partial_{pp}\varphi\rho^2]h + o(h).\end{aligned}$$

Thus, we have an approximation for the $q^{N,+}_{i,j}$ in (40):

$$\begin{aligned}q^{N,+}_{i,j} &\approx \frac{\varphi\Big(t+h, G\Big(t+h, x+\sqrt{h}\Big)\Big) - \varphi\Big(t, G\Big(t, x+2\sqrt{h}\Big)\Big).}{\varphi(t, G(t,x)) - \varphi\Big(t, G\Big(t, x+2\sqrt{h}\Big)\Big)}\\&= 1 + \frac{-\partial_p\varphi\rho\sqrt{h} + [\partial_t\varphi + \partial_p\varphi b\rho - \partial_p\varphi\partial_x\rho + \frac{1}{2}\partial_{pp}\varphi\rho^2]h + o(h)}{2\partial_p\varphi\rho\sqrt{h} + 2[\partial_p\varphi\partial_x\rho - \partial_{pp}\varphi\rho^2]h + o(h)}\\&= \frac{1}{2} + \frac{1}{2}\mu(t,x)\sqrt{h} + o\Big(\sqrt{h}\Big),\end{aligned} \tag{43}$$

where

$$\mu(t,x) := b(t,x) + \frac{\partial_t\varphi(t,G(t,x)) - \frac{1}{2}\partial_{pp}\varphi(t,G(t,x))\rho^2(t,x)}{\partial_p\varphi(t,G(t,x))\rho(t,x)}. \tag{44}$$

Next, note that

$$\mathbb{E}^{\mathbb{Q}_N}\left\{X^N_{t_{i+1}} - X^N_{t_i}\middle|X^N_{t_i} = x_{i,j}\right\} = \sqrt{h}\left[2q^{N,+}_{i,j} - 1\right] = \mu\left(t_i, x_{i,j}\right)h + o(h);$$
$$\mathbb{E}^{\mathbb{Q}_N}\left\{\left(X^N_{t_{i+1}} - X^N_{t_i} - \mu\left(t_i, x_{i,j}\right)h\right)^2\middle|X^N_{t_i} = x_{i,j}\right\} = h + o(h).$$

In other words, as $N \to \infty$, we expect that $\mathbb{Q}_N$ would converge to a probability measure $\mathbb{Q}$ such that, for some $\mathbb{Q}$-Brownian motion $\tilde{B}$, it holds that

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \tilde{B}_t, \quad \mathbb{Q}\text{-a.s.} \tag{45}$$

Moreover, formally one should be able to find a dynamic distortion function Φ satisfying:

$$\Phi\left(s,t,x;\mathbb{P}\left\{X_t \ge y|X_s = x\right\}\right) = \mathbb{Q}\left\{X_t \ge y|X_s = x\right\}, \quad 0 \le s < t \le T. \tag{46}$$

We shall note that, however, since $X_0 = x_0$ is degenerate, $\rho(0,\cdot)$ and hence $\mu(0,\cdot)$ do not exist, so the above convergence will hold only for $0 < s < t \le T$. It is also worth noting that asymptotically (33) should read:

$$u(t,x) \approx \frac{1}{2}\left[1 + \mu(t,x)\sqrt{h} + o\left(\sqrt{h}\right)\right]\left[u\left(t+h, x+\sqrt{h}\right) - u\left(t+h, x-\sqrt{h}\right)\right] + u\left(t+h, x-\sqrt{h}\right)$$
$$= u(t,x) + \left[\partial_t u + \frac{1}{2}\partial_{xx}u + \mu\partial_x u\right]h + o(h).$$

That is,

$$\mathscr{L}u(t,x) := \partial_t u + \frac{1}{2}\partial_{xx}u + \mu\partial_x u = 0. \tag{47}$$

4.2. Rigorous Results for the Continuous Time Model

We now substantiate the heuristic arguments in the previous section and derive the time-consistent dynamic distortion function and the distorted conditional expectation for the continuous time model. We first have the following tail estimates for the density of the diffusion (36). Since our main focus is the dynamic distortion function, we postpone the proof to Section 6.

Proposition 3. *Under* Assumption 1, *X_t has a density function $\rho(t,x)$ which is strictly positive and sufficiently smooth on $(0,T]\times\mathbb{R}$. Moreover, for any $0 < t_0 \le T$, there exists a constant C_0, possibly depending on t_0, such that*

$$\frac{|\partial_x\rho(t,x)|}{\rho(t,x)} \le C_0, \quad \frac{1}{C_0[1+|x|]} \le \frac{G(t,x)[1-G(t,x)]}{\rho(t,x)} \le C_0, \quad (t,x) \in [t_0,T]\times\mathbb{R}. \tag{48}$$

We next assume the following technical conditions on φ.

Assumption 2. *φ is continuous on $[0,T]\times[0,1]$ and is sufficiently smooth in $(0,T]\times(0,1)$ with $\partial_p\varphi > 0$. Moreover, for any $0 < t_0 < T$, there exists a constant $C_0 > 0$ such that for $(t,p) \in [t_0,T]\times(0,1)$ we have the following bounds:*

$$\left|\frac{\partial_{pp}\varphi(t,p)}{\partial_p\varphi(t,p)}\right| \le \frac{C_0}{p(1-p)}, \quad \left|\frac{\partial_{ppp}\varphi(t,p)}{\partial_p\varphi(t,p)}\right| \le \frac{C_0}{p^2(1-p)^2}, \tag{49}$$

$$\left|\frac{\partial_t\varphi(t,p)}{\partial_p\varphi(t,p)}\right| \le C_0 p(1-p), \quad \left|\frac{\partial_{tp}\varphi(t,p)}{\partial_p\varphi(t,p)}\right| \le C_0. \tag{50}$$

We note that, given the existence of $G(t,x)$ and $\rho(t,x)$ as well as the regularity of φ, the function $\mu(t,x)$ in (44) is well defined.

Remark 6.

(i) Note that in (43) and (44) only the composition $\varphi(t, G(t,x))$ is used, and obviously $0 < G(t,x) < 1$ for all $(t,x) \in (0,T] \times \mathbb{R}$. Therefore, we do not require the differentiability of φ at $p = 0, 1$. Moreover, since $\partial_p \varphi > 0$, the condition (49) involves only the singularities around $p \approx 0$ and $p \approx 1$.

(ii) The first line in (49) is not restrictive. For example, by straightforward calculation one can verify that all the following distortion functions commonly used in the literature (see, e.g., Huang et al. [12, section 4.2]) satisfy it: recalling that in the literature typically $\varphi(t,p) = \varphi(p)$ does not depend on t.

- Tversky and Kahneman [23]: $\varphi(p) = p^\gamma / [p^\gamma + (1-p)^\gamma]^{1/\gamma}$, $\gamma \in [\gamma_0, 1)$, where $\gamma_0 \approx 0.279$ so that φ is increasing.
- Tversky and Fox [22]: $\varphi(p) = \alpha p^\gamma / [\alpha p^\gamma + (1-p)^\gamma]$, $\alpha > 0, \gamma \in (0,1)$.
- Prelec [20]: $\varphi(p) = \exp(-\gamma(-\ln p)^\alpha)$, $\gamma > 0$, $\alpha \in (0,1)$.
- Wang [24]: $\varphi(p) = F(F^{-1}(p) + \alpha)$, $\alpha \in \mathbb{R}$, where F is the cdf of the standard normal.

As an example, we check the last one, which is less trivial. Set $q := F^{-1}(p)$. Then

$$\varphi(F(q)) = F(q+\alpha) \Longrightarrow \varphi'(F(q)) = \frac{F'(q+\alpha)}{F'(q)} \Longrightarrow \ln(\varphi'(F(q))) = \ln(F'(q+\alpha)) - \ln(F'(q)).$$

Note that $F'(q) = e^{-q^2/2}/\sqrt{2\pi}$. Then $\ln(F'(q)) = -\ln\sqrt{2\pi} - q/2$. Thus,

$$\ln(\varphi'(F(q))) = -\frac{(q+\alpha)^2}{2} + \frac{q^2}{2} \;\Longrightarrow\; \frac{\varphi''(F(q))}{\varphi'(F(q))} F'(q) = -\alpha.$$

This implies that, denoting by $G(q) := 1 - F(q)$ the survival function of the standard normal,

$$\frac{|\varphi''(p)|}{\varphi'(p)} p[1-p] = |\alpha| \frac{F(q)[1-F(q)]}{F'(q)} = |\alpha| \frac{G(q)[1-G(q)]}{F'(q)}.$$

Then by applying (48) on the standard normal (namely, $b = 0$ and $t = 1$ there) we obtain the desired estimate for $\frac{\partial_{pp}\varphi}{\partial_p \varphi}$. Similarly we may estimate $\frac{\partial_{ppp}\varphi}{\partial_p \varphi}$.

(iii) When $\varphi(t,p) \equiv \varphi(p)$ as in the standard literature, the second line in (49) is trivial. Another important example is the separable case: $\varphi(t,p) = f(t)\varphi_0(p)$. Assume f' is bounded. Then the second inequality here becomes trivial, and a sufficient condition for the first inequality is $\frac{\varphi_0(p)}{\varphi_0'(p)} \le \frac{C_0}{p(1-p)}$, which holds true for all the examples in (ii).

To have a better understanding about μ given by (44), we compute an example explicitly.

Example 4. Consider Wang's [24] distortion function: $\varphi(t,p) = F(F^{-1}(p) + \alpha)$, as in Remark 6(ii). Set $b = 0$. Then $\mu(t,x) = \alpha/2\sqrt{t}$.

Proof. First it is clear that $\partial_t \varphi = 0$ and $\partial_p \varphi(t,p) = F'(F^{-1}(p) + \alpha)/F'(F^{-1}(p))$. Then

$$\frac{\partial_{pp}\varphi(t,p)}{\partial_p \varphi(t,p)} = \partial_p[\ln(\partial_p \varphi(t,p))] = \frac{1}{F'(F^{-1}(p))}\left[\frac{F''(F^{-1}(p)+\alpha)}{F'(F^{-1}(p)+\alpha)} - \frac{F''(F^{-1}(p))}{F'(F^{-1}(p))}\right].$$

One can easily check that $F'(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $F''(x) = -xF'(x)$. Then

$$\frac{\partial_{pp}\varphi(t,p)}{\partial_p \varphi(t,p)} = \frac{1}{F'(F^{-1}(p))}\left[-[F^{-1}(p) + \alpha] + F^{-1}(p)\right] = -\frac{\alpha}{F'(F^{-1}(p))}.$$

Note that

$$G(t,x) = \mathbb{P}(B_t \ge x) = \mathbb{P}\left(B_1 \ge \frac{x}{\sqrt{t}}\right) = \mathbb{P}\left(B_1 \le -\frac{x}{\sqrt{t}}\right) = F\left(-\frac{x}{\sqrt{t}}\right).$$

Then

$$\mu(t,x) = \frac{\alpha\rho(t,x)}{F'(F^{-1}(G(t,x)))} = \frac{\alpha\rho(t,x)}{F'\left(-x/\sqrt{t}\right)} = \frac{\frac{\alpha}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(-x/\sqrt{t})^2}{2}}} = \frac{\alpha}{\sqrt{t}},$$

completing the proof.

We now give some technical preparations. Throughout the paper we shall use C to denote a generic constant which may vary from line to line.

Lemma 2. *Let* Assumptions 1 *and* 2 *hold.*

(i) The function μ *defined by* (44) *is sufficiently smooth in* $(0,T]\times\mathbb{R}$. *Moreover, for any* $0<t_0<T$, *there exists* $C_0>0$ *such that*

$$|\mu(t,x)|\le C_0[1+|x|],\quad |\partial_x\mu(t,x)|\le C_0\left[1+|x|^2\right],\quad \textit{for all } (t,x)\in[t_0,T]\times\mathbb{R}. \tag{51}$$

(ii) For any $(s,x)\in(0,T)\times\mathbb{R}$, *the following SDE on* $[s,T]$ *has a unique strong solution:*

$$\tilde X_t^{s,x}=x+\int_s^t\mu\left(r,\tilde X_r^{s,x}\right)dr+B_t^s,\quad \textit{where}\quad B_t^s:=B_t-B_s,\quad t\in[s,T],\ \mathbb{P}\text{-a.s.}. \tag{52}$$

Moreover, the following $M^{s,x}$ *is a true* $\mathbb{P}$*-martingale and* $\mathbb{P}\circ(\tilde X^{s,x})^{-1}=\mathbb{Q}^{s,x}\circ(X^{s,x})^{-1}$, *where*

$$X_t^{s,x}:=x+B_t^s,\quad M_t^{s,x}:=e^{\int_s^t\mu(r,X_r^{s,x})dB_r-\frac12\int_s^t|\mu(r,X_r^{s,x})|^2dr},\quad \frac{d\mathbb{Q}^{s,x}}{d\mathbb{P}}:=M_T^{s,x}. \tag{53}$$

(iii) Recall the process X *as in (36). Define*

$$G_t^{s,x}(y):=\mathbb{P}(X_t\ge y|X_s=x),\quad \tilde G_t^{s,x}(y):=\mathbb{P}\left(\tilde X_t^{s,x}\ge y\right),\ 0<s<t\le T,\ x,y\in\mathbb{R}. \tag{54}$$

Then $G_t^{s,x}$ *and* $\tilde G_t^{s,x}$ *are continuous, strictly decreasing in* y, *and enjoy the following properties:*

$$G_t^{s,x}(\infty):=\lim_{y\to\infty}G_t^{s,x}(y)=0,\qquad \tilde G_t^{s,x}(\infty):=\lim_{y\to\infty}\tilde G_t^{s,x}(y)=0;$$
$$G_t^{s,x}(-\infty):=\lim_{y\to-\infty}G_t^{s,x}(y)=1,\quad \tilde G_t^{s,x}(-\infty):=\lim_{y\to-\infty}\tilde G_t^{s,x}(y)=1.$$

Furthermore, $G_t^{s,x}$ *has a continuous inverse function* $(G_t^{s,x})^{-1}$ *on* $(0,1)$, *and by continuity we set* $(G_t^{s,x})^{-1}(0):=-\infty$, $(G_t^{s,x})^{-1}(1):=\infty$.

(iv) For any $g\in\mathcal{I}$ *fixed, let* $u(t,x):=\mathbb{E}^{\mathbb{P}}[g(\tilde X_T^{t,x})]$, $(t,x)\in(0,T]\times\mathbb{R}$. *Then* u *is bounded, is increasing in* x, *and is the unique bounded viscosity solution of the following PDE:*

$$\mathscr{L}u(t,x):=\partial_t u+\frac12\partial_{xx}u+\mu\partial_x u=0,\quad 0<t\le T;\quad u(T,x)=g(x). \tag{55}$$

(v) For the t_0 *and* C_0 *in (i), there exists* $\delta=\delta(C_0)>0$ *such that if* $g\in\mathcal{I}$ *is sufficiently smooth and* g' *has compact support, then* u *is sufficiently smooth on* $[T-\delta,T]\times\mathbb{R}$ *and there exists a constant* $C>0$, *which may depend on* g, *satisfying, for* $(t,x)\in[T-\delta,T]\times\mathbb{R}$,

$$\left|u(t,x)-g(-\infty)\right|\le Ce^{-x^2},x<0;\ |u(t,x)-g(\infty)|\le Ce^{-x^2},x>0;\ \partial_x u(t,x)\le Ce^{-x^2}. \tag{56}$$

Proof. (i) By our assumptions and Proposition 3, the regularity of μ follows immediately. For any $t\ge t_0$, by (49) and then (48) we have

$$\left|\frac{\partial_t\varphi(t,G(t,x))}{\partial_p\varphi(t,G(t,x))\rho(t,x)}\right|\le\frac{CG(t,x)[1-G(t,x)]}{\rho(t,x)}\le C;$$
$$\left|\frac{\partial_{pp}\varphi(t,G(t,x))\rho(t,x)}{\partial_p\varphi(t,G(t,x))}\right|\le\frac{C\rho(t,x)}{G(t,x)[1-G(t,x)]}\le C[1+|x|].$$

Then it follows from (44) that $|\mu(t,x)|\le C[1+|x|]$.

Moreover, note that

$$\partial_x\mu(t,x)=\partial_x b-\frac{\partial_{tp}\varphi}{\partial_p\varphi}+\frac{\partial_t\varphi\partial_{pp}\varphi}{(\partial_p\varphi)^2}-\frac{\partial_t\varphi\partial_x\rho}{\partial_p\varphi\rho^2}+\frac12\frac{\partial_{ppp}\varphi\rho^2}{\partial_p\varphi}-\frac12\frac{\partial_{pp}\varphi\partial_x\rho}{\partial_p\varphi}-\frac12\frac{(\partial_{pp}\varphi)^2\rho^2}{(\partial_p\varphi)^2}.$$

By (49) and (48) again one can easily verify that $|\partial_x \mu(t,x)| \le C_0[1+|x|^2]$.

(ii) Since μ is locally uniform Lipschitz continuous in x, by a truncation argument $\tilde{X}^{s,x}$ exists locally. Now the uniform linear growth (51) guarantees the global existence. Moreover, by [14, chapter 3, corollary 5.16] we see that $M^{t,x}$ is a true $\mathbb{P}$-martingale and thus $\mathbb{Q}^{t,x}$ is a probability measure.

(iii) Since the conditional law of X_t under $\mathbb{P}$ given $X_s = x$ has a strictly positive density, the statements concerning $G_t^{s,x}$ are obvious. Similarly, since the law of $X_t^{s,x}$ under $\mathbb{P}$ has a density and $d\mathbb{Q}^{s,x} \ll d\mathbb{P}$, the statements concerning $\tilde{G}_t^{s,x}$ are also obvious.

(v) We shall prove (v) before (iv). Let $\delta > 0$ be specified later, and let $t \in [T-\delta, T]$. Let $R > 0$ be such that $g'(x) = 0$ for $|x| \ge R$. Note that $u(t,x) = \mathbb{E}[M_T^{t,x} g(X_T^{t,x})]$. For $x > 2R$, we have

$$\begin{aligned}
|u(t,x) - g(\infty)| &= \left|\mathbb{E}\left[M_T^{t,x}\left[g\left(X_T^{t,x}\right) - g(\infty)\right]\right]\right| \le \mathbb{E}\left[M_T^{t,x}\left|g\left(X_T^{t,x}\right) - g(\infty)\right|\right] \\
&\le 2\|g\|_\infty \mathbb{E}\left[M_T^{t,x} 1_{\{X_T^{t,x} \le R\}}\right] = 2\|g\|_\infty \mathbb{E}\left[e^{\int_t^T \mu dB_r - \frac{3}{2}\int_t^T |\mu|^2 dr} e^{\int_t^T |\mu|^2 dr} 1_{\{X_T^{t,x} \le R\}}\right].
\end{aligned}$$

Then

$$|u(t,x) - g(\infty)|^3 \le \|g\|_\infty^3 \mathbb{E}\left[e^{3\int_t^T \mu dB_r - \frac{9}{2}\int_t^T |\mu|^2 dr}\right] \mathbb{E}\left[e^{3\int_t^T |\mu|^2 dr}\right] \mathbb{P}\left(X_T^{t,x} \le R\right).$$

By [14, chapter 3, corollary 5.16] again, we have $\mathbb{E}[e^{3\int_t^T \mu dB_r - \frac{9}{2}\int_t^T |\mu|^2 dr}] = 1$. By (51), we have

$$\mathbb{E}\left[e^{3\int_t^T |\mu|^2 dr}\right] \le \mathbb{E}\left[e^{C_0 \int_t^T \left[1+|x|^2+|B_r^t|^2\right] dr}\right] \le e^{C_0\delta\left[1+|x|^2\right]} \mathbb{E}\left[e^{C_0\delta \sup_{0\le s\le\delta} |B_s|^2}\right] \le e^{2C_0\delta\left[1+|x|^2\right]},$$

for δ small enough. Fix such a $\delta > 0$, and let $t \in [T-\delta, T]$. Note that we may choose δ independent from g. Henceforth, we let $C > 0$ be a generic constant. Moreover, since $x > 2R$,

$$\mathbb{P}\left(X_T^{t,x} \le R\right) \le \mathbb{P}\left(X_T^{t,x} \le \frac{x}{2}\right) = \mathbb{P}\left(B_T^t \le -\frac{x}{2}\right) \le \mathbb{P}\left(B_1 \le -\frac{x}{2\sqrt{\delta}}\right) \le Ce^{-\frac{x^2}{8\delta}}.$$

Putting these statements together, we have, for δ small enough,

$$|u(t,x) - g(\infty)|^3 \le Ce^{-\frac{x^2}{8\delta} + C\delta\left[1+|x|^2\right]} \le Ce^{-3x^2}.$$

This implies that $|u(t,x) - g(\infty)| \le Ce^{-x^2}$. Similarly, $|u(t,x) - g(-\infty)| \le Ce^{-x^2}$ for $x < 0$.

The preceding estimates allows us to differentiate inside the expectation, and we have

$$\begin{aligned}
&\partial_x u(t,x) = \mathbb{E}\left[g'\left(\tilde{X}_T^{t,x}\right)\nabla\tilde{X}_T^{t,x}\right], \\
&\text{where} \quad \nabla\tilde{X}_s^{t,x} = 1 + \int_t^s \partial_x \mu\left(r, \tilde{X}_r^{t,x}\right)\nabla\tilde{X}_r^{t,x} dr \quad \text{and thus} \quad \nabla\tilde{X}_T^{t,x} = e^{\int_t^T \partial_x \mu\left(s, \tilde{X}_s^{t,x}\right) ds} > 0.
\end{aligned}$$

Then $\partial_x u \ge 0$. Moreover, recalling (53) we have

$$\begin{aligned}
\partial_x u(t,x) &= \mathbb{E}\left[g'\left(\tilde{X}_T^{t,x}\right) e^{\int_t^T \partial_x \mu\left(s, \tilde{X}_s^{t,x}\right) ds}\right] = \mathbb{E}\left[M_T^{t,x} g'\left(\tilde{X}_T^{t,x}\right) e^{\int_t^T \partial_x \mu\left(s, \tilde{X}_s^{t,x}\right) ds}\right] \\
&\le C\mathbb{E}\left[M_T^{t,x} e^{\int_t^T \partial_x \mu\left(s, \tilde{X}_s^{t,x}\right) ds} 1_{\{|X_T^{t,x}| \le R\}}\right].
\end{aligned}$$

Then, by the estimate of $\partial_x \mu$ in (51), it follows from the same arguments as above that we can show that $|\partial_x u(t,x)| \le Ce^{-x^2}$.

Finally, we may apply the arguments further to show that u is sufficiently smooth, and then it follows from the flow property and the standard Itô formula that u satisfies PDE (55).

(iv) We shall only prove the results on $[T-\delta, T]$. Since $\delta > 0$ depends only on C_0 in (i), one may apply the results backwardly in time and extend the results to $[t_0, T]$. Then it follows from the arbitrariness of t_0 that the results hold true on $(0, T]$.

We now fix δ as in (v). The boundedness of u is obvious. Note that $u(t,x) = \mathbb{E}^{\mathbb{P}}[g(X_T^{t,x})M_T^{t,x}]$, and μ and g are continuous. Following similar arguments as in (v) one can show that u is continuous.

Next, for any $g \in \mathcal{I}$, there exist approximating sequences $\{g_n\}$ such that each g_n satisfies the conditions in (v). Let $u_n(t,x) := \mathbb{E}[g_n(\tilde{X}_T^{t,x})]$. Then u_n is increasing in x and is a classical solution to PDE (55) on $[T-\delta, T]$ with terminal condition g_n. It is clear that $u_n \to u$. Then u is also increasing in x, and its viscosity property follows from the stability of viscosity solutions.

The uniqueness of the viscosity solution follows from the standard comparison principle. We refer to the classical reference Crandall et al.[5] for the details of the viscosity theory. □

We are now ready for the main result of this section. Recall (54), and define

$$\Phi(s,t,x;p) := \tilde{G}_t^{s,x}\Big(\big(G_t^{s,x}\big)^{-1}(p)\Big), \quad s>0. \tag{57}$$

Theorem 2. *Let* Assumptions 1 *and* 2 *hold. Then* Φ *defined by* (57) *is a time-consistent dynamic distortion function which is consistent with the initial conditions:* $\Phi(0,t,x;p)=\varphi_t(p)$.

Proof. First, by Lemma 2(iv) it is straightforward to check that Φ satisfies Definition 2(i).
Next, for $0<s<t\le T$, note that the definition (57) of Φ implies the counterpart of (34):

$$\Phi\big(s,t,x;G_t^{s,x}(y)\big)=\tilde{G}_t^{s,x}(y). \tag{58}$$

Recall (16) and Lemma 2. One can easily see that $\mathscr{E}_{s,t}[g(X_t)]=u(s,X_s)$ for any $g\in\mathcal{I}$, where $u(s,x):=\mathbb{E}^{\mathbb{P}}[g(\tilde{X}_t^{s,x})]$ is increasing in x and is the unique viscosity solution of the PDE (55) on $[s,t]\times\mathbb{R}$ with terminal condition $u(t,x)=g(x)$. Then, either by the flow property of the solution to SDE (52) or the uniqueness of the PDE, we obtain the tower property (17) immediately for $0<r<s<t\le T$.
To verify the tower property at $r=0$, let $t_0>0$ and $\delta>0$ be as in Lemma 2(v). We first show that, for any g as in Lemma 2(v) and the corresponding u, we have

$$\mathscr{E}_{0,t_1}\big[u\big(t_1,X_{t_1}\big)\big]=\mathscr{E}_{0,t_2}\big[u\big(t_2,X_{t_2}\big)\big], \quad T-\delta\le t_1<t_2\le T. \tag{59}$$

Clearly the set of such g is dense in $\mathcal{I}$. Then (59) holds true for all $g\in\mathcal{I}$, where u is the viscosity solution to the PDE (55). Note that $u(t,X_t)=\mathscr{E}_{t,T}[g(X_T)]$. Then by setting $t_1=t$ and $t_2=T$ in (59) we obtain $\mathscr{E}_{0,t}[\mathscr{E}_{t,T}[g(X_T)]]=\mathscr{E}_{0,T}[g(X_T)]$ for $T-\delta\le t\le T$. Similarly we can verify the tower property over any interval $[t-\delta,t]\subset[t_0,T]$. Since $\mathscr{E}_{s,t}$ is already time consistent for $0<s<t$, we see the time consistency for any $t_0\le s<t\le T$. Now by the arbitrariness of $t_0>0$, we obtain the tower property at $r=0$ for all $0<s<t\le T$.
We now prove (59). Recall from (56) that $u(t,-\infty)=g(-\infty)=0$. Then, for $T-\delta\le t\le T$, similar to (10), we have

$$\mathscr{E}_{0,t}[u(t,X_t)]=\int_0^\infty\varphi(t,\mathbb{P}(u(t,X_t)\ge x))dx=\int_{\mathbb{R}}\varphi(t,G(t,x))\partial_x u(t,x)dx.$$

Let $\psi_m:\mathbb{R}\to[0,1]$ be smooth with $\psi_m(x)=1$, $|x|\le m$, and $\psi_m(x)=0$, $|x|\ge m+1$. Denote

$$\mathscr{E}_{0,t}^m[u(t,X_t)]:=\int_{\mathbb{R}}\varphi(t,G(t,x))\partial_x u(t,x)\psi_m(x)dx.$$

Then, recalling (42) and (55) and suppressing the variables when the context is clear, we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}_{0,t}^m[u(t,X_t)] &= \int_{\mathbb{R}}[[\partial_t\varphi+\partial_p\varphi\partial_t G]\partial_x u+\varphi\partial_{tx}u]\psi_m dx\\
&= \int_{\mathbb{R}}[[\partial_t\varphi+\partial_p\varphi\partial_t G]\partial_x u\psi_m+[\partial_p\varphi\rho\psi_m-\varphi\psi_m']\partial_t u]dx\\
&= \int_{\mathbb{R}}\bigg[\bigg[\partial_t\varphi+\partial_p\varphi\bigg[b\rho-\frac{1}{2}\partial_x\rho\bigg]\bigg]\psi_m\partial_x u-[\partial_p\varphi\rho\psi_m-\varphi\psi_m']\bigg[\frac{1}{2}\partial_{xx}u+\mu\partial_x u\bigg]\bigg]dx\\
&= \int_{\mathbb{R}}\bigg[\bigg[\partial_t\varphi+\partial_p\varphi\bigg[b\rho-\frac{1}{2}\partial_x\rho\bigg]-\partial_p\varphi\rho\mu\bigg]\psi_m\partial_x u+\varphi\psi_m'\mu\partial_x u\\
&\quad+\frac{1}{2}\partial_x u[\partial_p\varphi\partial_x\rho\psi_m-\partial_{pp}\varphi\rho^2\psi_m+2\partial_p\varphi\rho\psi_m'-\varphi\psi_m'']\bigg]dx\\
&= \int_{\mathbb{R}}\bigg[\bigg[\partial_t\varphi+\partial_p\varphi b\rho-\partial_p\varphi\rho\mu-\frac{1}{2}\partial_{pp}\varphi\rho^2\bigg]\psi_m+[\varphi\mu+\partial_p\varphi\rho]\psi_m'-\frac{1}{2}\varphi\psi_m''\bigg]\partial_x u dx\\
&= \int_{\mathbb{R}}\bigg[[\varphi\mu+\partial_p\varphi\rho]\psi_m'-\frac{1}{2}\varphi\psi_m''\bigg]\partial_x u dx,
\end{aligned} \tag{60}$$

where the last equality follows from (44). That is, for any $T-\delta \le t_1 < t_2 \le T$,

$$\mathscr{E}^m_{0,t_2}\left[u\left(t_2, X_{t_2}\right)\right] - \mathscr{E}^m_{0,t_1}\left[u\left(t_1, X_{t_1}\right)\right] = \int_{t_1}^{t_2}\int_{\mathbb{R}}\left[\left[\varphi\mu + \partial_p\varphi\rho\right]\psi'_m - \frac{1}{2}\varphi\psi''_m\right]\partial_x u dxdt. \tag{61}$$

It is clear that $\lim_{m\to\infty}\mathscr{E}^m_{0,t}[u(t,X_t)] = \mathscr{E}_{0,t}[u(t,X_t)]$. Note that, by (51) and (56), we have

$$|\mu| \le C[1+|x|], \quad \partial_p\varphi\rho \le \frac{C\rho}{G[1-G]} \le C[1+|x|], \quad |\partial_x u| \le Ce^{-x^2}.$$

Then, by sending $m \to \infty$ in (61) and applying the dominated convergence theorem, we obtain (59) and hence the theorem. □

Remark 7. In the definition of Φ (see (57)) we require that the initial time s is strictly positive. In fact, when $s = 0$ the distribution of X_s becomes degenerate, and thus, μ may have singularities. For example, assume $\varphi(t,\cdot) = \varphi(\cdot)$ is independent of t and $b \equiv 0$, $x_0 = 0$. Then

$$\mu(t,x) = -\frac{\varphi''(G(t,x))}{2\varphi'(G(t,x))}\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{2t}}.$$

It is not even clear if the following SDE is well posed in general:

$$\tilde{X}_t = \int_0^t \mu\left(s, \tilde{X}_s\right)ds + B_t.$$

Correspondingly, if we consider the following PDE on $(0,T]\times\mathbb{R}$:

$$\mathscr{L}u(t,x) = 0, \ (t,x) \in (0,T]\times\mathbb{R}, \quad u(T,x) = g(x),$$

then it is not clear whether $\lim_{(t,x)\to(0,0)} u(t,x)$ exists.

Unlike Theorem 1 in the discrete case, surprisingly here the time-consistent dynamic distortion function is *not* unique. Let $\check{\Phi}$ be an arbitrary time-consistent dynamic distortion function for $0 < s < t \le T$ (not necessarily consistent with φ_t when $s = 0$ at this point). Fix $0 < t \le T$. For any $s \in (0,t]$ and $g \in \mathcal{I}$, define

$$\check{u}(s,x) := \int_{\mathbb{R}} \check{\Phi}\left(s,t,x;\mathbb{P}\left(g(X_t) \ge y | X_s = x\right)\right)ds. \tag{62}$$

The corresponding $\{\check{\mathscr{E}}_{s,t}\}$ is time consistent, that is, the tower property holds. Suppose $\check{\Phi}$ defines via (58) a $\mathbb{Q}$-diffusion $\check{X}$ with coefficients $\check{\mu}$, $\check{\sigma}$, that is,

$$\check{\Phi}\left(s,t,x;\mathbb{P}\left(X_t \ge y | X_s = x\right)\right) = \mathbb{P}\left(\check{X}^{s,x}_t \ge y\right), \text{where} \quad \check{X}^{s,x}_t = x + \int_s^t \check{\mu}\left(r, \check{X}^{s,x}_r\right)dB_r + \int_s^t \check{\sigma}\left(r, \check{X}^{s,x}_r\right)dB_r, \quad \mathbb{P}\text{-a.s.}, \tag{63}$$

and $\check{u}$ satisfies the following PDE corresponding to the infinitesimal generator of $\check{X}$:

$$\partial_t\check{u} + \frac{1}{2}\check{\sigma}^2\partial_{xx}\check{u} + \check{\mu}\partial_x\check{u} = 0, \ (s,x) \in (0,t]\times\mathbb{R}; \quad \check{u}(t,x) = g(x). \tag{64}$$

We have the following more general result.

Theorem 3. *Let* Assumptions 1 *and* 2 *hold, and let $\check{\Phi}$ be an arbitrary smooth time-consistent (for $t > 0$) dynamic distortion function corresponding to* (63). *Suppose $\check{\mu}$ and $\check{\sigma}$ are sufficiently smooth such that $\check{u}$ is smooth and integration by parts in* (66) *below goes through. Then $\check{\Phi}$ is consistent with the initial condition $\check{\Phi}(0,t,x;p) = \varphi_t(p)$ if and only if*

$$\check{\mu} = b + \check{\sigma}\partial_x\check{\sigma} + \frac{1}{2}\left[\check{\sigma}^2 - 1\right]\frac{\partial_x\rho}{\rho} + \frac{\partial_t\varphi}{\partial_p\varphi\rho} - \frac{\check{\sigma}^2\partial_{pp}\varphi\rho}{2\partial_p\varphi}. \tag{65}$$

In particular, if we restrict to the case $\check{\sigma} = 1$, then $\check{\mu} = \mu$ and hence $\check{\Phi} = \Phi$ is unique.

Proof. The consistency of $\check{\Phi}$ with the initial condition $\Phi(0,t,x;p) = \varphi_t(p)$ is equivalent to (59) for $\check{u}$, where $\check{u}$ is the solution to PDE (64) on $(0,T]$ with terminal condition g. Similar to (60), we have

$$\begin{aligned}\frac{d}{dt}\mathscr{E}_{0,t}[\check{u}(t,X_t)] &= \int_{\mathbb{R}}[[\partial_t\varphi + \partial_p\varphi\partial_t G]\partial_x\check{u} + \varphi\partial_{tx}\check{u}]dx\\ &= \int_{\mathbb{R}}[[\partial_t\varphi + \partial_p\varphi\partial_t G]\partial_x\check{u} + \partial_p\varphi\rho\partial_t\check{u}]dx\\ &= \int_{\mathbb{R}}\left[\left[\partial_t\varphi + \partial_p\varphi\left[b\rho - \frac{1}{2}\partial_x\rho\right]\right]\partial_x\check{u} - \partial_p\varphi\rho\left[\frac{1}{2}\check{\sigma}^2\partial_{xx}\check{u} + \check{\mu}\partial_x u\right]\right]dx\\ &= \int_{\mathbb{R}}\left[\left[\partial_t\varphi + \partial_p\varphi\left[b\rho - \frac{1}{2}\partial_x\rho\right] - \partial_p\varphi\rho\check{\mu}\right]\partial_x\check{u}\right.\\ &\quad\left. + \frac{1}{2}\partial_x\check{u}[-\partial_{pp}\varphi\rho^2\check{\sigma}^2 + \partial_p\varphi\partial_x\rho\check{\sigma}^2 + 2\partial_p\varphi\rho\check{\sigma}\partial_x\check{\sigma}]\right]dx\\ &= \int_{\mathbb{R}}\left[b + \check{\sigma}\partial_x\check{\sigma} + \frac{\partial_t\varphi}{\partial_p\varphi\rho} - \frac{\check{\sigma}^2\partial_{pp}\varphi\rho}{2\partial_p\varphi} + \frac{1}{2}[\check{\sigma}^2 - 1]\frac{\partial_x\rho}{\rho} - \check{\mu}\right]\partial_p\varphi\rho\partial_x\check{u}dx. \end{aligned} \tag{66}$$

Since $\partial_p\varphi\rho > 0$ and since g (and hence $\check{u}$) is arbitrary, we get the equivalence of (59) and (65). □

Remark 8.

(i) When $\check{\sigma} \not\equiv 1$, the law of $\check{X}_t^{s,x}$ can be singular to the conditional law of X_t given $X_s = x$. That is, the agent may distort the probability so dramatically that the distorted probability is singular to the original one. For example, some event which is null under the original probability may be distorted into a positive or even full measure, so the agent could be worrying too much on something which could never happen, which does not seem to be reasonable in practice. Our result says that if we exclude this type of extreme distortion, then for given $\{\varphi_t\}$, the time-consistent dynamic distortion function Φ is unique.

(ii) In the discrete case in Section 3.2, because of the special structure of the binomial tree, we always have $|X_{t_{i+1}} - X_{t_i}|^2 = h$. Then for any possible $\mathbb{Q}$, we always have $\mathbb{E}^{\mathbb{Q}}[|X_{t_{i+1}} - X_{t_i}|^2 | X_{t_i} = x_{i,j}] = h$. This, in the continuous time model, means $\check{\sigma} \equiv 1$. This is why we can obtain the uniqueness in Theorem 1.

4.3. Rigorous Proof of the Convergence

We note that Theorem 2 already gives the definition of the desired time-consistent conditional expectation for the constant-diffusion case. Nevertheless, it is still worth asking whether the discrete system in Section 4.1 indeed converges to the continuous time system in Section 4.2, especially from the perspective of numerical approximations. We therefore believe that a detailed convergence analysis, which we now describe, is interesting in its own right.

For each N, denote $h := h_N := \frac{T}{N}$ and $t_i := t_i^N := ih$, $i = 0, \ldots, N$, as in Section 4.1. Consider the notation in (38) and (40), and denote

$$\rho_{i,j}^N := \mathbb{P}^N\left(X_{t_i}^N = x_{i,j}\right)/\left(2\sqrt{h}\right). \tag{67}$$

Proposition 4. *Under Assumption 1, for any sequence $(t_i^N, x_{i,j}^N) \to (t,x) \in (0,T] \times \mathbb{R}$, we have $G_{i,j}^N \to G(t,x)$ and $\rho_{i,j}^N \to \rho(t,x)$ as $N \to \infty$.*

Again we postpone this proof to Section 6.

Theorem 4. *Let Assumptions 1 and 2 hold, and let $g \in \mathcal{I}$. For each N, consider the notation in* (38) *and* (40), *and define by backward induction as in* (33):

$$u_N^N(x) := g(x),\ \ u_i^N\left(x_{i,j}\right) := q_{i,j}^{N,+}u_{i+1}^N\left(x_{i+1,j+1}\right) + q_{i,j}^{N,-}u_{i+1}^N\left(x_{i+1,j}\right),\ \ i = N-1, \ldots, 0. \tag{68}$$

Then, for any $(t,x) \in (0,T] \times \mathbb{R}$ and any sequence $(t_i^N, x_{i,j}^N) \to (t,x)$, we have

$$\lim_{N\to\infty} u_i^N\left(x_{i,j}\right) = u(t,x). \tag{69}$$

Proof. Define

$$\overline{u}(t,x) := \limsup_{N\to\infty, t_i\downarrow t, x_{i,j}\to x} u_i^N\left(x_{i,j}\right), \quad \underline{u}(t,x) := \liminf_{N\to\infty, t_i\downarrow t, x_{i,j}\to x} u_i^N\left(x_{i,j}\right).$$

We shall show that $\overline{u}$ is a viscosity subsolution and $\underline{u}$ a viscosity supersolution of PDE (55). By the comparison principle of the PDE (55) we have $\overline{u} = \underline{u} = u$, which implies (69) immediately.

We shall only prove that $\overline{u}$ is a viscosity subsolution. The viscosity supersolution property of $\underline{u}$ can be proved similarly. Fix $(\bar{t}, \bar{x}) \in (0, T] \times \mathbb{R}$. Let w be a smooth test function at $(\bar{t}, \bar{x})$ such that $[w - \overline{u}](\bar{t}, \bar{x}) = 0 \le [w - u](t, x)$ for all $(t, x) \in [\bar{t}, T] \times \mathbb{R}$ satisfying $t - \bar{t} \le \delta^2, |x - \bar{x}| \le \delta$ for some $\delta > 0$. Introduce

$$\tilde{w}(t, x) := w(t, x) + \delta^{-5}\big[|t - \bar{t}|^2 + |x - \bar{x}|^4\big]. \tag{70}$$

Then

$$[\tilde{w} - \overline{u}](\bar{t}, \bar{x}) = 0 < \frac{1}{C\delta} \le \inf_{\frac{\delta^2}{2} \le |t - \bar{t}| + |x - \bar{x}|^2 \le \delta^2} [\tilde{w} - \overline{u}](t, x).$$

By the definition of $\overline{u}(\bar{t}, \bar{x})$, by otherwise choosing a subsequence of N, without loss of generality we assume there exist (i_N, j_N) such that $t_{i_N} \downarrow \bar{t}$, $x_{i_N, j_N} \to \bar{x}$, and $\lim_{N\to\infty} u^N_{i_N}(x_{i_N, j_N}) = \overline{u}(\bar{t}, \bar{x})$. Since $\overline{u}$ and u^N are bounded, for δ small, we have

$$c_N := \big[\tilde{w} - u^N\big](t_{i_N}, x_{i_N, j_N}) < \frac{1}{2C\delta} \le \inf_{\frac{\delta^2}{2} \le |t_i - \bar{t}| + |x_{i,j} - \bar{x}|^2 \le \delta^2} \big[\tilde{w} - u^N\big](t_i, x_{i,j}).$$

Denote

$$c_N^* := \inf_{t_{i_N} \le t_i \le \bar{t} + \frac{\delta^2}{2}, |x_{i,j} - \bar{x}|^2 \le \delta^2} \big[\tilde{w} - u^N\big](t_i, x_{i,j}) = \big[\tilde{w} - u^N\big]\Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) \le c_N.$$

Then clearly $|t_{i_N^*} - \bar{t}| + |x_{i_N^*, j_N^*} - \bar{x}| < \delta^2/2$. Moreover, by a compactness argument, by otherwise choosing a subsequence, we may assume $(t_{i_N^*}, x_{i_N^*, j_N^*}) \to (t_*, x_*)$. Then

$$\begin{aligned}0 = \lim_{N\to\infty} c_N &\ge \limsup_{N\to\infty} \big[\tilde{w} - u^N\big]\Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) = \tilde{w}(t_*, x_*) - \liminf_{N\to\infty} u^N\Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) \\ &\ge \tilde{w}(t_*, x_*) - \overline{u}(t_*, x_*) \ge \delta^{-5}\big[|t_* - \bar{t}|^2 + |x_* - \bar{x}|^4\big].\end{aligned}$$

That is, $(t_*, x_*) = (\bar{t}, \bar{x})$, namely,

$$\lim_{N\to\infty} \Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) = (\bar{t}, \bar{x}). \tag{71}$$

Note that

$$\begin{aligned}\tilde{w}\Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) &= u^N\Big(t_{i_N^*}, x_{i_N^*, j_N^*}\Big) + c_N^* \\ &= q^{N,+}_{i_N^*, j_N^*} u^N\Big(t_{i_N^*+1}, x_{i_N^*+1, j_N^*+1}\Big) + q^{N,-}_{i_N^*, j_N^*} u^N\Big(t_{i_N^*+1}, x_{i_N^*+1, j_N^*}\Big) + c_N^* \\ &\le q^{N,+}_{i_N^*, j_N^*} \tilde{w}\Big(t_{i_N^*+1}, x_{i_N^*+1, j_N^*+1}\Big) + q^{N,-}_{i_N^*, j_N^*} \tilde{w}\Big(t_{i_N^*+1}, x_{i_N^*+1, j_N^*}\Big).\end{aligned}$$

Then, denoting $(i, j) := (i_N^*, j_N^*)$ for notational simplicity, we have

$$\begin{aligned}0 &\le q^{N,+}_{i,j}\big[\tilde{w}(t_{i+1}, x_{i+1,j+1}) - \tilde{w}(t_i, x_{i,j})\big] + q^{N,-}_{i,j}\big[\tilde{w}(t_{i+1}, x_{i+1,j}) - \tilde{w}(t_i, x_{i,j})\big] \\ &= q^{N,+}_{i,j}\Big[\partial_t \tilde{w}(t_i, x_{i,j})h + \partial_x \tilde{w}(t_i, x_{i,j})\sqrt{h} + \frac{1}{2}\partial_{xx}\tilde{w}(t_i, x_{i,j})h\Big] \\ &\quad + q^{N,-}_{i,j}\Big[\partial_t \tilde{w}(t_i, x_{i,j})h - \partial_x \tilde{w}(t_i, x_{i,j})\sqrt{h} + \frac{1}{2}\partial_{xx}\tilde{w}(t_i, x_{i,j})h\Big] + o(h) \\ &= \Big[\partial_t \tilde{w}(t_i, x_{i,j}) + \frac{1}{2}\partial_{xx}\tilde{w}(t_i, x_{i,j})\Big]h + \Big[q^{N,+}_{i,j} - q^{N,-}_{i,j}\Big]\partial_x \tilde{w}(t_i, x_{i,j})\sqrt{h} + o(h).\end{aligned} \tag{72}$$

Note that

$$q_{i,j}^{N,+} - q_{i,j}^{N,-} = 1 + 2\frac{\varphi_{t_{i+1}}\left(G_{i+1,j+1}^N\right) - \varphi_{t_i}\left(G_{i,j}^N\right)}{\varphi_{t_i}\left(G_{i,j}^N\right) - \varphi_{t_i}\left(G_{i,j+1}^N\right)};$$

$$\varphi_{t_i}\left(G_{i,j}^N\right) - \varphi_{t_i}\left(G_{i,j+1}^N\right) = \varphi_{t_i}\left(G_{i,j}^N\right) - \varphi_{t_i}\left(G_{i,j}^N - 2\rho_{i,j}^N\sqrt{h}\right) = \partial_p\varphi\left(t_i, G_{i,j}^N\right)2\rho_{i,j}^N\sqrt{h} + o\left(\sqrt{h}\right);$$

$$\begin{aligned}\varphi_{t_{i+1}}\left(G_{i+1,j+1}^N\right) - \varphi_{t_i}\left(G_{i,j}^N\right) &= \varphi_{t_{i+1}}\left(G_{i,j}^N - 2\rho_{i,j}^N\sqrt{h}p_{i,j}^-\right) - \varphi_{t_i}\left(G_{i,j}^N\right)\\ &= \partial_t\varphi_{t_i}\left(G_{i,j}^N\right)h - \partial_p\varphi_{t_i}\left(G_{i,j}^N\right)2\rho_{i,j}^N\sqrt{h}p_{i,j}^- + \frac{1}{2}\partial_{pp}\varphi_{t_i}\left(G_{i,j}^N\right)\left[2\rho_{i,j}^N\sqrt{h}p_{i,j}^-\right]^2 + o(h)\\ &= \partial_t\varphi_{t_i}\left(G_{i,j}^N\right)h - \partial_p\varphi_{t_i}\left(G_{i,j}^N\right)\rho_{i,j}^N\sqrt{h}\left[1 - b_{i,j}\sqrt{h}\right] + \frac{1}{2}\partial_{pp}\varphi_{t_i}\left(G_{i,j}^N\right)\left[\rho_{i,j}^N\right]^2 h + o(h).\end{aligned}$$

Then, denoting $G_{i,j} := G(t_i, x_{i,j})$ and $\rho_{i,j} := \rho(t_i, x_{i,j})$ and by Proposition 4, we have

$$\begin{aligned} q_{i,j}^{N,+} - q_{i,j}^{N,-} &= \frac{\partial_t\varphi_{t_i}\left(G_{i,j}^N\right)h + \partial_p\varphi_{t_i}\left(G_{i,j}^N\right)\rho_{i,j}^N b_{i,j}h - \frac{1}{2}\partial_{pp}\varphi_{t_i}\left(G_{i,j}^N\right)\left[\rho_{i,j}^N\right]^2 h + o(h)}{\partial_p\varphi\left(t_i, G_{i,j}^N\right)\rho_{i,j}^N\sqrt{h} + o\left(\sqrt{h}\right)}\\ &= \left[b_{i,j} + \frac{\partial_t\varphi_{t_i}\left(G_{i,j}^N\right) - \frac{1}{2}\partial_{pp}\varphi_{t_i}\left(G_{i,j}^N\right)\left[\rho_{i,j}^N\right]^2}{\partial_p\varphi\left(t_i, G_{i,j}^N\right)\rho_{i,j}^N} + o(1)\right]\sqrt{h}\\ &= \left[b_{i,j} + \frac{\partial_t\varphi_{t_i}(G_{i,j}) - \frac{1}{2}\partial_{pp}\varphi_{t_i}(G_{i,j})[\rho_{i,j}]^2}{\partial_p\varphi(t_i, G_{i,j})\rho_{i,j}} + o(1)\right]\sqrt{h}\\ &= \left[\mu\left(t_i, x_{i,j}\right) + o(1)\right]\sqrt{h}.\end{aligned}$$

Thus, by (72) and (71),

$$\begin{aligned} 0 &\le \left[\partial_t\tilde{w}\left(t_i, x_{i,j}\right) + \frac{1}{2}\partial_{xx}\tilde{w}\left(t_i, x_{i,j}\right) + \mu\left(t_i, x_{i,j}\right)\partial_x\tilde{w}\left(t_i, x_{i,j}\right)\right]h + o(h)\\ &= \left[\partial_t\tilde{w}(\bar{t}, \bar{x}) + \frac{1}{2}\partial_{xx}\tilde{w}(\bar{t}, \bar{x}) + \mu(\bar{t}, \bar{x})\partial_x\tilde{w}(\bar{t}, \bar{x})\right]h + o(h).\end{aligned}$$

This implies $\mathscr{L}\tilde{w}(\bar{t}, \bar{x}) \ge 0$. By (70), it is clear that $\mathscr{L}w(\bar{t}, \bar{x}) = \mathscr{L}\tilde{w}(\bar{t}, \bar{x})$. Then $\mathscr{L}w(\bar{t}, \bar{x}) \ge 0$; thus, $\bar{u}$ is a viscosity subsolution at $(\bar{t}, \bar{x})$ □

5. The General Diffusion Case

In this section we consider a general diffusion process given by the SDE:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad \mathbb{P}\text{-almost surely (a.s.)} \tag{73}$$

Provided that σ is nondegenerate, this problem can be transformed back to (36):

$$\hat{X}_t := \psi(t, X_t), \quad \hat{x}_0 := \psi(0, x_0), \quad \text{where} \quad \psi(t, x) := \int_0^x \frac{dy}{\sigma(t, y)}. \tag{74}$$

Then, by a simple application of Itô's formula, we have

$$\hat{X}_t = \hat{x}_0 + \int_0^t \hat{b}\left(s, \hat{X}_s\right)ds + B_t, \quad \text{where} \quad \hat{b}(t, x) := \left[\partial_t\psi + \frac{b}{\sigma} - \frac{1}{2}\partial_x\sigma\right]\left(t, \psi^{-1}(t, x)\right). \tag{75}$$

Here ψ^{-1} is the inverse mapping of $x \mapsto \psi(t, x)$. Denote

$$G(t, x) := \mathbb{P}(X_t \ge x), \quad \rho := -\partial_x G, \quad \hat{G}(t, x) := \mathbb{P}\left(\hat{X}_t \ge x\right), \quad \hat{\rho} := -\partial_x\hat{G}. \tag{76}$$

To formulate a rigorous statement, we shall make the following assumption.

Assumption 3. *The functions b, σ are sufficiently smooth, and both b, σ and the required derivatives are bounded. Moreover, $\sigma \geq c_0 > 0$.*

The following result is immediate and we omit the proof.

Lemma 3. *Under Assumption 3, we have*
(i) *the $\hat{b}$ defined in* (75) *satisfies Assumption 1;*
(ii) $G(t,x) = \hat{G}(t, \psi(t,x)), \rho(t,x) = \hat{\rho}(t,\psi(t,x))/\sigma(t,x)$ are sufficiently smooth and satisfy (48).

Here is the main result of this section.

Theorem 5. *Assume* Assumptions 3 *and* 2 *hold. Let $\check{\Phi}$ be a time-consistent dynamic distortion function determined by* (63) *for $0 < s < t \leq T$, where $\check{\sigma}$ and $\check{\mu}$ satisfy the same technical requirements as in* Theorem 3. *Then $\check{\Phi}$ is consistent with initial condition $\check{\Phi}(0, t, x_0; p) = \varphi_t(p)$ if and only if*

$$\check{\mu} = b - \sigma\partial_x\sigma + \check{\sigma}\partial_x\check{\sigma} + \frac{1}{2}\left[\check{\sigma}^2 - \sigma^2\right]\frac{\partial_x\rho}{\rho} + \frac{\partial_t\varphi(t, G(t,x))}{\partial_p\varphi(t, G(t,x))\rho} - \frac{\check{\sigma}^2\rho\partial_{pp}\varphi(t, G(t,x))}{2\partial_p\varphi(t, G(t,x))}. \tag{77}$$

In particular, if we require $\check{\sigma} = \sigma$, then $\check{\Phi}$ is unique with

$$\check{\mu}(t,x) = \mu(t,x) := b(t,x) + \frac{\partial_t\varphi(t, G(t,x)) - \frac{1}{2}\partial_{pp}\varphi(t, G(t,x))\rho^2\sigma^2(t,x)}{\partial_p\varphi(t, G(t,x))\rho(t,x)}. \tag{78}$$

Proof. Let $g \in \mathcal{I}$, and let $\check{u}$ be the solution to PDE (64) on $(0, T] \times \mathbb{R}$ with terminal condition g. Then $\check{\Phi}$ is consistent with initial condition $\check{\Phi}(0, t, x_0; p) = \varphi_t(p)$ means the mapping $t \in (0, T] \mapsto \mathscr{E}_{0,t}[\check{u}(t, X_t)]$ is a constant. Note that

$$\mathscr{E}_{0,t}[\check{u}(t, X_t)] = \int_{\mathbb{R}} \varphi_t(\mathbb{P}(X_t \geq x))\partial_x\check{u}(t,x)dx = \int_{\mathbb{R}} \varphi_t\big(\mathbb{P}\big(\hat{X}_t \geq \psi(t,x)\big)\big)\partial_x\check{u}(t,x)dx.$$

Denote $\hat{x} := \psi(t,x)$. Then

$$\mathscr{E}_{0,t}[\check{u}(t, X_t)] = \int_{\mathbb{R}} \varphi_t\big(\mathbb{P}\big(\hat{X}_t \geq \hat{x}\big)\big)\partial_{\hat{x}}\hat{u}(t,\hat{x})d\hat{x}, \quad \text{where} \quad \hat{u}(t,\hat{x}) := \check{u}\big(t, \psi^{-1}(t,\hat{x})\big). \tag{79}$$

Note that $\check{u}(t,x) = \hat{u}(t, \psi(t,x))$. Then

$$\partial_t\check{u} = \partial_t\hat{u} + \partial_{\hat{x}}\hat{u}\partial_x\psi, \quad \partial_x\check{u} = \partial_{\hat{x}}\hat{u}\partial_x\psi, \quad \partial_{xx}\check{u} = \partial_{\hat{x}\hat{x}}\hat{u}\big(\partial_x\psi\big)^2 + \partial_{\hat{x}}\hat{u}\partial_{xx}\psi,$$

and thus, PDE (64) implies

$$\begin{aligned} 0 &= \left[\partial_t\hat{u} + \partial_{\hat{x}}\hat{u}\partial_t\psi\right] + \frac{1}{2}\check{\sigma}^2\left[\partial_{\hat{x}\hat{x}}\hat{u}\big(\partial_x\psi\big)^2 + \partial_{\hat{x}}\hat{u}\partial_{xx}\psi\right] + \check{\mu}\partial_{\hat{x}}\hat{u}\partial_x\psi \\ &= \partial_t\hat{u} + \frac{1}{2}\big(\check{\sigma}\partial_x\psi\big)^2\partial_{\hat{x}\hat{x}}\hat{u} + \left[\partial_t\psi + \frac{1}{2}\check{\sigma}^2\partial_{xx}\psi + \check{\mu}\partial_x\psi\right]\partial_{\hat{x}}\hat{u}. \end{aligned}$$

Recall (75) and (79), and note that $G(t,x) = \hat{G}(t, \psi(t,x))$. Applying Theorem 3 we see that the required time consistency is equivalent to

$$\begin{aligned} &\partial_t\psi + \frac{1}{2}\check{\sigma}^2\partial_{xx}\psi + \check{\mu}\partial_x\psi \\ &\quad = \hat{b} + \big(\check{\sigma}\partial_x\psi\big)\partial_{\hat{x}}\big(\check{\sigma}\partial_x\psi\big) + \frac{1}{2}\left[\big(\check{\sigma}\partial_x\psi\big)^2 - 1\right]\frac{\partial_{\hat{x}}\hat{\rho}}{\hat{\rho}} + \frac{\partial_t\varphi(t, G(t,x))}{\partial_p\varphi(t, G(t,x))\hat{\rho}} - \frac{\big(\check{\sigma}\partial_x\psi\big)^2\partial_{pp}\varphi(t, G(t,x))\hat{\rho}}{2\partial_p\varphi(t, G(t,x))}. \end{aligned} \tag{80}$$

Note that

$$\begin{aligned} &\partial_x\psi\partial_{\hat{x}}\big(\check{\sigma}\partial_x\psi\big) = \partial_{\hat{x}}\big(\check{\sigma}\partial_x\psi\big), \quad \partial_x\psi = \frac{1}{\sigma}, \quad \partial_{xx}\psi = -\frac{\partial_x\sigma}{\sigma^2}, \\ &\hat{\rho}\big(t, \psi(t,x)\big) = \rho\sigma(t,x), \quad \partial_{\hat{x}}\hat{\rho} = \left[\partial_x\rho\sigma + \rho\partial_x\sigma\right]\sigma. \end{aligned}$$

Then (80) is equivalent to

$$\partial_t\psi - \frac{\check\sigma^2\partial_x\sigma}{2\sigma^2} + \frac{\check\mu}{\sigma} = \left[\partial_t\psi + \frac{b}{\sigma} - \frac{1}{2}\partial_x\sigma\right] + \left[\frac{\check\sigma\partial_x\check\sigma}{\sigma} - \frac{\check\sigma^2\partial_x\sigma}{\sigma^2}\right] + \frac{1}{2}\left[\left(\frac{\check\sigma}{\sigma}\right)^2 - 1\right]\left[\frac{\partial_x\rho\sigma}{\rho} + \partial_x\sigma\right] + \frac{\partial_t\varphi(t,\psi(t,x))}{\partial_p\varphi(t,G(t,x))\rho\sigma} - \frac{\check\sigma^2\rho\partial_{pp}\varphi(t,G(t,x))}{2\partial_p\varphi(t,G(t,x))\sigma}.$$

This implies (77) immediately. □

Remark 9. In this remark we investigate possible discretization for the general SDE (73), in the spirit of Section 4.1. Note that

$$X_{t_{i+1}} \approx X_{t_i} + b(t_i, X_{t_i})h + \sigma(t_i, X_{t_i})[B_{t_{i+1}} - B_{t_i}].$$

For a desired approximation X^N, we would expect

$$\mathbb{E}\left[X^N_{t_{i+1}} - X^N_{t_i}\middle|X^N_{t_i} = x\right] = b(t_i,x)h + o(h),\ \mathbb{E}\left[\left(X^N_{t_{i+1}} - X^N_{t_i}\right)^2\middle|X^N_{t_i} = x\right] = \sigma^2(t_i,x)h + o(h). \tag{81}$$

However, for the binomial tree in Figure 3, at each node $x_{i,j}$ there is only one parameter $p^+_{i,j}$, and in general, we are not able to match both the drift and the volatility. To overcome this, we have three natural choices:

(i) The first one is to use trinomial tree approximation: assuming $0 < \sigma \le C_0$, we have

$$x_{i,j} = C_0 j\sqrt{h},\ j = -i,\dots,i,\quad \mathbb{P}\left(X^N_{t_{i+1}} = x_{i+1,j+1}\middle|X^N_{t_i} = x_{i,j}\right) = p^+_{i,j},$$
$$\mathbb{P}\left(X^N_{t_{i+1}} = x_{i+1,j-1}\middle|X^N_{t_i} = x_{i,j}\right) = p^-_{i,j},\quad \mathbb{P}\left(X^N_{t_{i+1}} = x_{i+1,j}\middle|X^N_{t_i} = x_{i,j}\right) = p^0_{i,j} := 1 - p^+_{i,j} - p^-_{i,j}.$$

See the left figure in Figure 4 for the case $N = 2$. Then, by choosing appropriate $p^+_{i,j}, p^-_{i,j}$, one may achieve (81). However, note that the trinomial tree has crossing edges, and they may destroy the crucial monotonicity property we used in the previous section, as we saw in Remark 4(iii) and Example 3.

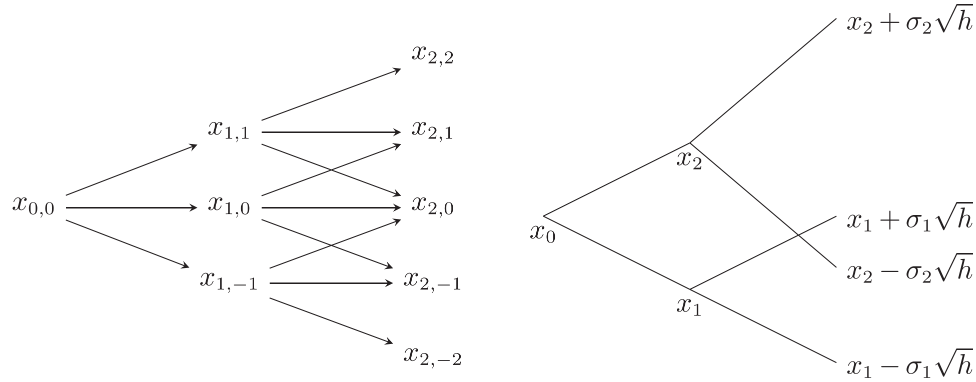
(ii) The second choice is to use the binary tree approximation (see the right figure in Figure 4 for the case $N = 2$), where $x_1 = x_0 - \sigma(t_0,x_0)\sqrt{h}$, $x_2 = x_0 + \sigma(t_0,x_0)\sqrt{h}$, $\sigma_1 = \sigma(t_1,x_1)$, and $\sigma_2 = \sigma(t_1,x_2)$. But again there are crossing edges, and thus, the monotonicity property is violated.

(iii) The third choice, which indeed works well, is to utilize the transformation (74). Let $\hat X^N$ be the discretization for $\hat X$ in (75), as introduced in Section 4.1. Then $X^N := \psi^{-1}_{t_i}(\hat X^N_{t_i})$ will serve for our purpose. We skip the details here.

6. Analysis of the Density

In this section we prove Propositions 3 and 4. The estimates rely on the following representation formula for ρ by using the Brownian bridge. The result is a direct consequence of Karatzas-Shreve [14, section 5.6, exercise 6.17] and holds true in the multidimensional case as well.

Figure 4. Left: trinomial tree. Right: binary tree.



Proposition 5. *Assume b is bounded. Then we have the following representation formula:*

$$\rho(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-x_0)^2}{2t} + I(t,x)\right), \quad t>0, \quad \textit{where}$$
$$\bar{M}_s^t := \int_0^s \frac{dB_r}{t-r}, \quad \bar{X}_s^{t,x} := x_0 + [x-x_0]\frac{s}{t} + [t-s]\bar{M}_s^t, \quad 0 \le s < t;$$
$$e^{I(t,x)} := \mathbb{E}\left[e^{\int_0^t b(s,\bar{X}_s^{t,x})dB_s + \int_0^t \left[(x-x_0)b(s,\bar{X}_s^{t,x}) - b(s,\bar{X}_s^{t,x})\bar{M}_s^t - \frac{1}{2}|b(s,\bar{X}_s^{t,x})|^2\right]ds}\right]. \tag{82}$$

Proof. Since we will use the arguments, in particular, that for (84), in the proof of Proposition 4, we provide a detailed proof here. For notational simplicity, let us assume $t=1$ and $x_0=0$. Then (82) becomes:

$$\rho(1,x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + I(x)\right), \quad \text{where}$$
$$\bar{M}_s := \int_0^s \frac{dB_r}{1-r}, \quad \bar{X}_s^x := xs + [1-s]\bar{M}_s, 0 \le s < 1;$$
$$e^{I(x)} := \mathbb{E}\left[e^{\int_0^1 b(s,\bar{X}_s^x)dB_s + \int_0^1 \left[xb(s,\bar{X}_s^x) - b(s,\bar{X}_s^x)\bar{M}_s - \frac{1}{2}|b(s,\bar{X}_s^x)|^2\right]ds}\right]. \tag{83}$$

We first show that the right-hand side of the last line in (83) is integrable. Since b is bounded, it suffices to prove the following (stronger) claim: for any $C>0$ and $\alpha \in (0,2)$,

$$\mathbb{E}\left[e^{C\int_0^1 |\bar{M}_s|^\alpha ds}\right] < \infty. \tag{84}$$

Indeed, by time change $s = \frac{t}{1+t}$, we have $\int_0^1 |\bar{M}_s|^\alpha ds = \int_0^\infty |\bar{M}_{t/(1+t)}|^\alpha (1+t)^2 dt$. Since

$$\mathbb{E}\left[|\bar{M}_{\frac{t}{1+t}}|^2\right] = \int_0^{\frac{t}{1+t}} \frac{dr}{(1-r)^2} = t,$$

by Levy's characterization we see that $t \mapsto \bar{M}_{t/(1+t)}$ is a Brownian motion. Then

$$\mathbb{E}\left[e^{C\int_0^1 |\bar{M}_s|^\alpha ds}\right] = \mathbb{E}\left[e^{C\int_0^\infty \frac{|B_t|^\alpha}{(1+t)^2}dt}\right] = \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\left(\int_0^\infty \frac{|B_t|^\alpha}{(1+t)^2}dt\right)^n\right].$$

Note that

$$\int_0^\infty \frac{|B_t|^\alpha}{(1+t)^2}dt \le \sup_{t\ge 0}\frac{|B_t|^\alpha}{(1+t)^{\frac{2+\alpha}{4}}}\int_0^\infty \frac{dt}{(1+t)^{1+\frac{2-\alpha}{4}}} = \frac{4}{2-\alpha}\sup_{t\ge 0}\frac{|B_t|^\alpha}{(1+t)^{\frac{2+\alpha}{4}}}.$$

Then, for a generic constant C,

$$\begin{aligned}
\mathbb{E}\left[e^{C\int_0^1 |\bar{M}_s|^\alpha ds}\right] &\le \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{t\ge 0}\frac{|B_t|^{n\alpha}}{(1+t)^{\frac{n(2+\alpha)}{4}}}\right] \\
&\le \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{0\le t\le 1}|B_t|^{n\alpha} + \sum_{m=0}^\infty \sup_{2^m\le t\le 2^{m+1}}\frac{|B_t|^{n\alpha}}{(1+t)^{\frac{n(2+\alpha)}{4}}}\right] \\
&\le \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{0\le t\le 1}|B_t|^{n\alpha} + \sum_{m=0}^\infty 2^{-\frac{mn(2+\alpha)}{4}}\sup_{0\le t\le 2^{m+1}}|B_t|^{n\alpha}\right] \\
&= \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{0\le t\le 1}|B_t|^{n\alpha} + \sum_{m=0}^\infty 2^{-\frac{mn(2+\alpha)}{4}+\frac{(m+1)n\alpha}{2}}\sup_{0\le t\le 1}|B_t|^{n\alpha}\right] \\
&\le \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{0\le t\le 1}|B_t|^{n\alpha}\right]\sum_{m=0}^\infty 2^{-\frac{mn(2-\alpha)}{4}} \le \sum_{n=0}^\infty \frac{C^n}{n!}\mathbb{E}\left[\sup_{0\le t\le 1}|B_t|^{n\alpha}\right] \\
&= \mathbb{E}\left[e^{C\sup_{0\le t\le 1}|B_t|^\alpha}\right].
\end{aligned}$$

This implies (84) immediately.

We now prove (83). By [14, section 5.6.B], conditional on $\{B_1 = x\}$, B is a Brownian bridge and its conditional law is equal to the law of $\bar{X}^x$. Then, by the Girsanov theorem,

$$
\begin{aligned}
G(1,x) = \mathbb{P}(X_1 \geq x) &= \mathbb{E}\Big[e^{\int_0^1 b(s,B_s)dB_s - \frac{1}{2}\int_0^1 |b(s,B_s)|^2 ds} 1_{\{B_1 \geq x\}}\Big] \\
&= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbb{E}\Big[e^{\int_0^1 b(s,B_s)dB_s - \frac{1}{2}\int_0^1 |b(s,B_s)|^2 ds}\Big| B_1 = y\Big] dy \\
&= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbb{E}\Big[e^{\int_0^1 b(s,\bar{X}_s^y) d\bar{X}_s^y - \frac{1}{2}\int_0^1 |b(s,X_s^y)|^2 ds}\Big] dy.
\end{aligned}
$$

This, together with the fact that $d\bar{X}_s^x = xds - \bar{M}_s ds + dB_s$, implies (83) immediately. □

Proof of Proposition 3. Again we shall only prove the case that $t = 1, x_0 = 0$.

We first show that, for the I in (83),

$$|I'(x)| \leq C. \tag{85}$$

This, together with (82), implies immediately the first estimate in (48).

Indeed, denote $\bar{b}(t,x) := \int_0^x b(t,y)dy$. Applying the Itô formula we have

$$\bar{b}(1,x) = \bar{b}(1,\bar{X}_1^x) - \bar{b}(0,\bar{X}_0^x) = \int_0^1 \Big[\partial_t \bar{b}(t,\bar{X}_t^x) + \frac{1}{2}\partial_x b(t,\bar{X}_t^x)\Big] dt + \int_0^1 b(t,\bar{X}_t^x) d\bar{X}_t^x.$$

Then

$$e^{I(x)} = \mathbb{E}\Big[e^{\bar{b}(1,x) - \int_0^1 [\partial_t \bar{b}(t,\bar{X}_t^x) + \frac{1}{2}\partial_x b(t,\bar{X}_t^x)] dt}\Big]. \tag{86}$$

Differentiating with respect to x and noting that $\partial_x \bar{X}_t^x = t$, we have

$$e^{I(x)} I'(x) = \mathbb{E}\Big[e^{\bar{b}(1,x) - \int_0^1 [\partial_t \bar{b}(t,\bar{X}_t^x) + \frac{1}{2}\partial_x b(t,\bar{X}_t)] dt} \Big[b(1,x) - \int_0^1 t\Big[\partial_t b(t,\bar{X}_t^x) + \frac{1}{2}\partial_{xx} b(t,\bar{X}_t)\Big] dt\Big]\Big].$$

This implies

$$e^{I(x)} |I'(x)| \leq C\mathbb{E}\Big[e^{\bar{b}(1,x) - \int_0^1 [\partial_t \bar{b}(t,\bar{X}_t^x) + \frac{1}{2}\partial_x b(t,\bar{X}_t)] dt}\Big] = Ce^{I(x)}, \quad \text{and thus} \quad |I'(x)| \leq C.$$

We next verify the second part of (48) for $x > 0$. The case $x < 0$ can be proved similarly. Clearly it suffices to verify it for x large. Note that

$$\frac{G(1,x)}{\rho(1,x)} = \int_0^\infty \frac{\rho(1,x+y)}{\rho(1,x)} dy = \int_0^\infty e^{I(x+y) - \frac{1}{2}(x+y)^2 + \frac{x^2}{2} - I(x)} dy = \int_0^\infty e^{I(x+y) - I(x) - xy - \frac{1}{2}y^2} dy.$$

Then, for $x > C + 1$, where C is the bound of I',

$$
\begin{aligned}
\frac{G(1,x)}{\rho(1,x)} &\leq \int_0^\infty e^{Cy - xy} dy = \frac{1}{x - C} \leq 1; \\
\frac{G(1,x)}{\rho(1,x)} &\geq \int_0^1 e^{-Cy - xy - \frac{1}{2}y^2} dy \geq e^{-\frac{1}{2}} \frac{1 - e^{-x-C}}{x + C} \geq \frac{c}{x},
\end{aligned}
$$

completing the proof. □

Proof of Proposition 4. The convergence of G^N is standard and is also implied by the convergence of ρ^N, so we shall only prove the latter. Assume for simplicity that $T = 1$. Note that ρ is locally uniformly continuous in $(0,T] \times \mathbb{R}$. Without loss of generality we shall only estimate $|\rho^N(1,x) - \rho(1,x)|$ for x in the range of X_1^N. We remark that we shall assume $|x| \leq R$ for some constant $R > 0$, and in the following proof, the generic constant C may depend on R.

Let ξ_i^N, $i = 1, \ldots, N$, be independent and identically distributed with $\mathbb{P}(\xi_i^N = 1/\sqrt{N}) = \mathbb{P}(\xi_i^N = -1/\sqrt{N}) = \frac{1}{2}$, $B_{t_0}^N = 0$, and $B_{t_{i+1}}^N := B_{t_i}^N + \xi_{i+1}^N$, and denote $b_i^N := b(t_i, B_{t_i}^N)$. Introduce the conditional expectation:

$$\mathbb{E}_x[\cdot] := \mathbb{E}\big[\cdot | B_1^N = x\big].$$

Then we see that

$$\begin{aligned}\rho^N(1,x) &= \mathbb{P}\big(X_1^N = x\big)/\Big(2\sqrt{h}\Big) = \mathbb{E}\Big[\Pi_{i=0}^{N-1}\big[1 + b_i^N \xi_{i+1}^N\big]1_{\{B_1^N = x\}}\Big]/\Big(2\sqrt{h}\Big)\\ &= \mathbb{E}_x\left[\prod_{i=0}^{N-1}\big[1 + b_i^N \xi_{i+1}^N\big]\right]\mathbb{P}\big(B_1^N = x\big)/\Big(2\sqrt{h}\Big)\\ &= \mathbb{E}_x\left[e^{\sum_{i=0}^{N-1}\left[b_i^N \xi_{i+1}^N - \frac{1}{2}|b_i^N|^2 h\right]}[1 + o(1)]\right]\mathbb{P}\big(B_1^N = x\big)/\Big(2\sqrt{h}\Big).\end{aligned}$$

One can easily show that $\lim_{N\to\infty}\mathbb{P}(B_1^N = x)/(2\sqrt{h}) = e^{-x^2/2}/\sqrt{2\pi}$, by an elementary argument using Stirling's approximation. Then it remains to establish the limit

$$\mathbb{E}_x\left[e^{\sum_{i=0}^{N-1}\left[b_i^N \xi_{i+1}^N - \frac{1}{2}|b_i^N|^2 h\right]}\right] \longrightarrow e^{I(x)}. \tag{87}$$

We proceed in three steps, and for simplicity we assume $N = 2n$ and $x = 2k/\sqrt{2n}$.

Step 1. Fix $t \in (0,1)$, and assume $t = t_i$ for some even $i = 2m$. (More rigorously we shall consider $t_{2m} \le t < t_{2m+2}$.) For any bounded and smooth test function f,

$$\begin{aligned}\mathbb{E}_x\Big[f\Big(B_{t_i}^N\Big)\Big] &= \sum_j f(x_{ij})\frac{\mathbb{P}\Big(B_{t_i}^N = x_{ij}, B_1^N = x\Big)}{\mathbb{P}\big(B_1^N = x\big)} = \sum_j f(x_{ij})\frac{\mathbb{P}\Big(B_{t_i}^N = x_{ij}, B_1^N - B_{t_i}^N = x - x_{ij}\Big)}{\mathbb{P}\big(B_1^N = x\big)}\\ &= \sum_l f\Big(2l\sqrt{h}\Big)\frac{\mathbb{P}\Big(B_{t_i}^N = 2l\sqrt{h}\Big)\mathbb{P}\Big(B_1^N - B_{t_i}^N = 2(k-l)\sqrt{h}\Big)}{\mathbb{P}\Big(B_1^N = 2k\sqrt{h}\Big)}.\end{aligned}$$

Note that $m/n = t$ and $k/n = x\sqrt{h}$, and denote $y := \frac{2l}{\sqrt{2n}} = 2l\sqrt{h}$. By Stirling's formula we have

$$\begin{aligned}\mathbb{E}_x\Big[f\Big(B_{t_i}^N\Big)\Big] &= \sum_l f\Big(2l\sqrt{h}\Big)\frac{\frac{(2m)!}{(m+l)!(m-l)!}\frac{(2n-2m)!}{(n-m+k-l)!(n-m-k+l)!}}{\frac{(2n)!}{(n+k)!(n-k)!}}\\ &= [1 + o(1)]\sum_l f\Big(2l\sqrt{h}\Big)\sqrt{\frac{2m(n-m)(n^2-k^2)}{2\pi n(m^2-l^2)\big((n-m)^2-(k-l)^2\big)}}\times\\ &\qquad\frac{m^{2m}(n-m)^{2(n-m)}(n+k)^{n+k}(n-k)^{n-k}}{(m+l)^{m+l}(m-l)^{m-l}(n-m+k-l)^{n-m+k-l}(n-m-k+l)^{n-m-k+l}n^{2n}}\\ &= [1 + o(1)]\sum_l f\Big(2l\sqrt{h}\Big)\sqrt{\frac{2t(1-t)(1-x^2h)}{2\pi n\big(t^2-y^2h\big)\Big((1-t)^2-\big(x-y\big)^2h\Big)}}\frac{A_1}{A_2A_3},\end{aligned}$$

where

$$
\begin{aligned}
A_1 &:= \left(1 + x\sqrt{h}\right)^{n\left(1+x\sqrt{h}\right)}\left(1 - x\sqrt{h}\right)^{n\left(1-x\sqrt{h}\right)};\\
A_2 &:= \left(1 + \frac{y}{t}\sqrt{h}\right)^{n\left(t+y\sqrt{h}\right)}\left(1 - \frac{y}{t}\sqrt{h}\right)^{n\left(t-y\sqrt{h}\right)};\\
A_3 &:= \left(1 + \frac{x-y}{1-t}\sqrt{h}\right)^{n\left(1-t+(x-y)\sqrt{h}\right)}\left(1 - \frac{x-y}{1-t}\sqrt{h}\right)^{n\left(1-t-(x-y)\sqrt{h}\right)}.
\end{aligned}
\tag{88}
$$

Note that, for any $0 < z < 1$,

$$
e^{z^2} \le (1+z)^{1+z}(1-z)^{1-z} \le e^{z^2+\frac{2z^3}{3}}.
$$

Then, by noting that $n = 1/2h$,

$$
\begin{aligned}
\frac{A_1}{A_2A_3} &\le e^{n\left[x^2h+\frac{2}{3}x^3h^{\frac{3}{2}}-\frac{y^2h}{t}-\frac{(x-y)^2h}{1-t}\right]} = e^{-\frac{(tx-y)^2}{2t(1-t)}+\frac{1}{3}x^3\sqrt{h}};\\
\frac{A_1}{A_2A_3} &\ge e^{n\left[x^2h-\frac{y^2h}{t}-\frac{2|y|^3h^{\frac{3}{2}}}{3t^3}-\frac{(x-y)^2h}{1-t}-\frac{2|x-y|^3h^{\frac{3}{2}}}{3(1-t)^3}\right]} = e^{-\frac{(tx-y)^2}{2t(1-t)}-\frac{1}{3}\left[\frac{|y|^3}{t^3}+\frac{|x-y|^3}{(1-t)^3}\right]\sqrt{h}}.
\end{aligned}
\tag{89}
$$

Then, by denoting $x \approx y$ as $x = y[1 + o(1)]$ for $h \to 0$, we have

$$
\mathbb{E}_x\left[f\left(B^N_{t_i}\right)\right] \approx \sum_l f\left(2l\sqrt{h}\right)\frac{2\sqrt{h}}{\sqrt{2\pi t(1-t)}}e^{-\frac{(tx-y)^2}{2t(1-t)}} \approx \int f(y)\frac{1}{\sqrt{2\pi t(1-t)}}e^{-\frac{(tx-y)^2}{2t(1-t)}}dy.
$$

That is, for $t_i = t$, the conditional law of B^N_t given $B^N_1 = x$ asymptotically has density

$$
\frac{1}{\sqrt{2\pi t(1-t)}}e^{-\frac{(tx-y)^2}{2t(1-t)}}dy,
$$

which is exactly the density of the $\bar{X}^x_t$ defined in (83).

Step 2. Again assume for simplicity that $i = 2m$ is even. Note that, for each l,

$$
\begin{aligned}
&\mathbb{P}\left(\xi^N_{i+1} = \sqrt{h}|B^N_{t_i} = 2l\sqrt{h}, B^N_1 = x\right)\\
&\quad= \mathbb{P}\left(\xi^N_{i+1} = \sqrt{h}|B^N_{t_i} = 2l\sqrt{h}, B^N_1 - B^N_{t_i} = 2(k-l)\sqrt{h}\right)\\
&\quad= \frac{\mathbb{P}\left(B^N_{t_i} = 2l\sqrt{h}, \xi^N_{i+1} = \sqrt{h}, B^N_1 - B^N_{t_{i+1}} = (2k-2l-1)\sqrt{h}\right)}{\mathbb{P}\left(B^N_{t_i} = 2l\sqrt{h}, B^N_1 - B^N_{t_i} = (2k-2l)\sqrt{h}\right)}\\
&\quad= \frac{\mathbb{P}\left(\xi^N_{i+1} = \sqrt{h}\right)\mathbb{P}\left(B^N_1 - B^N_{t_{i+1}} = (2k-2l-1)\sqrt{h}\right)}{\mathbb{P}\left(B^N_1 - B^N_{t_i} = (2k-2l)\sqrt{h}\right)}\\
&\quad= \frac{(2n-2m-1n-m+k-l-1)}{(2n-2mn-m+k-l)} = \frac{n-m+k-l}{2(n-m)}.
\end{aligned}
$$

Note further that, given $B^N_{t_i}$, $(B^N_{t_1}, \ldots, B^N_{t_{i-1}})$ and (ξ^N_{i+1}, B^N_1) are conditionally independent. Then

$$
\mathbb{P}\left(\xi^N_{i+1} = \sqrt{h}|\mathcal{F}^N_{t_i}, B^N_1 = x\right) = \frac{n-m+k-\frac{B^N_{t_i}}{2\sqrt{h}}}{2(n-m)} = \frac{1}{2} - \frac{B^N_{t_i} - x}{2(1-t_i)}\sqrt{h},
$$

where $\mathcal{F}^N_{t_i} := \sigma(B^N_{t_1},\ldots,B^N_{t_i})$. This implies

$$\mathbb{E}_x\Big[\xi^N_{i+1}|\mathcal{F}^N_{t_i}\Big] = \sqrt{h}\bigg[\frac{1}{2}-\frac{B^N_{t_i}-x}{2(1-t_i)}\sqrt{h}\bigg] - \sqrt{h}\bigg[\frac{1}{2}+\frac{B^N_{t_i}-x}{2(1-t_i)}\sqrt{h}\bigg] = -\frac{B^N_{t_i}-x}{1-t_i}h.$$

Now denote, for $i < N-1$,

$$\bar{\xi}^N_{i+1} := \xi^N_{i+1} - \mathbb{E}_x\Big[\xi^N_{i+1}|\mathcal{F}^N_{t_i}\Big] = \frac{1-t_i}{1-t_{i+1}}\xi^N_{t_{i+1}} + \frac{h}{1-t_{i+1}}\Big[B^N_{t_i}-x\Big]. \tag{90}$$

Then $\mathcal{F}^N_i = \sigma(\bar{\xi}_1,\ldots,\bar{\xi}_i)$, and

$$|\bar{\xi}^N_{i+1}| \le \sqrt{h}, \quad \mathbb{E}_x\Big[\bar{\xi}^N_{i+1}|\mathcal{F}^N_{t_i}\Big] = 0. \tag{91}$$

By induction one can easily verify

$$B^N_{t_i} = xt_i + (1-t_i)\bar{M}^N_{t_i}, \quad \text{where} \quad \bar{M}^N_{t_i} := \sum_{j=0}^{i-1}\frac{\bar{\xi}^N_{j+1}}{1-t_j}. \tag{92}$$

By (91) we see that $\bar{M}^N$ is a martingale under the conditional expectation $\mathbb{E}_x$, and thus,

$$\mathbb{E}_x\Big[|\bar{M}^N_{t_i}|^2\Big] = \sum_{j=0}^{i-1}\frac{\mathbb{E}_x\Big[|\bar{\xi}^N_{j+1}|^2\Big]}{\big(1-t_j\big)^2} \le \sum_{j=0}^{i-1}\frac{h}{\big(1-t_j\big)^2} \le \int_0^{t_i}\frac{dt}{(1-t)^2} = \frac{t_i}{1-t_i}. \tag{93}$$

Clearly $\overline{M}^n_{t_i} = (B^N_{t_i} - xt_i)/(1-t_i)$. For any $C>0$, by setting $f(y) = e^{C(y-xt_i)/(1-t_i)}$ and applying the first inequality in (89), we have

$$\mathbb{E}_x\Big[e^{C\overline{M}^n_{t_i}}\Big] \le [1+o(1)]\int e^{C\frac{y-xt_i}{1-t_i}}\frac{1}{\sqrt{2\pi t_i(1-t_i)}}e^{-\frac{(t_ix-y)^2}{2t_i(1-t_i)}+\frac{1}{3}x^3\sqrt{h}}dy = [1+o(1)]e^{\frac{Ct_i}{1-t_i}}.$$

Similarly,

$$\mathbb{E}_x\Big[e^{-C\overline{M}^n_{t_i}}\Big] \le \big[1+o(1)\big]e^{\frac{Ct_i}{1-t_i}}.$$

Applying the Doob's maximum inequality on the martingale $\overline{M}^N$ we have that, for any $l \ge 2$,

$$\mathbb{E}_x\Bigg[\sum_{0\le j\le i}|\overline{M}^N_{t_j}|^l\Bigg] \le \bigg(\frac{l}{l-1}\bigg)^l\mathbb{E}_x\Big[|\overline{M}^N_{t_i}|^l\Big] \le C\mathbb{E}_x\Big[|\overline{M}^N_{t_i}|^l\Big].$$

This implies

$$\begin{aligned}\mathbb{E}_x\Bigg[e^{C\sup\limits_{0\le j\le i}|\overline{M}^N_{t_j}|}\Bigg] &= \sum_{l=0}^{\infty}\frac{C^l}{l!}\mathbb{E}_x\Bigg[\sup_{0\le j\le i}|\overline{M}^N_{t_j}|^l\Bigg] \le \sum_{l=0}^{\infty}\frac{C^l}{l!}\mathbb{E}_x\Big[|\overline{M}^N_{t_i}|^l\Big]\\ &= \mathbb{E}_x\Big[e^{C|\overline{M}^N_{t_i}|}\Big] \le \mathbb{E}_x\Big[e^{C\overline{M}^N_{t_i}} + e^{-C\overline{M}^N_{t_i}}\Big] \le Ce^{\frac{Ct_i}{1-t_i}}.\end{aligned}$$

Now following the arguments for (84), one can show that

$$\mathbb{E}_x\Bigg[e^{Ch\sum_{i=1}^{N-1}|\bar{M}^N_{t_i}|}\Bigg] \le C.$$

Moreover, note that

$$e^{\sum_{i=0}^{N-1}b_i\xi^N_{i+1}} = e^{b_{N-1}\xi^N_N + \sum_{i=0}^{N-2}b_i\Big[\bar{\xi}^N_{i+1} - h\bar{M}^N_{t_{i+1}} + xh\Big]} \le Ce^{\sum_{i=0}^{N-2}b_i\bar{\xi}^N_{i+1}}e^{Ch\sum_{i=1}^{N-1}|\bar{M}^N_{t_i}|}.$$

By (91) one can easily show that $\mathbb{E}_x[e^{C\sum_{i=0}^{N-2} b_i\bar{\xi}_{i+1}^N}] \le C$. Then we have

$$\mathbb{E}_x\left[e^{C\sum_{i=0}^{N-1}\left[b_i^N\xi_{i+1}^N - \frac{1}{2}|b_i^N|^2 h\right]}\right] \le C. \tag{94}$$

Step 3. Fix m, and set $s_j = j/m$. Similar to Step 1, we see that the conditional law of $(B_{s_1}^N, \dots, B_{s_m}^N)$ given $B_1^N = x$ is asymptotically equal to the law of $(\bar{X}_{s_1}^x, \dots, \bar{X}_{s_m}^x)$. Assume for simplicity that $N = nm$. (More rigorously we shall consider $nm \le N < (n+1)m$.) Then

$$\mathbb{E}_x\left[e^{\sum_{j=0}^{m-1}\left[b_{nj}^N\left(B_{s_{j+1}}^N - B_{s_j}^N\right) - \frac{1}{2m}|b_{nj}^N|^2\right]}\right] \approx \mathbb{E}\left[e^{\sum_{j=0}^{m-1}\left[b\left(s_j, \bar{X}_{s_j}^x\right)\left(\bar{X}_{s_{j+1}}^x - \bar{X}_{s_j}^x\right) - \frac{1}{2m}|b\left(s_j, \bar{X}_{s_j}^x\right)|^2\right]}\right]. \tag{95}$$

Send $m \to \infty$. Clearly the right-hand side of (95) converges to $e^{I(x)}$.

It remains to estimate the difference between the left-hand side of (87) and that of (95). Denote

$$\delta_{m,1}^N := \mathbb{E}_x\left[\left|\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{h}{2}\left|\left|b_{t_{nj+i}}^N\right|^2 - \left|b_{t_{nj}}^N\right|^2\right|\right|\right], \delta_{m,2}^N := \mathbb{E}_x\left[\left|\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left[\left[b_{t_{nj+i}}^N - b_{t_{nj}}^N\right]\left(B_{t_{nj+i+1}}^N - B_{t_{nj+i}}^N\right)\right]\right|\right]. \tag{96}$$

For any $R > |x|$, note that b is uniformly continuous on $[0, T] \times [-R, R]$ with some modulus of continuity function ρ_R. Then, for $j = 0, \dots, m-1$, $i = 0, \dots, n-1$,

$$\begin{aligned}
\delta_{m,i,j}^N &:= \mathbb{E}_x\left[\left|b^N\left(t_{nj+i}, B_{t_{nj+i}}^N\right) - b^N\left(t_{nj}, B_{t_{nj}}^N\right)\right|\right] \\
&\le C\mathbb{E}_x\left[\left[|B_{t_{nj+i}}^N - B_{t_{nj}}^N| + \rho_R\left(\frac{1}{m}\right) + 1_{\left\{|B_{t_{nj}}^N|>R\right\}} + 1_{\left\{|B_{t_{nj+i}}^N|>R\right\}}\right]\right] \\
&\le C\rho_R\left(\frac{1}{m}\right) + \frac{C}{R}\mathbb{E}_x\left[|B_{t_{nj}}^N| + |B_{t_{nj+i}}^N|\right] + C\mathbb{E}_x\left[|B_{t_{nj+i}}^N - B_{t_{nj}}^N|\right].
\end{aligned} \tag{97}$$

Recalling (93), we have

$$\begin{aligned}
&\mathbb{E}_x\left[|B_{t_i}^N|^2\right] \le C|xt_i|^2 + C(1-t_i)^2\mathbb{E}_x\left[|\bar{M}_{t_i}^N|^2\right] \le C(xt_i)^2 + Ct_i(1-t_i) \le C; \\
&\mathbb{E}_x\left[|B_{t_{nj+i}}^N - B_{t_{nj}}^N|^2\right] = \mathbb{E}_x\left[|\left[t_{nj+i} - t_{nj}\right]\left[x - \bar{M}_{t_{nj+i}}^N\right] + \left(1 - t_{nj}\right)\left[\bar{M}_{t_{nj+i}}^N - \bar{M}_{t_{nj}}^N\right]|^2\right] \\
&\le \frac{C}{m^2}\left[|x|^2 + \mathbb{E}_x\left[|\bar{M}_{t_{nj+i}}^N|^2\right]\right] + Ch\left(1 - t_{nj}\right)^2\sum_{l=0}^{i-1}\frac{1}{\left(1 - t_{nj+l}\right)^2} \\
&\le \frac{C}{m^2}\left[|x|^2 + \frac{1}{1 - t_{nj+i}}\right] + \frac{C}{m}\frac{1 - t_{nj}}{1 - t_{nj+i}} \le \frac{C}{m\left[1 - t_{nj+i}\right]}.
\end{aligned}$$

Then

$$\delta_{m,i,j}^N \le C\rho_R\left(\frac{1}{m}\right) + \frac{C}{R} + \frac{C}{\sqrt{m\left[1 - t_{nj+i}\right]}}. \tag{98}$$

Thus,

$$\delta_{m,1}^N \le Ch\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\delta_{m,i,j}^N \le C\rho_R\left(\frac{1}{m}\right) + \frac{C}{R} + \sum_{i=0}^{N-1}\frac{Ch}{\sqrt{m(1-t_i)}} \le C\left[\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m}}\right]. \tag{99}$$

Moreover,

$$\begin{aligned}\delta^N_{m,2} &= \mathbb{E}_x\left[\left|\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left[b^N_{t_{nj+i}} - b^N_{t_{nj}}\right]\left[h\left[x - \bar{M}^N_{t_{nj+i+1}}\right] + \bar{\xi}^N_{nj+i+1}\right]\right|\right]\\
&\le Ch\mathbb{E}_x\left[\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}|b^N_{t_{nj+i}} - b^N_{t_{nj}}||x - \bar{M}^N_{t_{nj+i+1}}|\right] + C\left(\mathbb{E}_x\left[\left|\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left[b^N_{t_{nj+i}} - b^N_{t_{nj}}\right]\bar{\xi}^N_{nj+i+1}\right|^2\right]\right)^{\frac{1}{2}}\\
&\le Ch\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left(\delta^N_{m,i,j}\right)^{\frac{1}{2}}\left(\mathbb{E}_x\left[|x - M^N_{t_{nj+i+1}}|^2\right]\right)^{\frac{1}{2}} + C\left(h\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\delta^N_{m,i,j}\right)^{\frac{1}{2}}\\
&\le Ch\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left(\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m(1-t_{nj})}}\right)^{\frac{1}{2}}\left(x^2 + \frac{1}{1-t_{nj+i+1}}\right)^{\frac{1}{2}}\\
&\quad + C\left(h\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left[\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m(1-t_{nj})}}\right]\right)^{\frac{1}{2}}\\
&\le C\left(\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m}}\right)^{\frac{1}{2}}.\end{aligned} \tag{100}$$

We now estimate the desired difference between (87) and (95). Denote

$$\xi_1 := \sum_{i=0}^{N-1}\left[b^N_i\xi^N_{i+1} - \frac{1}{2}|b^N_i|^2 h\right], \quad \xi_2 := \sum_{j=0}^{m-1}\left[b^N_{nj}\left(B^N_{s_{j+1}} - B^N_{s_j}\right) - \frac{1}{2m}|b^N_{nj}|^2\right].$$

Then, by (96), (99), and (100), we have

$$\mathbb{E}_x[|\xi_1 - \xi_2|] \le C\left[\delta^N_{m,1} + \delta^N_{m,2}\right] \le C\left(\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m}}\right)^{\frac{1}{2}}.$$

Moreover, similar to (94), we have

$$\mathbb{E}_x\left[e^{C\xi_1} + e^{-C\xi_1} + e^{C\xi_2} + e^{-C\xi_2}\right] \le C.$$

One can easily check that $|e^z - 1| \le C\sqrt{|z|}[e^{2z} + e^{-2z}]$. Then

$$\begin{aligned}\left|\mathbb{E}_x\left[e^{\xi_1}\right] - \mathbb{E}_x\left[e^{\xi_2}\right]\right| &= \mathbb{E}_x\left[e^{\xi_2}\left|e^{\xi_1-\xi_2} - 1\right|\right] \le C\mathbb{E}_x\left[e^{\xi_2}\sqrt{|\xi_1 - \xi_2|}\left[e^{2[\xi_1-\xi_2]} + e^{2[\xi_2-\xi_1]}\right]\right]\\
&\le C(\mathbb{E}_x[|\xi_1 - \xi_2|])^{\frac{1}{2}}\left(\mathbb{E}_x\left[e^{4\xi_1-2\xi_2} + e^{5\xi_2-4\xi_1}\right]\right)^{\frac{1}{2}} \le C\left(\rho_R\left(\frac{1}{m}\right) + \frac{1}{R} + \frac{1}{\sqrt{m}}\right)^{\frac{1}{4}}.\end{aligned}$$

By first sending $m \to \infty$ and then $R \to \infty$, we obtain the desired convergence. □

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