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Dynamic Set Values for Nonzero-Sum Games with Multiple Equilibriums

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Abstract. Nonzero sum games typically have multiple Nash equilibriums (or no equilibrium), and unlike the zero-sum case, they may have different values at different equilibriums. Instead of focusing on the existence of individual equilibriums, we study the set of values over all equilibriums, which we call the set value of the game. The set value is unique by nature and always exists (with possible value $\emptyset$). Similar to the standard value function in control literature, it enjoys many nice properties, such as regularity, stability, and more importantly, the dynamic programming principle. There are two main features in order to obtain the dynamic programming principle: (i) we must use closed-loop controls (instead of open-loop controls); and (ii) we must allow for path dependent controls, even if the problem is in a state-dependent (Markovian) setting. We shall consider both discrete and continuous time models with finite time horizon. For the latter, we will also provide a duality approach through certain standard PDE (or path-dependent PDE), which is quite efficient for numerically computing the set value of the game.

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Keywords: nonzero sum game • Nash equilibrium • set value • dynamic programming principle • closed-loop controls • path dependent PDE

1. Introduction

In a standard stochastic control problem, the value function is well defined and is the unique (viscosity) solution of the associated HJB equation or the path-dependent HJB equation in a path-dependent setting. The existence and/or uniqueness of optimal controls often require stronger conditions (typically certain compactness and/or convexity conditions). We remark that the value exists even if there is no optimal control; additionally, when there are multiple optimal controls, they share the same value. Similar results hold for two-person, zero-sum games under the Isaacs condition, where one may study the unique game value without requiring the existence or uniqueness of the equilibriums (saddle points). We refer to the book by Mertens et al. [30] for a general exposition of the theory and section 2 of Possamai et al. [33] for a literature review on continuous-time, two-person, zero-sum stochastic differential games. The situation is quite different for nonzero-sum stochastic differential games. There have been many works on the existence of Nash equilibriums by using either the PDE method or BSDE method; see, for example, Bensoussanand Frehse [4], Buckdahn et al. [6], Cardaliaguet and Plaskacz [8], El-Karoui and Hamadene [13], Friedman [18], Hamadene [19], Hamadene et al. [23], Hamadene and Mannucci [20], Hamadene and Mu [21, 22], Lin [26], Mannucci [28, 29], Olsder [31], Rainer [34], Sun and Young [37], Uchida [38], and Wu [39], to mention a few. We emphasize that, unlike stochastic control problems or zero-sum games, in the nonzero-sum case, different equilibriums could lead to different values, which makes it difficult to study the game value in a standard manner when there are multiple equilibriums. On the other hand, when there is no equilibrium, it becomes inconvenient even to define the game value.

We shall define the game value as the set of the values of the game over all equilibriums, which we call the set value of the game. For general set valued analysis, we refer to the book by Aubin and Frankowska [2]. With the empty set as a possible set value, both the existence and uniqueness of the set value of the game is always guaranteed by definition. It turns out that this set value behaves benignly as the (real-valued) value function in stochastic control theory; it enjoys the regularity, stability, and most importantly, the Dynamic Programming Principle (DPP for short) in an appropriate sense. When the set value is a singleton, for example, in two-person, zero-sum games or in stochastic control problems (a "game" with only one player), it reduces to a standard value function (real or vector valued) and satisfies a (path-dependent) PDE.

Our idea of studying the set value for nonzero sum games follows the line of, among others, Abreu et al. [1] and Sannikov [36]. The work of Abreu et al. [1] considers the set value of an infinitely repeated game in discrete time over all sequential equilibriums. Because of the homogeneousness of the game, its set value is time and state invariant and thus is actually a fixed set or, say, a set valued constant. It is shown in Abreu et al. [1] that this set value satisfies the so-called factorization and self-generation, which is exactly in the same spirit of our DPP. The work of Sannikov [36] considers a similar game, but in continuous time models. The set value is again a fixed set, and the main focuses of Sannikov [36] are the characterization and geometric properties of this set as well as their economic implications. Another highly related work is Cardaliaguet et al. [9], which uses viability theory. The main focus of Cardaliaguet et al. [9] is the numerical approximation for the set of initial states satisfying some required properties. Our goal is to study standard nonzero-sum games in finite time horizon both in discrete time and in continuous time models, and we shall investigate systematically the dynamic set value of the game over all Nash equilibriums.

In Section 2, we study the discrete time model. Besides establishing the DPP, in the spirit of Abreu et al. [1], our main contribution is to show that, even in the state-dependent (or, say, Markovian) setting, the DPP would fail if one restricts to state-dependent equilibriums. Consequently, it is necessary to consider path-dependent controls in order to have the DPP, which is not the case for stochastic control problems and zero-sum games and is due to the nonuniqueness of the values (although the set of values is always unique). Although already studied in the literature in various contexts, we also show that DPP would fail if we restrict to Pareto optimal equilibriums and discuss how to choose an "optimal" equilibrium by introducing a central planner. Another highly relevant problem, although not discussed in this paper, is to estimate the model parameters with the presence of multiple equilibriums, for which we refer to Section 2 of the survey paper by Ho and Rosen [24] and the references therein. We shall also remark that, as already observed in Pham and Zhang [32], through Buckdahn's counterexample for zero-sum games, to ensure the DPP for the game value we need to consider closed-loop controls rather than open-loop controls.

In Section 3, we study our main object: a continuous time model in a path-dependent setting. It is in general difficult to study the true equilibriums in this model. Motivated by Buckdahn et al. [6] and chapter VII.4 of Mertens et al. [30], we relax the set value of the game to the limit of the value sets over all ε-equilibriums. Then, the set value will be compact and nonempty as long as there exist ε-equilibriums for all $\varepsilon > 0$, which is a much weaker requirement than the existence of true equilibriums (see e.g., Frei and dos Reis [17] for an example) and is sufficient for practical purposes in most applications. This is exactly in the spirit of the stochastic control problems, where the value is the limit of the values over ε-optimal controls. Indeed, for stochastic control problems and zero-sum game problems, the (standard) value function corresponds to this relaxed set value, not the original one from true equilibriums when an optimal control or saddle point does not exist. We believe this approach of the values could be efficient in more general control/game problems, where the optimal control/equilibrium may not exist or is hard to analyze.

Our next result is the regularity (sensitivity with respect to the state process) and stability (sensitivity with respect to the coefficients) of the set value under mild regularity assumptions on the coefficients. These results have fundamental importance in applications. As a consequence, we obtain the measurability of the set value in terms of the state. Our result is in the direction of Feinstein [15], except that Feinstein [15] studies the set of the equilibriums instead of the values.

The main result of this paper is the DPP for the set value, which can be viewed as a type of time consistency and justifies that the set value is an appropriate object for our dynamic model. Although natural in light of its counterpart in the discrete model, the result is much more involved in the continuous time model and requires several approximations. The pathwise setting adds the technical difficulty. As already observed in Section 2, the pathwise structure is intrinsically needed even in the state-dependent setting.

Finally, we provide a duality result, motivated by Ma and Yong [27] and Karnam et al. [25], which is in the same spirit of the level set approach; see, for example, Barles et al. [3]. We introduce an auxiliary control problem on an enlarged state space, where the additional state corresponds to the possible values of the game. The value function of the new control problem is a viscosity solution of a standard path-dependent HJB equation, for which we refer to Ekren et al. [11, 12] and Ren et al. [35]. Then, the set value of the game is characterized as the nodal set of this new value function. This approach is related to the viability approach in Cardaliaguet [9] and is quite efficient in terms of numerical computation of the set value.

2. The Discrete Model

In this section, we study a discrete model with finite time horizon, which is introduced in Section 2.2. The DPP for the set value is similar to Abreu et al. [1] and is presented in Section 2.3. The results in Section 2.4 concerning the

Table 1. Costs of static nonzero-sum game for Example 1.

$J(a)$	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$(0,1)$	$(2,2)$
$a_1 = 1$	$(3,3)$	$(1,0)$

state-dependent case are new, to the best of our knowledge. The observations in Sections 2.1, 2.5, and 2.6 are interesting but not surprising in the game literature. We nevertheless present them here because the same properties hold in the continuous-time model in the next section, but it is easier for the readers to include them in this section.

2.1. A Static Game

In this subsection, we consider a simple static game with N players and present some basic observations about Nash equilibriums. Player i's control takes values in a Borel measurable set A_i in some arbitrary topological space. For $a = (a_1, \dots, a_N) \in A := A_1 \times \dots \times A_N$, $J_i(a)$ is the player i's cost function that they player seeks to minimize, and $J := (J_1, \dots, J_N) : A \to \mathbb{R}^N$. We say that $a^* \in A$ is a Nash equilibrium if

$$J_i(a^*) \le J_i(a^{*,-i}, a_i) \quad \text{for all } \ a_i \in A_i,$$

where $(a^{*,-i}, a_i)$ is the same as a^*, except that its i-th component is replaced by a_i.

Note that there might be multiple equilibriums or no equilibriums. We emphasize that the nonzero-sum game could have different values $J(a^*)$ at different equilibriums a^*, as we see in Example 1 below. We thus introduce the set value of the game:

$$\mathbb{V} := \{J(a^*) : \text{for all equilibriums } \ a^*\} \subset \mathbb{R}^N.$$

Example 1. Set $N = 2$, $A_1 = A_2 = \{0,1\}$, and $J(a)$ as in Table 1 below. Then the game has two equilibriums, $a^* = (0,0)$ and $a^* = (1,1)$, and the set value is $\mathbb{V} = \{(0,1),(1,0)\}$.

Remark 1. The existence of Nash equilibrium is not guaranteed. However, we emphasize that in this case our set value is still well defined with $\mathbb{V} = \emptyset$. Moreover, our set value is by definition unique, even if there are multiple equilibriums.

Remark 2.

i. Nash equilibriums may not be Pareto optimal among all controls. Again, set $N = 2$, $A_1 = A_2 = \{0,1\}$, and let $J(a)$ be as in the left side of Table 2; then, clearly there is a unique equilibrium $a^* = (1,1)$ with value $J(a^*) = (3,3)$. However, we note that $J_i(0,0) = 1 < 3 = J_i(a^*)$ for both $i = 1,2$.

ii. In general, the comparison principle does not hold for the game value. Consider the $\tilde{J}$ on the right side of Table 2. There is a unique equilibrium $\tilde{a}^* = (0,0)$ with value $\tilde{J}(\tilde{a}^*) = (2,2)$. Note that $J_i(a) < \tilde{J}_i(a)$ for all $a \in A$ and $i = 1,2$, but $J_i(a^*) = 3 > 2 = \tilde{J}_i(\tilde{a}^*)$ for both $i = 1,2$.

2.2. The Set Value in a Dynamic Setting

We now consider a dynamic setting. In this section, we assume that both the time and the state are discrete. Let $\mathbb{T} := \{0,1,\dots,T\}$ denote the set of discrete times, and for each $t \in \mathbb{T}$, $\mathbb{S}_t$ the set of discrete states at t with $|\mathbb{S}_t| < \infty$. For the reason we will explain in Section 2.4 below, we shall consider a path-dependent setting: $\mathbb{S}^{\mathbb{T}} := \{\mathbf{x} = (\mathbf{x}_0, \dots, \mathbf{x}_T) : \mathbf{x}_t \in \mathbb{S}_t, t \in \mathbb{T}\}$. Set $\Omega := \mathbb{S}^{\mathbb{T}}$ as the sample space, $\mathcal{F} := 2^{\Omega}$, $X_t : \Omega \to \mathbb{S}_t$ the canonical process: $X_t(\mathbf{x}) = \mathbf{x}_t$, and $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T} = \mathbb{F}^X$, the natural filtration generated by X. Clearly, all the functions involved will be $\mathcal{F}$-measurable. Throughout this section, all of the time-dependent functions φ will be required to be adapted in the sense that $\varphi(t,\mathbf{x})$ depends only on $(t, \mathbf{x}_0, \cdots, \mathbf{x}_t)$. We shall denote

$$\mathbf{x} =_t \tilde{\mathbf{x}} \quad \text{if} \quad \mathbf{x}_s = \tilde{\mathbf{x}}_s \ \text{ for all } \ s = 0, \cdots, t, \quad \text{and} \quad \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}} := \{\tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}} : \tilde{\mathbf{x}} =_t \mathbf{x}\}.$$

There are N players, where the set of admissible controls $\mathcal{A}_i$ of the i-th player consists of adapted mappings $\alpha_i : \mathbb{T} \times \mathbb{S}^{\mathbb{T}} \to A_i$. Denote $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$ and $\alpha := (\alpha_1, \cdots, \alpha_N)$. For any $(t, \mathbf{x}, a) \in \mathbb{T} \times \mathbb{S}^{\mathbb{T}} \times A$, $q(t, \mathbf{x}, a; \cdot) : \mathbb{S}_{t+1} \to (0,1]$ is a transition probability function: $\Sigma_{x \in \mathbb{S}_{t+1}} q(t, \mathbf{x}, a; x) = 1$. Let $\mathbb{P}^{t,\mathbf{x},\alpha}$ denote the probability measure such that

$$\mathbb{P}^{t,\mathbf{x},\alpha}(X =_t \mathbf{x}) = 1, \quad \text{and}$$
$$\mathbb{P}^{t,\mathbf{x},\alpha}(X_{s+1} = x \,|\, X =_s \tilde{\mathbf{x}}) = q(s, \tilde{\mathbf{x}}, \alpha(s, \tilde{\mathbf{x}}); x) \quad \forall \ \ s \ge t, \ \ \tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}, \ \ x \in \mathbb{S}_{s+1}.$$

Table 2. Costs of static nonzero-sum games for Remark 2.

$J(a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(1,1)$	$(4,0)$
$a_1=1$	$(0,4)$	$(3,3)$

$\tilde{J}(a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(2,2)$	$(5,5)$
$a_1=1$	$(5,5)$	$(6,6)$

Now for $i=1,\cdots,N$, let $g_i:\mathbb{S}^{\mathbb{T}}\to\mathbb{R}$ and $f_i:=\mathbb{T}\times\mathbb{S}^{\mathbb{T}}\times A_i\to\mathbb{R}$ be adapted and measurable in $a_i\in A_i$ (the measurability in $(t,\mathbf{x})$ is trivial since the space $\mathbb{T}\times\mathbb{S}^{\mathbb{T}}$ is finite). The i-th player's cost function is defined as:

$$J_i(t,\mathbf{x},\alpha):=\mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\alpha}}\left[g_i(X)+\sum_{s=t}^{T-1}f_i\left(s,X,\alpha_i(s,X)\right)\right].$$

We shall always denote

$$J(t,\mathbf{x},\alpha):=(J_1(t,\mathbf{x},\alpha),\cdots,J_N(t,\mathbf{x},\alpha))\in\mathbb{R}^N.$$

Definition 1. Fix $(t,\mathbf{x})\in\mathbb{T}\times\mathbb{S}^{\mathbb{T}}$. We say that $\alpha^*\in\mathcal{A}$ is a Nash equilibrium of the game at $(t,\mathbf{x})$, denoted as $\alpha^*\in NE(t,\mathbf{x})$, if, for each $i=1,\cdots,N$,

$$J_i(t,\mathbf{x},\alpha^*)\le J_i(t,\mathbf{x},\alpha^{*,-i},\alpha_i)\ \text{ for all }\alpha_i\in\mathcal{A}_i.$$

As we saw in Example 1, the game could have different values $J(t,\mathbf{x},\alpha^*)$ at different equilibriums α^*. Our main object is the following set value over all equilibriums:

$$\mathbb{V}(t,\mathbf{x}):=\left\{J(t,\mathbf{x},\alpha^*):\alpha^*\in NE(t,\mathbf{x})\right\}\subset\mathbb{R}^N,$$

which is the counterpart of the value function in the standard control literature. As mentioned in Remark 1, $\mathbb{V}(t,\mathbf{x})$ always exists (with possible value $\emptyset$) and is by nature unique.

Remark 3. For the ease of presentation in this section, we restrict to the case $|\mathbb{S}_t|<\infty$, but all of the results can be easily extended to the case that $\mathbb{S}_t$ is countable. When $\mathbb{S}_t$ is uncountable, although intuitively the results will still hold true, we will encounter some very subtle measurability issue, as we will see in the next section.

Remark 4. For two-person, zero-sum games under the Isaacs condition and other technical conditions, even if there are multiple equilibriums, their values J will always be the same; namely, $\mathbb{V}(t,\mathbf{x})=\{V(t,\mathbf{x})\}$ is a singleton, and in the continuous-time setting the value function V would satisfy a (path-dependent) Isaacs equation.

We also remark that, by considering mixed strategies, the Isaacs condition will always hold (under very mild conditions); see, for example, Mertens et al. [30] for discrete time models and Buckdahn et al. [7] for continuous-time models, and hence, the set value for these zero-sum games is a singleton. It will be interesting to study the set value of nonzero-sum games under mixed strategies, which we leave for future research.

We note that, although $\mathbb{S}^{\mathbb{T}}$ is finite, unless we assume that A is also finite, in general, $\mathbb{V}(t,\mathbf{x})$ may not be finite. The following basic property is interesting in its own right.

Proposition 1. *If q and f are continuous in a and A is compact, then $\mathbb{V}(t,\mathbf{x})$ is compact.*

Proof. Under our assumption, $g(\mathbf{x})$ and $f(t,\mathbf{x},a)$ are bounded, and thus obviously $\mathbb{V}(t,\mathbf{x})$ is bounded. Now let $y_n=J(t,\mathbf{x},\alpha_n^*)\in\mathbb{V}(t,\mathbf{x})$ for some $\alpha_n^*\in NE(t,\mathbf{x})$ and $y_n\to y$. Because A is compact, for any $(s,\tilde{\mathbf{x}})\in\mathbb{T}\times\mathbb{S}^{\mathbb{T}}$, $\{\alpha_n^*(s,\tilde{\mathbf{x}})\}_{n\ge1}$ has a convergent subsequence. Note further that $\mathbb{S}^{\mathbb{T}}$ is finite; then, without loss of generality, we may assume that there exists $\alpha^*\in\mathcal{A}$ such that $\alpha_n^*(s,\tilde{\mathbf{x}})\to\alpha^*(s,\tilde{\mathbf{x}})$ for all $(s,\tilde{\mathbf{x}})\in\mathbb{T}\times\mathbb{S}^{\mathbb{T}}$. Now, for any i and $\alpha_i\in\mathcal{A}_i$, we have

$$J_i(t,\mathbf{x},\alpha_n^*)\le J_i\left(t,\mathbf{x},\alpha_n^{*,-i},\alpha_i\right).$$

By the continuity of q and f in α, one can easily check that $J_i(t,\mathbf{x},\alpha_n^*)\to J_i(t,\mathbf{x},\alpha^*)$ and $J_i(t,\mathbf{x},\alpha_n^{*,-i},\alpha_i)\to J_i(t,\mathbf{x},\alpha^{*,-i},\alpha_i)$. Then, $J_i(t,\mathbf{x},\alpha^*)\le J_i(t,\mathbf{x},\alpha^{*,-i},\alpha_i)$. This implies that $\alpha^*\in NE(t,\mathbf{x})$, and thus $y=J(t,\mathbf{x},\alpha^*)\in\mathbb{V}(t,\mathbf{x})$. So $\mathbb{V}(t,\mathbf{x})$ is closed and hence, compact. Q.E.D.

2.3. Dynamic Programming Principle for the Set Value

Given an $\mathbb{F}$-stopping time τ and an $\mathcal{F}_\tau$-measurable function $\psi : \mathbb{S}^{\mathbb{T}} \to \mathbb{R}^N$ (namely $\psi(\mathbf{x}) = \psi(\mathbf{x}_{\tau(\mathbf{x})\wedge \cdot})$), consider the game with terminal time τ and terminal condition ψ:

$$J_i(\tau, \psi; t, \mathbf{x}, \alpha) := \mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\alpha}}\left[\psi_i(X) + \sum_{s=t}^{\tau-1} f_i(s, X, \alpha_i(s, X))\right].$$

Define the equilibrium at $(\tau, \psi; t, \mathbf{x})$ in the obvious way, and denote its set $NE(\tau, \psi; t, \mathbf{x})$. Our main result of this section is the following dynamic programming principle.

Theorem 1. *For any $(t, \mathbf{x}) \in \mathbb{T} \times \mathbb{S}^{\mathbb{T}}$ and any $\mathbb{F}$-stopping time τ with $\tau(\mathbf{x}) > t$,*

$$\mathbb{V}(t, \mathbf{x}) = \left\{J(\tau, \psi; t, \mathbf{x}, \alpha^*) : \text{for all } \psi \text{ and } \alpha^* \text{ satisfying } \quad \psi(\tilde{\mathbf{x}}) \in \mathbb{V}\big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}\big),\ \forall \tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}, \text{and } \ \alpha^* \in NE\big(\tau, \psi; \mathrm{t}, \mathbf{x}\big)\right\}. \tag{1}$$

Proof. Let $\tilde{\mathbb{V}}(t, \mathbf{x})$ denote the right side of (1).

Step 1. We first prove $\subset$. For any $y = J(t, \mathbf{x}, \alpha^*) \in \mathbb{V}(t, \mathbf{x})$ with $\alpha^* \in NE(t, \mathbf{x})$, denote

$$\psi(\tilde{\mathbf{x}}) := J\big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}, \alpha^*\big), \ \text{ for all } \ \tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}.$$

Now for any i and $\alpha_i \in \mathcal{A}_i$, denote $\tilde{\alpha}_i := \alpha_i \mathbf{1}_{\{s<\tau\}} + \alpha^*_i \mathbf{1}_{\{s\geq\tau\}} \in \mathcal{A}_i$. Then

$$\begin{aligned} J_i(\tau, \psi; t, \mathbf{x}, \alpha^{*,-i}, \alpha_i) &= \mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\alpha^{*,-i},\alpha_i}}\left[\psi_i(X) + \sum_{s=t}^{\tau-1} f_i(s, X, \alpha_i(s, X))\right] \\ &= \mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\alpha^{*,-i},\tilde{\alpha}_i}}\left[g_i(X) + \sum_{s=t}^{T-1} f_i(s, X, \tilde{\alpha}_i(s, X))\right] = J_i\big(t, \mathbf{x}, \alpha^{*,-i}, \tilde{\alpha}_i\big). \end{aligned}$$

By setting $\alpha_i = \alpha^*_i$, we also have $J_i(\tau, \psi; t, \mathbf{x}, \alpha^*) = J_i(t, \mathbf{x}, \alpha^*)$. Because $\alpha^* \in NE(t, \mathbf{x})$, then $J_i(\tau, \psi; t, \mathbf{x}, \alpha^{*,-i}, \alpha_i) \geq J_i(\tau, \psi; t, \mathbf{x}, \alpha^*)$. That is, $\alpha^* \in NE(\tau, \psi; t, \mathbf{x})$.

Moreover, for any $\tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}$, denote

$$\hat{\alpha}_i(s, \hat{\mathbf{x}}) := \alpha_i(s, \hat{\mathbf{x}}) \mathbf{1}_{\{s\geq\tau(\hat{\mathbf{x}})\}\cap\{\hat{\mathbf{x}} =_{\tau(\hat{\mathbf{x}})} \tilde{\mathbf{x}}\}} + \alpha^*_i(s, \hat{\mathbf{x}}) \mathbf{1}_{(\{s\geq\tau(\hat{\mathbf{x}})\}\cap\{\hat{\mathbf{x}} =_{\tau(\hat{\mathbf{x}})} \tilde{\mathbf{x}}\})^c} \in \mathcal{A}_i. \tag{2}$$

Similarly, we have

$$0 \leq J_i\big(t, \mathbf{x}, \alpha^{*,-i}, \hat{\alpha}_i\big) - J_i(t, \mathbf{x}, \alpha^*) \ = \mathbb{P}^{t,\mathbf{x},\alpha^*}\left(X =_{\tau(\tilde{\mathbf{x}})} \tilde{\mathbf{x}}\right)\left[J_i\big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}, \alpha^{*,-i}, \alpha_i\big) - \psi_i(\tilde{\mathbf{x}})\right].$$

Note that $q > 0$ and thus $\mathbb{P}^{t,\mathbf{x},\alpha^*}(X =_{\tau(\tilde{\mathbf{x}})} \tilde{\mathbf{x}}) > 0$. This implies that $\alpha^* \in NE(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}})$, and then $\psi(\tilde{\mathbf{x}}) \in \mathbb{V}(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}})$. Therefore, it follows from (1) that $y \in \tilde{\mathbb{V}}(t, \mathbf{x})$.

Step 2. On the other hand, let $y = J(\tau, \psi; t, \mathbf{x}, \alpha^*) \in \tilde{\mathbb{V}}(t, \mathbf{x})$ for some desired ψ and α^*. For each $\tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}$, we have $\psi(\tilde{\mathbf{x}}) \in \mathbb{V}(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}})$, and thus there exists $\alpha^*_{\tilde{\mathbf{x}}} \in NE(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}})$ such that $\psi(\tilde{\mathbf{x}}) = J\big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}, \alpha\tilde{\mathbf{x}}^*\big)$. Define

$$\hat{\alpha}^*(s, \hat{\mathbf{x}}) := \alpha^*(s, \hat{\mathbf{x}}) \mathbf{1}_{\{s<\tau(\hat{\mathbf{x}})\}} + \sum_{\tilde{\mathbf{x}}\in\mathbb{S}^{\mathbb{T}}} \alpha^*_{\tilde{\mathbf{x}}}(s, \hat{\mathbf{x}}) \mathbf{1}_{\{s\geq\tau(\hat{\mathbf{x}})\}\cap\{\tilde{\mathbf{x}} =_{\tau(\hat{\mathbf{x}})} \hat{\mathbf{x}}\}} \in \mathcal{A}.$$

Note that $\tau(\tilde{\mathbf{x}}) = \tau(\hat{\mathbf{x}})$ when $\tilde{\mathbf{x}} =_{\tau(\hat{\mathbf{x}})} \hat{\mathbf{x}}$. Then, for any i and any $\alpha_i \in \mathcal{A}_i$, denoting $\tilde{\alpha}_i := \alpha_i \mathbf{1}_{\{s<\tau\}} + \hat{\alpha}^* \mathbf{1}_{\{s\geq\tau\}} \in \mathcal{A}_i$,

$$\begin{aligned} & J_i\big(t, \mathbf{x}, \hat{\alpha}^{*,-i}, \alpha_i\big) - J_i(t, \mathbf{x}, \hat{\alpha}^*) \\ &\quad = J_i\big(t, \mathbf{x}, \hat{\alpha}^{*,-i}, \alpha_i\big) - J_i\big(t, \mathbf{x}, \hat{\alpha}^{*,-i}, \tilde{\alpha}_i\big) + J_i\big(t, \mathbf{x}, \hat{\alpha}^{*,-i}, \tilde{\alpha}_i\big) - J_i(t, \mathbf{x}, \hat{\alpha}^*) \\ &\quad = \sum_{\tilde{\mathbf{x}}\in\mathbb{S}^{\mathbb{T}}} \mathbb{P}^{t,\mathbf{x},\alpha^{*,-i},\alpha_i}\left(X =_{\tau(\tilde{\mathbf{x}})} \tilde{\mathbf{x}}\right)\left[J_i\Big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}, \alpha^{*,-i}_{\tilde{\mathbf{x}}}, \alpha_i\Big) - J_i\big(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}, \alpha^*_{\tilde{\mathbf{x}}}\big)\right] \\ &\qquad\quad + J_i\big(\tau, \psi; t, \mathbf{x}, \alpha^{*,-i}, \alpha_i\big) - J_i\big(\tau, \psi; t, \mathbf{x}, \alpha^*\big) \\ &\quad \geq 0. \end{aligned}$$

This implies $\hat{\alpha}^* \in NE(t, \mathbf{x})$, and thus $y = J_i(t, \mathbf{x}, \hat{\alpha}^*) \in \mathbb{V}(t, \mathbf{x})$. Q.E.D.

Remark 5. The condition $q > 0$, implying that $\mathbb{P}^{t,\mathbf{x},\alpha}$ are all equivalent for different α, seems crucial in the proof of Theorem 1. This condition is also used in [1] and is interpreted as that no player can infer the other players' controls through the observed state process.

When q is only nonnegative, we can prove the partial DPP: $\tilde{\mathbb{V}}(t,\mathbf{x}) \subset \mathbb{V}(t,\mathbf{x})$, where $\tilde{\mathbb{V}}(t,\mathbf{x})$ again denotes the right side of (1), and the inclusion could be strict. However, when the measures are singular, it is too strong to require $\psi(\tilde{\mathbf{x}}) \in \mathbb{V}(\tau,\tilde{\mathbf{x}})$ for all $\tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}$. It will be very interesting to see whether it is possible to weaken this requirement in an appropriate way so that the DPP will hold true.

Remark 6. It is crucial that the control is a closed loop: $\alpha = \alpha(X_\cdot)$. If one uses open-loop controls, then DPP typically fails even for zero-sum games. See Buckdahn's counterexample in Pham and Zhang [32] in a continuous-time setting; see also remark 4.4.(ii) in Possamai et al. [33]. Below, we present a counterexample in the discrete time setting.

We recall that open-loop controls do not depend on the state X. In this case, the value of X, instead of its distribution, will depend on the control.

Example 2. Consider a two-player game with open-loop controls as follows. Fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Set $\mathbb{T} := \{t_0,t_1,t_2\} := \{0,1,2\}$ and ξ_1,ξ_2 are independent one-dimensional random variables with $\mathbb{E}[\xi_i] = 0$, $Var(\xi_i) = 1$, the filtration is $\mathbb{F} = \{\mathcal{F}_{t_j}\}_{j=0,1,2}$ with $\mathcal{F}_{t_0} := \{\emptyset,\Omega\}$, $\mathcal{F}_{t_1} := \sigma(\xi_1)$, and $\mathcal{F}_{t_2} := \sigma(\xi_1,\xi_2)$, the controls $\alpha = (\alpha^1,\alpha^2)$ are $\mathbb{F}$-adapted and take values in $A_1 = A_2 := \mathbb{R}$, the state process is for some constant $\sigma \geq 0$,

$$X^\alpha_{t_0} := 0, \quad X^\alpha_{t_1} := \alpha^1_{t_0} + \alpha^2_{t_0} + \sigma\xi_1, \quad X^\alpha_{t_2} := \left[\alpha^1_{t_1} + \alpha^2_{t_1}\right]X_{t_1} + \sigma\xi_2,$$

and the cost functions are $g_i(x) := -x$, $f_i(t_1,a) := \frac{1}{2}|a|^2$, and $f_i(t_0,a) := 4|a|^2 + 2a$; that is,

$$J_i(t_0,0,\alpha) := \mathbb{E}\left[\frac{1}{2}|\alpha^i_{t_1}|^2 + 4|\alpha^i_{t_0}|^2 + 2\alpha^i_{t_0} - X^\alpha_{t_2}\right], \quad i = 1,2.$$

We note that the game is symmetric for the two players. However, DPP fails for this game:

$$\begin{aligned}\mathbb{V}(t_0,0) &= \left\{\left(-\frac{3}{2}[|\sigma|^2+1],\ -\frac{3}{2}[|\sigma|^2+1]\right)\right\},\\ \tilde{\mathbb{V}}(t_0,0) &= \left\{\left(-\left[\frac{3}{2}|\sigma|^2+4\right],\ -\left[\frac{3}{2}|\sigma|^2+4\right]\right)\right\}.\end{aligned} \tag{3}$$

We note that, when $\sigma = 0$, the above game is deterministic.

We first show that the two-period game has a unique equilibrium: $\alpha^{*,i}_{t_0} = -\frac{1}{2}$, $\alpha^{*,i}_{t_1} = \sigma\xi_1 - 1$, and $i = 1,2$. Then, $J_i(t_0,0,\alpha^*) = -\frac{3}{2}[|\sigma|^2 + 1]$, and thus we obtain the $\mathbb{V}(t_0,0)$ in (3). Indeed, assume that α^* is an arbitrary equilibrium. Fix $\alpha^{*,2}$. Note that

$$J_1(t_0,0,\alpha^1,\alpha^{*,2}) = \mathbb{E}\left[\frac{1}{2}|\alpha^1_{t_1}|^2 + 4|\alpha^1_{t_0}|^2 + 2\alpha^1_{t_0} - \left[\alpha^1_{t_1} + \alpha^{*,2}_{t_1}\right]\left[\alpha^1_{t_0} + \alpha^{*,2}_{t_0} + \sigma\xi_1\right]\right].$$

One can easily see that the unique optimal $\alpha^1_{t_1}$ satisfies $\alpha^{*,1}_{t_1} = \alpha^1_{t_0} + \alpha^{*,2}_{t_0} + \sigma\xi_1$. Then,

$$J_1\left(t_0,0,\alpha^1_{t_0},\alpha^{*,1}_{t_1},\alpha^{*,2}\right) = \mathbb{E}\left[4|\alpha^1_{t_0}|^2 + 2\alpha^1_{t_0} - \frac{1}{2}\left[\alpha^1_{t_0} + \alpha^{*,2}_{t_0} + \sigma\xi_1\right]^2 - \alpha^{*,2}_{t_1}\left[\alpha^1_{t_0} + \alpha^{*,2}_{t_0} + \sigma\xi_1\right]\right].$$

This is strictly convex in $\alpha^1_{t_0}$. By the first-order condition, we have

$$0 = \mathbb{E}\left[8\alpha^{*,1}_{t_0} + 2 - \left[\alpha^{*,1}_{t_0} + \alpha^{*,2}_{t_0} + \sigma\xi_1\right] - \alpha^{*,2}_{t_1}\right] = 7\alpha^{*,1}_{t_0} + 2 - \alpha^{*,2}_{t_0} - \mathbb{E}\left[\alpha^{*,2}_{t_1}\right].$$

Similarly, we have $\alpha^{*,2}_{t_1} = \alpha^{*,2}_{t_0} + \alpha^{*,1}_{t_0} + \sigma\xi_1$. Then, $\mathbb{E}[\alpha^{*,2}_{t_1}] = \alpha^{*,2}_{t_0} + \alpha^{*,1}_{t_0}$, and thus

$$0 = 7\alpha^{*,1}_{t_0} + 2 - \alpha^{*,2}_{t_0} - \left[\alpha^{*,2}_{t_0} + \alpha^{*,1}_{t_0}\right] = 6\alpha^{*,1}_{t_0} - 2\alpha^{*,2}_{t_0} + 2.$$

Similarly we have $6\alpha^{*,2}_{t_0} - 2\alpha^{*,1}_{t_0} + 2 = 0$. Then one can easily obtain $\alpha^{*,1}_{t_0} = \alpha^{*,2}_{t_0} = -\frac{1}{2}$. This implies that $\alpha^{*,1}_{t_1} = \alpha^{*,2}_{t_1} = \sigma\xi_1 - 1$.

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We next compute $\tilde{\mathbb{V}}(t_0, 0)$. Note that

$$J_i(t_1, x, \alpha_{t_1}) = \mathbb{E}\left[\frac{1}{2}|\alpha^i_{t_1}|^2 - \left[\alpha^1_{t_1} + \alpha^2_{t_1}\right]x - \xi_2\right] = \mathbb{E}\left[\frac{1}{2}|\alpha^i_{t_1}|^2 - \left[\alpha^1_{t_1} + \alpha^2_{t_1}\right]x\right].$$

For fixed x, one can easily see that the unique equilibrium is $\tilde{\alpha}^{*,1}_{t_1} = \tilde{\alpha}^{*,2}_{t_1} = x$ (which, for fixed x, is deterministic and hence, is an open-loop control for the game at the second period). Then, $J_i(t_1, x, \tilde{\alpha}^*_{t_1}) = -\frac{3}{2}x^2$, and thus $\mathbb{V}(t_1, x) = \{(-\frac{3}{2}x^2, -\frac{3}{2}x^2)\}$. Now consider the game at the first period with terminal $\psi(x) := (-\frac{3}{2}x^2, -\frac{3}{2}x^2)$:

$$J_i(t_1, \psi; t_0, 0, \alpha_{t_0}) = \mathbb{E}\left[4|\alpha^i_{t_0}|^2 + 2\alpha^i_{t_0} - \frac{3}{2}\left[\alpha^1_{t_0} + \alpha^2_{t_0} + \sigma\xi_1\right]^2\right].$$

By first-order conditions, we see that the equilibrium satisfies

$$8\tilde{\alpha}^{*,i}_{t_0} + 2 - 3\left[\tilde{\alpha}^{*,1}_{t_0} + \tilde{\alpha}^{*,2}_{t_0}\right] = 0, \quad i = 1, 2.$$

This implies that $\tilde{\alpha}^{*,1}_{t_0} = \tilde{\alpha}^{*,2}_{t_0} = -1$, and then $J_i(t_1, \psi; t_0, 0, \tilde{\alpha}^*_{t_0}) = -[\frac{3}{2}|\sigma|^2 + 4]$.

Remark 7. Motivated by the mean field equilibriums, we call an equilibrium α^* at $(t, \mathbf{x})$ symmetric if $\alpha^{*,1} = \cdots = \alpha^{*,N}$. Denote

$$\mathbb{V}_{symmetric}(t, \mathbf{x}) := \left\{J(t, \mathbf{x}, \alpha^*) : \text{for all symmetric equilibriums } \alpha^*\right\}.$$

Then, following the same arguments $\mathbb{V}_{symmetric}$ also satisfies DPP:

$$\mathbb{V}_{symmetric}(t, \mathbf{x}) = \Big\{J(\tau, \psi; t, \mathbf{x}, \alpha^*) : \text{for all } \psi \text{ and } \alpha^* \text{ such that } \alpha^* \text{ is a symmetric equilibrium at } (\tau, \psi; t, \mathbf{x}) \text{ and } \psi(\tilde{\mathbf{x}}) \in \mathbb{V}_{symmetric}(\tau(\tilde{\mathbf{x}}), \tilde{\mathbf{x}}) \text{ for all } \tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}\Big\}.$$

2.4. The State-Dependent Case

In this subsection, we consider a state-dependent (i.e., Markovian) model:

$$q(t, \mathbf{x}, a; x) = q(t, \mathbf{x}_t, a; x), \quad g(\mathbf{x}) = g(\mathbf{x}_T), \quad f_i(t, \mathbf{x}, a) = f_i(t, \mathbf{x}_t, a). \tag{4}$$

We shall call a function φ on $\mathbb{T} \times \mathbb{S}^{\mathbb{T}}$ state dependent if $\varphi(t, \mathbf{x}) = \varphi(t, \tilde{\mathbf{x}})$ whenever $\mathbf{x}_t = \tilde{\mathbf{x}}_t$, and in this case it is natural to abuse the notation and denote it as $\varphi(t, \mathbf{x}_t)$.

We first remark that in this case we may still have path-dependent equilibriums, whose value is different from those of state-dependent equilibriums.

Example 3. Set $T = 3$, $N = 2$, and $A_1 = A_2 = \{0, 1\}$, and $\mathbb{S}^{\mathbb{T}}$ takes values as in Figure 1.

That is, $\mathbb{S}_0 = \{s_0\}$, $\mathbb{S}_1 = \{s_{10}, s_{11}\}$, $\mathbb{S}_2 = \{s_2\}$, and $\mathbb{S}_3 = \{s_{30}, s_{31}\}$. For the first two periods and for g, we set

$$f(0, \cdot) = f(1, \cdot) = 0, \quad q(0, \cdot) = \frac{1}{2}, \quad q(1, \cdot) = 1, \quad g(s_{30}) = (1, 1), \quad g(s_{31}) = (0, 0),$$

Then, the game at $(0, s_0)$ does not depend on $\alpha(0, \cdot)$ and $\alpha(1, \cdot)$. Indeed,

$$J(0, s_0, \alpha) = \frac{1}{2}\left[\bar{J}(\alpha(2, (s_0, s_{10}, s_2))) + \bar{J}(\alpha(2, (s_0, s_{11}, s_2)))\right], \quad \text{where } \bar{J}_i(a) = f_i(2, s_2, a_i) + q(2, s_2, a; s_{30}), \quad i = 1, 2. \tag{5}$$

Let us assume that the game for $\bar{J}(a)$, which corresponds to the last period of the original game, has two equilibriums a^* and $\tilde{a}^*$. Then, we may construct a path-dependent equilibrium, noting that $X_0 \equiv s_0$ and $X_2 \equiv s_2$ are deterministic,

$$\alpha^*(2, X) := a^*\mathbf{1}_{\{X_1 = s_{10}\}} + \tilde{a}^*\mathbf{1}_{\{X_1 = s_{11}\}}. \tag{6}$$

Figure 1. States for Example 3.

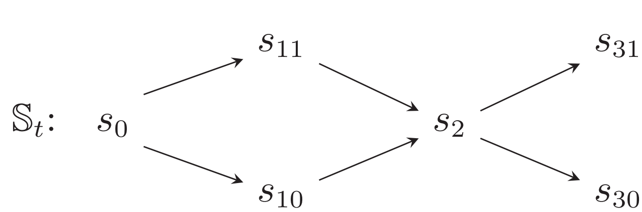


Table 3. Cost matrices and transition probabilities for Example 3.

$f(2,s_2,a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(-\frac{1}{4},0)$	$(-\frac{1}{4},-\frac{1}{4})$
$a_1=1$	$(0,0)$	$(0,-\frac{1}{4})$

$q(2,s_2,a;s_{30})$	$a_2=0$	$a_2=1$
$a_1=0$	$\frac{1}{4}$	$\frac{3}{4}$
$a_1=1$	$\frac{3}{4}$	$\frac{1}{4}$

For this purpose, we set $f(2,s_2,a)$ and $q(2,s_2,a;s_{30})$ for $a\in A$ as in Table 3. Then by (5) we see that $4\bar{J}$ is the same as Table 1, and thus there are two equilibriums, $a^*=(0,0)$ and $\tilde{a}^*=(1,1)$, with corresponding values $\bar{J}(a^*)=(0,1/4)$ and $\bar{J}(\tilde{a}^*)=(1/4,0)$.

We now come back to the original game $J(0,s_0,\alpha)$. Note that, by (5), the only relevant control is $\alpha(2,(s_0,X_1,s_2))$. If α is state dependent, then $\alpha(2,(s_0,X_1,s_2))=\alpha(2,s_2)$ is deterministic. This implies $J(0,s_0,\alpha)=\bar{J}(\alpha(2,s_2))$, and thus there are only two equilibriums with values $(0,1/4)$ and $(1/4,0)$. However, we can construct a path-dependent equilibrium α^* by (6), whose corresponding value is $J(0,s_0,\alpha^*)=\bar{J}(a^*)/2+\bar{J}(\tilde{a}^*)/2=(1/8,1/8)$.

In view of Example 3, nevertheless, $\mathbb{V}$ is still state dependent if we restrict to the state dependent model (4).

Proposition 2. *Under* (4), $\mathbb{V}(t,\mathbf{x})=\mathbb{V}(t,\mathbf{x}_t)$ *is state dependent.*

Proof. Assume that $\mathbf{x}_t=\mathbf{x}'_t$. For any $\alpha\in\mathcal{A}$ and $\tilde{\mathbf{x}}'\in\mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}'}$, introduce α' by $\alpha'(s,\tilde{\mathbf{x}}'):=\alpha(s,\tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}}_s:=\mathbf{x}_s\mathbf{1}_{\{s\le t\}}+\tilde{\mathbf{x}}'_s\mathbf{1}_{\{s>t\}}$. Then, one can easily check that $J(t,\mathbf{x},\alpha)=J(t,\mathbf{x}',\alpha')$. Such correspondence is one to one, and thus it is clear that $\mathbb{V}(t,\mathbf{x})=\mathbb{V}(t,\mathbf{x}')$. Q.E.D.

From now on, in the state-dependent case, we may write the set value as $\mathbb{V}(t,x)$. The following DPP is an immediate consequence of Theorem 1.

Corollary 1. *Under* (4), *for any* $(t,\mathbf{x})\in\mathbb{T}\times\mathbb{S}^{\mathbb{T}}$ *and* $\mathbb{F}$*-stopping time* τ *with* $\tau(\mathbf{x})\ge t$,

$$\mathbb{V}(t,x)=\Big\{J(\tau,\psi;t,\mathbf{x},\alpha^*): \textit{for all } \psi,\alpha^*,\mathbf{x} \textit{ such that } \mathbf{x}_t=x,\\ \psi(\tilde{\mathbf{x}})\in\mathbb{V}\big(\tau(\tilde{\mathbf{x}}),\tilde{\mathbf{x}}_{\tau(\tilde{\mathbf{x}})}\big) \textit{ for all } \tilde{\mathbf{x}}\in\mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}}, \textit{ and } \alpha^*\in NE(\tau,\psi;t,\mathbf{x})\Big\}.$$

We emphasize that, although our model is state dependent here, the DPP above involves path-dependent ψ and α^*. In fact, if we restrict to state-dependent functions ψ and/or α^*, then the DPP may fail, as we explain next. For simplicity, below we consider only deterministic time: $\tau\equiv T_0$ for some $T_0>t$.

We first investigate the case that ψ is state dependent but that α^* can be still path dependent. In this case, by Corollary 1, the following partial DPP is obvious:

$$\mathbb{V}(t,x)\supset\Big\{J(T_0,\psi;t,\mathbf{x},\alpha^*): \text{for all state dependent } \psi \text{ and } \alpha^*\in\mathcal{A},\mathbf{x}\in\mathbb{S}^{\mathbb{T}}\\ \text{such that } \mathbf{x}_t=x,\ \psi(\tilde{x})\in\mathbb{V}(T_0,\tilde{x}),\ \forall\tilde{x}\in\mathbb{S}_{T_0},\ \text{and } \alpha^*\in NE(T_0,\psi;t,\mathbf{x})\Big\}. \tag{7}$$

However, the above inclusion can be strict.

Example 4. Consider Example 3 and set $T_0=2$. By Example 3, we see that

$$\mathbb{V}(2,s_2)=\{\bar{J}(a^*),\bar{J}(\tilde{a}^*)\}=\left\{\left(0,\frac{1}{4}\right),\left(\frac{1}{4},0\right)\right\}.$$

If ψ is state dependent, then there are only two possible functions: $\psi_1(s_2)=(0,1/4)$ and $\psi_2(s_2)=(1/4,0)$. Recalling that $f(0,\cdot)=f(1,\cdot)=0$, then $J(T_0,\psi;0,s_0,\alpha)=\psi(s_2)$ for all α. Thus the right side of (7) is $\{(0,1/4),(1/4,0)\}$. However, by Example 3, we know that $\mathbb{V}(0,s_0)$ contains at least one more value, $(1/8,1/8)$.

We next investigate the case that both ψ and $\alpha\in\mathcal{A}$ are state dependent; then, obviously $J(t,\mathbf{x},\alpha)$ and $J(T_0,\psi;t,\mathbf{x},\alpha)$ are also state dependent. Define

$$\mathcal{A}_{state}:=\{\alpha\in\mathcal{A}:\ \alpha \text{ is state dependent}\};\\ \mathbb{V}_{state}(t,x):=\{J(t,x,\alpha^*):\ \alpha^*\in\mathcal{A}_{state} \text{ is an equilibrium among all } \alpha\in\mathcal{A}_{state}\}.$$

We emphasize that here all controls are required to be state dependent; in particular, the above $\alpha^*\in\mathcal{A}_{state}$ may not be an equilibrium among all controls $\alpha\in\mathcal{A}$. Consequently, $\mathbb{V}_{state}(t,x)$ may not be a subset of $\mathbb{V}(t,x)$. Again, $\mathbb{V}_{state}$ does not satisfy the DPP.

Proposition 3. *Under* (4), $\mathbb{V}_{state}$ *satisfies a partial DPP,*

$$\mathbb{V}_{state}(t,x) \subset \Big\{J(T_0,\psi;t,x,\alpha^*) : \textit{for all state dependent } \psi \textit{ and } \alpha^* \in \mathcal{A}_{state}\ s.t.\\ \psi(\tilde{x}) \in \mathbb{V}_{state}(T_0,\tilde{x}),\ \forall \tilde{x} \in \mathbb{S}_{T_0},\ \textit{and } \alpha^* \textit{ is an equilibrium in } \mathcal{A}_{state} \textit{ at } (T_0,\psi;t,x)\Big\}, \tag{8}$$

but the inclusion could be strict.

We remark that the inclusions in (7) and (8) have opposite directions.

Proof. Let $\tilde{V}_{state}(t,x)$ denote the right side of (8). We shall prove $\mathbb{V}_{state} \subset \tilde{\mathbb{V}}_{state}$, and see in Example 5 below that $\mathbb{V}_{state} \neq \tilde{\mathbb{V}}_{state}$. We follow the arguments in Theorem 1, Step 1 and proceed in two steps.

Step 1. Let $\alpha^* \in \mathcal{A}_{state}$ be an equilibrium in $\mathcal{A}_{state}$ at (t,x). Denote

$$\psi(\tilde{x}) := J(T_0,\tilde{x},\alpha^*),\ \textit{for all}\ \tilde{x} \in \mathbb{S}_{T_0}.$$

For any i and $\alpha_i \in \mathcal{A}_{state,i}$, note that $\tilde{\alpha}_i := \alpha_i \mathbf{1}_{\{s<T_0\}} + \alpha_i^* \mathbf{1}_{\{s\geq T_0\}}$ is also in $\mathcal{A}_{state,i}$. Then, following the same arguments as in Theorem 1, Step 1 we see that α^* is an equilibrium in $\mathcal{A}_{state}$ at $(T_0,\psi;t,x)$.

Step 2. It remains to show that $\psi(\tilde{x}) \in \mathbb{V}_{state}(T_0,\tilde{x})$ for all $\tilde{x} \in \mathbb{S}_{T_0}$. That is,

$$J_i(T_0,\tilde{x},\alpha^{*,-i},\alpha_i) \geq J_i(T_0,\tilde{x},\alpha^*), \quad \textit{for all } i, \text{all } \tilde{x} \in \mathbb{S}_{T_0},\ \textit{and all } \alpha_i \in \mathcal{A}_{state,i}. \tag{9}$$

We emphasize that the $\hat{\alpha}_i$ constructed in 2 is not in $\mathcal{A}_{state,i}$, even when the α^* and α_i there are state dependent, so a more careful argument is required. We shall prove (9) by backward induction on T_0.

First, if $T_0 = T-1$, then the counterpart of 2 becomes, for any fixed $\tilde{x} \in \mathbb{S}_{T_0}$,

$$\hat{\alpha}_i(s,\hat{x}) := \alpha_i(s,\tilde{x})\mathbf{1}_{\{s=T_0\}\cap\{\hat{x}=\tilde{x}\}} + \alpha_i^*(s,\hat{x})\mathbf{1}_{\{s<T_0\}\cup\{\hat{x}\neq\tilde{x}\}},$$

which is in $\mathcal{A}_{state,i}$. Then, (9) follows from the same arguments in Theorem 1, Step 1.

Assume that (9) holds true for T_0+1. Now, for T_0, note that

$$\begin{aligned} &J_i(T_0,\tilde{x},\alpha^{*,-i},\alpha_i)\\ &= f_i(T_0,\tilde{x},\alpha_i(T_0,\tilde{x})) + \sum_{\hat{x}\in\mathbb{S}_{T_0+1}} q(T_0,\tilde{x},(\alpha^{*,-i},\alpha_i)(T_0,\tilde{x}),\hat{x})J_i(T_0+1,\hat{x},\alpha^{*,-i},\alpha_i)\\ &\geq f_i(T_0,\tilde{x},\alpha_i(T_0,\tilde{x})) + \sum_{\hat{x}\in\mathbb{S}_{T_0+1}} q(T_0,\tilde{x},(\alpha^{*,-i},\alpha_i)(T_0,\tilde{x}),\hat{x})J_i(T_0+1,\hat{x},\alpha^*), \end{aligned} \tag{10}$$

where the last inequality is due to the induction assumption. Fix $\tilde{x} \in \mathbb{S}_{T_0}$ and define

$$\hat{\alpha}_i(s,\hat{x}) := \alpha_i(s,\hat{x})\mathbf{1}_{\{s=T_0\}\cap\{\hat{x}=\tilde{x}\}} + \alpha_i^*(s,\hat{x})\mathbf{1}_{\{s\neq T_0\}\cup\{\hat{x}\neq\tilde{x}\}},$$

which is again state dependent. Then, denoting $\mathbb{P}^{t,x,\alpha}$ in the obvious way,

$$\begin{aligned} 0 \leq J_i(t,x,\alpha^{*,-i},\hat{\alpha}_i) - J_i(t,x,\alpha^*) = \mathbb{P}^{t,x,\alpha^*}(X_{T_0}=\tilde{x}) \times\\ \Big[f_i(T_0,\tilde{x},\alpha_i(T_0,\tilde{x})) + \sum_{\hat{x}\in\mathbb{S}_{T_0+1}} q(T_0,\tilde{x},(\alpha^{*,-i},\alpha_i)(T_0,\tilde{x}),\hat{x})J_i(T_0+1,\hat{x},\alpha^*)\\ - f_i(T_0,\tilde{x},\alpha_i^*(T_0,\tilde{x})) - \sum_{\hat{x}\in\mathbb{S}_{T_0+1}} q(T_0,\tilde{x},\alpha^*(T_0,\tilde{x}),\hat{x})J_i(T_0+1,\hat{x},\alpha^*)\Big]. \end{aligned}$$

Note that $\mathbb{P}^{t,x,\hat{\alpha}^*}(X_{T_0}=\tilde{x}) > 0$. Then, together with 10, the above implies

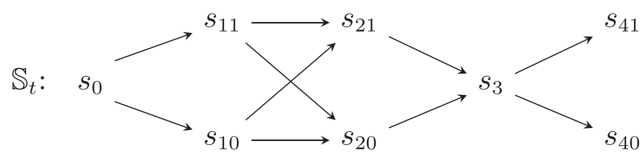
$$\begin{aligned} J_i(T_0,\tilde{x},\alpha^{*,-i},\alpha_i) &\geq f_i(T_0,\tilde{x},\alpha_i^*(T_0,\tilde{x})) - \sum_{\hat{x}\in\mathbb{S}_{T_0+1}} q(T_0,\tilde{x},\alpha^*(T_0,\tilde{x}),\hat{x})J_i(T_0+1,\hat{x},\alpha^*)\\ &= J_i(T_0,\tilde{x},\alpha^*). \end{aligned}$$

This proves (9), hence (8). Q.E.D.

We now construct a counterexample such that the inclusion in (8) is strict. This is again due to the nonuniqueness of equilibriums.

Example 5. Let $T=4$, $N=2$, and $A_1 = A_2 = \{0,1\}$, and $\mathbb{S}^{\mathbb{T}}$ takes values as in Figure 2.

Figure 2. States for Example 5.



We shall construct an equilibrium whose value is in $\tilde{\mathbb{V}}_{state}(0,s_0)\backslash\mathbb{V}_{state}(0,s_0)$. Set

$$T_0 = 1, \quad q(0,\cdot) = \frac{1}{2}, \quad f(0,\cdot) = 0.$$

Given a desired ψ, for any $\alpha \in \mathcal{A}_{state}$, clearly $J(1,\psi;0,s_0,\alpha) = \frac{1}{2}\big[\psi(s_{10}) + \psi(s_{11})\big]$, and thus

$$\tilde{\mathbb{V}}_{state}(0,s_0) = \left\{\frac{1}{2}\big[\psi(s_{10}) + \psi(s_{11})\big] : \textit{for all } \psi \text{ s.t. } \psi\,(s_{1i}) \in \mathbb{V}_{state}\,(1,s_{1i}),\ i = 0,1\right\}. \tag{11}$$

Note that $\mathbb{V}_{state}(1,s_{10})$ and $\mathbb{V}_{state}(1,s_{11})$ are two different three-period games. Let the (3-period) subgames at branch $X_1 = s_{10}$ and at branch $X_1 = s_{11}$ be exactly as in Example 3. Becayse we consider only $\alpha \in \mathcal{A}_{state}$, by (5) we have

$$J(1,s_{1i},\alpha) = \bar{J}(\alpha(3,s_3)), \quad i = 0,1.$$

Then, by Example 3,

$$\mathbb{V}_{state}(1,s_{10}) = \mathbb{V}_{state}(1,s_{11}) = \left\{\left(0,\frac{1}{4}\right),\left(\frac{1}{4},0\right)\right\},$$

with corresponding equilibriums $\alpha(3,s_3) = (0,0)$ and $\alpha(3,s_3) = (1,1)$ (the other values of $\alpha(t,\mathbf{x})$ are irrelevant or, say, can be arbitrary). Then, by (11),

$$\tilde{\mathbb{V}}_{state}(0,s_0) = \left\{\left(0,\frac{1}{4}\right),\left(\frac{1}{4},0\right),\left(\frac{1}{8},\frac{1}{8}\right)\right\}.$$

On the other hand, because $q(0,\cdot) = \frac{1}{2}$ and $f(0,\cdot) = 0$, for any $\alpha \in \mathcal{A}_{state}$, we have

$$J(0,s_0,\alpha) = \frac{1}{2}\big[J(1,s_{10},\alpha) + J(1,s_{11},\alpha)\big] = \bar{J}(\alpha(3,s_3)).$$

So, $\mathbb{V}_{state}(0,s_0) = \{(0,1/4),(1/4,0)\}$; therefore, $(1/8,1/8) \in \tilde{\mathbb{V}}_{state}(0,s_0)\backslash\mathbb{V}_{state}(0,s_0)$.

2.5. Pareto Equilibriums

For $y,\tilde{y} \in \mathbb{R}^N$, we say that $y \le \tilde{y}$ if $y_i \le \tilde{y}_i$ for $i = 1,\cdots,N$, and $y < \tilde{y}$ if we assume further that $y_i < \tilde{y}_i$ for some i. As we saw in Remark 2 (ii), for a nonzero-sum game, typically the comparison principle fails in the sense: for equilibriums $\alpha^*,\tilde{\alpha}^*$, for games $J,\tilde{J}$, respectively,

$$J(\alpha) \le \tilde{J}(\alpha) \ \text{ for all } \alpha, \text{but } \ J(\alpha^*) > \tilde{J}(\tilde{\alpha}^*).$$

A consequence of the above property is that DPP would fail, in general, if one restricts to the so called Pareto equilibriums.

Definition 2. We say that $\alpha^* \in NE(t,x)$ is a Pareto equilibrium if there does not exist another equilibrium $\tilde{\alpha} \in NE(t,\mathbf{x})$ such that $J(t,\mathbf{x},\tilde{\alpha}) < J(t,\mathbf{x},\alpha^*)$.

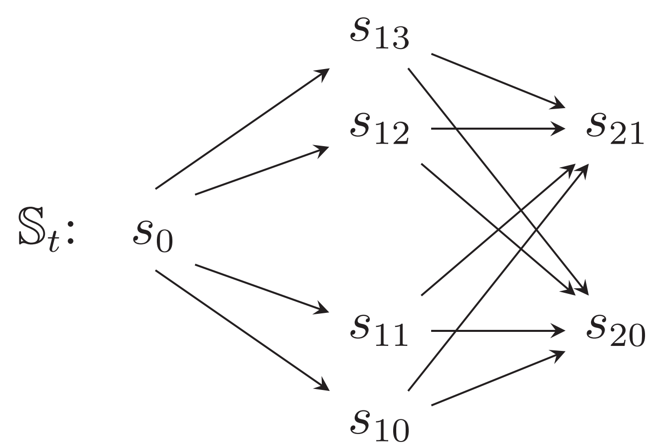
Define

$$\mathbb{V}_{Pareto}(t,\mathbf{x}) := \{J(t,\mathbf{x},\alpha^*) : \text{for all Pareto equilibriums } \alpha^* \in NE(t,\mathbf{x})\}.$$

As the following example shows, even the partial DPPs fail in general:

$$\begin{aligned} \mathbb{V}_{Pareto}(t,\mathbf{x}) \ne \Big\{ & J(T_0,\psi;t,\mathbf{x},\alpha^*) : \text{for all } \psi \text{ and } \alpha^* \text{ such that} \\ & \psi(\tilde{\mathbf{x}}) \in \mathbb{V}_{Pareto}(T_0,\tilde{\mathbf{x}}),\ \forall\, \tilde{\mathbf{x}} \in \mathbb{S}^{\mathbb{T}}_{t,\mathbf{x}},\ \text{and } \alpha^* \text{ is a Pareto equilibrium at } (T_0,\psi;t,\mathbf{x})\Big\}. \end{aligned} \tag{12}$$

Figure 3. States for Example 6.



Example 6. As usual, let $\tilde{\mathbb{V}}_{Pareto}(t, \mathbf{x})$ denote the right side of 12. Let $T = 2$, $N = 2$, and $A_1 = A_2 = \{0, 1\}$, and $\mathbb{S}^{\mathbb{T}}$ takes values as in Figure 3.

We first consider the subgame $\mathbb{V}(1, \mathbf{x})$. Set

$$g(\mathbf{x})|_{\mathbf{x}_2 = s_{21}} = (0, 0).$$

Let $f(1, x) := f(1, (s_0, x), a)$ (independent of a), $g(x) := g(s_0, x, s_{20})$, and $q(1, x, a) := q(1, \mathbf{x}, a; s_{20})$ (independent of $\mathbf{x}$) be as in Table 4. Then $J(1, x, a) := J(1, (s_0, x), a)$ is as in Table 5. This implies that

$$\mathbb{V}(1, \mathbf{x}) = \left\{\psi^*(\mathbf{x}_1), \tilde{\psi}^*(\mathbf{x}_1)\right\}, \quad \mathbb{V}_{Pareto}(1, \mathbf{x}) = \{\psi^*(\mathbf{x}_1)\},$$

where ψ^* and $\tilde{\psi}^*$ are given in Table 6.

We now consider $J(1, \psi; 0, s_0, \alpha)$ for $\psi = \psi^*, \tilde{\psi}^*$. Fix some $\varepsilon > 0$ to be small enough. Set

$$f(0, \cdot) = (0, 0), \quad q(0, s_0, a; s_{1j}) = 1 - 3\varepsilon \text{ if } j = I(a) \quad \text{and} \quad q(0, s_0, a; s_{1j}) = \varepsilon \text{ if } j \neq I(a),$$

where

$$I(0, 0) = 0, \quad I(1, 0) = 1, \quad I(0, 1) = 2, \quad I(1, 1) = 3.$$

and all other $q(0, s_0, a; x) = \varepsilon$. Then,

$$J(1, \psi; 0, s_0, a) = \sum_{j=0}^{3} q(0, s_0, a; s_{1j})\psi(s_{1j}) = \psi(s_{1I(a)}) + O(\varepsilon).$$

That is, $J(1, \psi; 0, s_0, a)$ is approximately equal to $\psi(s_{1I(a)})$, and, when ε is small enough, the two subgames have the same equilibrium. In particular, recalling the J and $\tilde{J}$ in Example 6, we see that

$$J(1, \psi^*; 0, s_0, a) = J(a) + O(\varepsilon), \quad J\left(1, \tilde{\psi}^*; 0, s_0, a\right) = \tilde{J}(a) + O(\varepsilon).$$

Then, by Theorem 1,

$$\tilde{\mathbb{V}}_{Pareto}(0, s_0) = \{(4, 4) + O(\varepsilon)\}, \quad \mathbb{V}(0, s_0) = \{(3, 3) + O(\varepsilon), (4, 4) + O(\varepsilon)\},$$

and thus $\mathbb{V}_{Pareto}(0, s_0) = \{(3, 3) + O(\varepsilon)\}$. This implies that $\tilde{\mathbb{V}}_{Pareto}(0, s_0)$ and $\mathbb{V}_{Pareto}(0, s_0)$ do not include each other; namely, partial DPP fails in both directions.

Table 4. Cost and transition functions for Example 6.

x	s_{10}	s_{11}	s_{12}	s_{13}
$f(1, x)$	$(1, 1)$	$(-4, 4)$	$(4, -4)$	$(1, 1)$
$g(x)$	$(4, 4)$	$(20, 4)$	$(4, 20)$	$(12, 12)$

$q(1, a)$	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$\frac{1}{2}$	$\frac{3}{4}$
$a_1 = 1$	$\frac{3}{4}$	$\frac{1}{4}$

Table 5. Cost matrices for Example 6.

$J(1,s_{10},a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(3,3)$	$(4,4)$
$a_1=1$	$(4,4)$	$(2,2)$

$J(1,s_{11},a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(6,6)$	$(11,7)$
$a_1=1$	$(11,7)$	$(1,5)$

$J(1,s_{12},a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(6,6)$	$(7,11)$
$a_1=1$	$(7,11)$	$(5,1)$

$J(1,s_{11},a)$	$a_2=0$	$a_2=1$
$a_1=0$	$(7,7)$	$(10,10)$
$a_1=1$	$(10,10)$	$(4,4)$

Remark 8. We emphasize that in Definition 2, a Pareto equilibrium α^* is compared only with other equilibriums. In general, it is possible that there exists another control $\alpha \in \mathcal{A}$ (not an equilibrium) such that $J(t,\mathbf{x},\alpha) < J(t,\mathbf{x},\alpha^*)$; see Remark 2 (i). We may call an equilibrium $\alpha^* \in \mathcal{A}$ a strong Pareto equilibrium if there is no such control $\alpha \in \mathcal{A}$. Denote

$$\mathbb{V}^{strong}_{Pareto}(t,\mathbf{x}) := \{J(t,\mathbf{x},\alpha^*):\ \text{for all strong Pare to equilibriums } \alpha^*\}.$$

In general, DPP fails for $\mathbb{V}^{strong}_{Pareto}$, too.

2.6. Optimal Equilibriums

We now fix $x_0 \in \mathbb{S}_0$ and consider $\mathbb{V}(0,x_0)$. In practice, it is important to determine which equilibrium to implement. For this purpose, we introduce a central planner and assume the central planner is interested in minimizing

$$V_0 := \inf_{y\in\mathbb{V}(0,x_0)} \sum_{i=1}^N \lambda_i y_i = \inf\left\{\sum_{i=1}^N \lambda_i J_i(0,x_0,\alpha^*) : \alpha^* \in NE(0,x_0)\right\}. \tag{13}$$

where $\lambda_i \ge 0$ with $\Sigma_{i=1}^N \lambda_i = 1$. Such problems are natural, say, for social welfares. By Proposition 1, the problem (13) has an optimizer $y^* \in \mathbb{V}(0,x_0)$, and correspondingly there exists $\alpha^* \in NE(0,x_0)$. Note that, when $\lambda_i > 0$ for all i, such α^* is automatically a Pareto equilibrium. We remark that in general neither y^* nor α^* is unique; however, the central planer is indifferent to them and thus can pick an arbitrary one. More importantly, in practice it is quite easy to implement such an equilibrium, as we explain below.

Remark 9.

i. Assume that the central planner picks an optimal equilibrium α^*, and recommend it to the players. As long as each player believes that the others would follow the recommended one, it is in the player's best interest to follow the same α^*, since it is an equilibrium. Moreover, because α^* is a Pareto optimal one (assuming $\lambda_i > 0$ for all i), the players are unlikely to make a collective decision to choose a different equilibrium.

ii. The problem is quite different from a "dictatorship" scenario, where the dictator wants to minimize

$$\tilde{V}_0 := \inf_{\alpha\in\mathcal{A}} \sum_{i=1}^N \lambda_i J_i(0,x_0,\alpha).$$

Assume that the problem $\tilde{V}_0$ has an optimal argument $\tilde{\alpha}^*$ and the dictator forces the players to follow it. However, because $\tilde{\alpha}^*$ is (in general) not an equilibrium, the individual players have no incentive to follow it, even if

Table 6. Values of the game in Example 6 at time 1.

x	s_{10}	s_{11}	s_{12}	s_{13}
$\psi^*(x)$	$(2,2)$	$(1,5)$	$(5,\ 1)$	$(4,4)$
$\tilde{\psi}^*(x)$	$(3,3)$	$(6,6)$	$(6,\ 6)$	$(7,\ 7)$

they believe the others would do so. Consequently, the dictator has to use regulation/penalty (or other means) to force them to implement this strategy, which adds to the social cost.

Remark 10. Because DPP fails for the Pareto equilibriums, as detailed in Section 2.5, the dynamic version of (13) will generally be time inconsistent. In particular, this implies that there need not, and typically will not, exist a moving scalarization (a moving objective parameterized by an adapted process λ), as in Feinstein and Rudloff [16], so that α^* is a consistent equilibrium for this problem. Therefore, time inconsistency implies that although a central planner may dictate a socially beneficial equilibrium at time 0, at some time t this may no longer be an optimal equilibrium for the subgame over $[t,T]$.

3. The Continuous-Time Model

In this section, we extend our results to a continuous time setting. We shall consider a diffusion model with drift controls only. In this case, all of the involved probability measures are equivalent. The case with volatility controls may require new insights, especially in light of Remark 5, and is left for future research.

3.1. The Nonzero-Sum Game

Let $[0,T]$ be the time horizon, $(\Omega, \mathbb{F}=\{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}_0)$ a filtered probability space, and B a d-dimensional $\mathbb{P}_0$-Brownian motion. Consider a game with N players. Let $A = A_1\times\cdots\times A_N$ be a convex domain in a Euclidean space and $\mathcal{A}=\mathcal{A}_1\times\cdots\times\mathcal{A}_N$ the set of $\mathbb{F}$-progressively measurable A-valued processes. The data of the game satisfy the following basic properties, where the boundedness assumption is mainly for simplicity.

Assumption 1. *$(b,f):[0,T]\times\Omega\times A\to\mathbb{R}^d\times\mathbb{R}^N$ is $\mathbb{F}$-progressively measurable and bounded, and $\xi:\Omega\to\mathbb{R}^N$ is $\mathcal{F}_T$-measurable and bounded.*

As usual, we omit the variable ω in b, f, ξ. For each $\alpha\in\mathcal{A}$, define

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}_0} := M_T^\alpha := \exp\left(\int_0^T b(s,\alpha_s)\cdot dB_s - \frac{1}{2}\int_0^T |b(s,\alpha_s)|^2 ds\right).$$

At time t, each player has the value defined through the conditional expectation

$$J_i(t,\alpha) := \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\xi_i + \int_t^T f_i\left(s,\alpha_s^i\right)ds\right], \quad i=1,\cdots,N.$$

We remark that we may replace the above expectation with some nonlinear operator through BSDEs; see Remark 15 (ii) below. We say that $\alpha^*\in\mathcal{A}$ is a Nash equilibrium at t if

$$J_i(t,\alpha^*) \le J_i\left(t,\alpha^{*,-i},\alpha^i\right), \quad \mathbb{P}_0\text{−a.s. for all } i \text{ and all } \alpha^i\in\mathcal{A}_i,$$

and we introduce the set value

$$\mathcal{V}_t := \left\{J(t,\alpha^*):\ \text{for all Nash equilibrium } \alpha^* \text{at } t\right\}.$$

We remark that the elements of $\mathcal{V}_t$ are $\mathcal{F}_t$-measurable, $\mathbb{R}^N$-valued random variables, and we shall consider the localization in $\mathbb{R}^N$ in the next subsection.

Given T_0 and $\eta\in\mathbb{L}^\infty(\mathcal{F}_{T_0};\mathbb{R}^N)$, denote

$$J_i(T_0,\eta;t,\alpha) := \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\eta_i + \int_t^{T_0} f_i\left(s,\alpha_s^i\right)ds\right], \quad i=1,\cdots,N,$$

and we define Nash equilibrium at $(T_0,\eta;t)$ in the obvious way. As such, we then have the following DPP. We remark that this result does not even require the right continuity of $\mathbb{F}$.

Theorem 2. *Under Assumption 1, for any $0\le t<T_0\le T$, it holds*

$$\mathcal{V}_t := \left\{J(T_0,\eta;t,\alpha^*): \textit{for all } \eta\in\mathcal{V}_{T_0} \textit{ all Nash equilibrium } \alpha^* \textit{ at } (T_0,\eta;t)\right\}. \tag{1}$$

Proof. Let $\tilde{\mathcal{V}}_t$ denote the right side of (1). First, for $J(t,\alpha^*)\in\mathcal{V}_t$, denote $\eta := J(T_0,\alpha^*)$. For any i and $\alpha^i\in\mathcal{A}_i$, denote $\hat{\alpha}^i := \alpha^i\mathbf{1}_{[0,T_0]} + \alpha^*\mathbf{1}_{(T_0,T]}$. It is clear that

$$J_i\left(T_0,\eta;t,\alpha^{*,-i},\alpha^i\right) = J_i\left(t,\alpha^{*,-i},\hat{\alpha}^i\right) \ge J_i(t,\alpha^*) = J_i(T_0,\eta;t,\alpha^*).$$

That is, α^* is a Nash equilibrium at $(T_0,\eta;t)$. Moreover, assume by contradiction that $\eta\notin\mathcal{V}_{T_0}$, and then there exist i and $\alpha^i\in\mathcal{A}_i$ such that $\mathbb{P}_0(E_i)>0$, where $E_i:=\{J_i(T_0,\alpha^{*,-i},\alpha^i)<J_i(T_0,\alpha^*)\}$. Denote $\hat\alpha^i:=\alpha^*\mathbf{1}_{[0,T_0]}+\mathbf{1}_{(T_0,T]}[\alpha^i\mathbf{1}_{E_i}+\alpha^*\mathbf{1}_{E_i^c}]$. Then,

$$J_i\big(t,\alpha^{*,-i},\hat\alpha^i\big)=\mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[\int_t^{T_0}f_i\Big(s,\alpha_s^{*,i}\Big)ds+J_i\big(T_0,\alpha^{*,-i},\alpha^i\big)\mathbf{1}_{E_i}+J_i(T_0,\alpha^*)\mathbf{1}_{E_i^c}\right]$$
$$<\mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[\int_t^{T_0}f_i\Big(s,\alpha_s^{*,i}\Big)ds+J_i(T_0,\alpha^*)\right]=J_i(t,\alpha^*).$$

This contradicts with the assumption that α^* is an equilibrium at t. Thus $\eta\in\mathcal{V}_{T_0}$, and therefore, $J(t,\alpha^*)\in\tilde{\mathcal{V}}_t$.

Next, let $J(T_0,\eta;t,\alpha^*)\in\tilde{\mathcal{V}}_t$ with desired (η,α^*). Because $\eta\in\mathcal{V}_{T_0}$, $\eta=J(T_0,\tilde\alpha^*)$ for some equilibrium $\tilde\alpha^*$ at T_0. Denote $\hat\alpha^*:=\alpha^*\mathbf{1}_{[0,T_0]}+\tilde\alpha^*\mathbf{1}_{(T_0,T]}$, and for any i and $\alpha^i\in\mathcal{A}_i$, denote $\hat\alpha^i:=\alpha^i\mathbf{1}_{[0,T_0]}+\tilde\alpha^*\mathbf{1}_{(T_0,T]}$. Then,

$$\begin{aligned}&J_i\big(t,\hat\alpha^{*,-i},\alpha^i\big)-J_i(t,\hat\alpha^*)\\&=\Big[J_i\big(t,\hat\alpha^{*,-i},\alpha^i\big)-J_i\big(t,\hat\alpha^{*,-i},\hat\alpha^i\big)\Big]+\Big[J_i\big(t,\hat\alpha^{*,-i},\hat\alpha^i\big)-J_i(t,\hat\alpha^*)\Big]\\&=\mathbb{E}_t^{\mathbb{P}^{\hat\alpha^{*,-i},\alpha^i}}\Big[J_i\big(T_0,\tilde\alpha^{*,-i},\alpha^i\big)-J_i(T_0,\tilde\alpha^*)\Big]+\Big[J_i\big(T_0,\eta;t,\alpha^{*,-i},\alpha^i\big)-J_i(T_0,\eta;t,\alpha^*)\Big].\end{aligned}$$

The second term above is nonnegative by the requirement of α^*. Moreover, note that $J_i(T_0,\tilde\alpha^{*,-i},\alpha^i)\ge J_i(T_0,\tilde\alpha^*)$, $\mathbb{P}_0$-a.s., and $\mathbb{P}^{\hat\alpha^{*,-i},\alpha^i}$ are equivalent to $\mathbb{P}_0$, and then $J_i(T_0,\tilde\alpha^{*,-i},\alpha^i)\ge J_i(T_0,\tilde\alpha^*)$, $\mathbb{P}^{\hat\alpha^{*,-i},\alpha^i}$-a.s. This implies $J_i(t,\hat\alpha^{*,-i},\alpha^i)\ge J_i(t,\hat\alpha^*)$. So α^* is an equilibrium at t, and thus $J(T_0,\eta;t,\alpha^*)=J(t,\hat\alpha^*)\in\mathcal{V}_t$. Q.E.D.

3.2. The Localization

While Theorem 2 is quite simple, as mentioned, $\mathcal{V}_t$ is a set of random variables rather than value sets in $\mathbb{R}^N$ as in Section 2, which is not desirable in applications. In this subsection, we localize the random variables in a pointwise sense. For this purpose, it is more convenient to use the canonical space.

For the rest of this section, let $\Omega:=\{\omega\in C([0,T];\mathbb{R}^d):\omega_0=0\}$ be the canonical space, B the canonical process, $B_t(\omega)=\omega_t$, $\mathbb{P}_0$ the Wiener measure, and $\mathbb{F}=\{\mathcal{F}_t\}_{0\le t\le T}:=\mathbb{F}^B$ the $\mathbb{P}_0$-augmented filtration generated by B. Denote

$$\|\omega\|:=\sup_{0\le t\le T}|\omega_t|,\quad \mathbf{d}\Big((t,\omega),(\tilde t,\tilde\omega)\Big):=\sqrt{|t-\tilde t|}+\|\omega_{t\wedge\cdot}-\tilde\omega_{\tilde t\wedge\cdot}\|.$$

Then $(\Omega,\|\cdot\|)$ is a Polish space. For $t\in[0,T]$, $\omega,\tilde\omega\in\Omega$, and $\xi\in\mathbb{L}^0(\mathcal{F}_T)$, $\zeta\in\mathbb{L}^0(\mathbb{F})$, denote

$$(\omega\otimes_t\tilde\omega)_s:=\omega_s\mathbf{1}_{[0,t]}(s)+[\omega_t+\tilde\omega_{s-t}]\mathbf{1}_{[t,T]}(s),$$
$$\xi^{t,\omega}(\tilde\omega):=\xi(\omega\otimes_t\tilde\omega),\quad \zeta_s^{t,\omega}(\tilde\omega):=\zeta_{t+s}(\omega\otimes_t\tilde\omega).$$

Let $A,\mathcal{A},b,f,\xi$ be as in the previous subsection. For $(t,\omega)\in[0,T]\times\Omega$ and $\alpha\in\mathcal{A}$, define

$$\begin{aligned}&\frac{d\mathbb{P}^{t,\omega,\alpha}}{d\mathbb{P}_0}:=M_{T-t}^{t,\omega,\alpha}:=\exp\left(\int_0^{T-t}b^{t,\omega}(s,B_\cdot,\alpha_s)\cdot dB_s-\frac{1}{2}\int_0^{T-t}|b^{t,\omega}(s,B_\cdot,\alpha_s)|^2ds\right);\\&J_i(t,\omega,\alpha):=\mathbb{E}^{\mathbb{P}^{t,\omega,\alpha}}\left[\xi_i^{t,\omega}(B_\cdot)+\int_0^{T-t}f_i^{t,\omega}\Big(s,B_\cdot,\alpha_s^i\Big)ds\right],\quad i=1,\cdots,N.\end{aligned}\tag{2}$$

We say that $\alpha^*\in\mathcal{A}$ is a Nash equilibrium at (t,ω), denoted as $\alpha^*\in NE(t,\omega)$, if

$$J_i(t,\omega,\alpha^*)\le J_i\big(t,\omega,\alpha^{*,-i},\alpha^i\big),\quad\text{for all } i \text{ and all } \alpha^i\in\mathcal{A}_i,$$

and we introduce the set value

$$\mathbb{V}_0(t,\omega):=\big\{J(t,\omega,\alpha^*):\alpha^*\in NE(t,\omega)\big\}\subset\mathbb{R}^N.$$

Intuitively, $\eta\in\mathcal{V}_t$ means $\eta(\omega)\in\mathbb{V}_0(t,\omega)$ for $\mathbb{P}_0$-a.e. ω. This is indeed true in the setting of Section 2 if we introduce the corresponding $\mathcal{V}_t$. However, in the continuous time model, we encounter some serious measurability issues. Because the state space Ω is uncountable, the measurability or even certain regularity of the set value will be required. Note that $\mathcal{A}$ is typically not compact, so the arguments in Proposition 1 do not work here. In fact, in this case neither the (Borel or analytic) measurability of the set $\mathbb{V}_0(t,\omega)\subset\mathbb{R}^N$ for fixed (t,ω) nor the $\mathbb{F}$-progressive measurability of the mapping $(t,\omega)\to\mathbb{V}_0(t,\omega)$ is clear to us. To get around of this difficulty, we relax the equilibriums to approximating ones, which are usually sufficient in practice.

Definition 3. We say that $\alpha^\varepsilon\in\mathcal{A}$ is an ε-equilibrium at (t,ω), denoted as $\alpha^\varepsilon\in NE_\varepsilon(t,\omega)$, if

$$J_i(t,\omega,\alpha^\varepsilon)\le J_i\big(t,\omega,\alpha^{\varepsilon,-i},\alpha^i\big)+\varepsilon,\quad \text{for all } i \text{ and all } \alpha^i\in\mathcal{A}_i.$$

Denote $O_\varepsilon(y):=\{\tilde y\in\mathbb{R}^N:|\tilde y-y|<\varepsilon\}\subset\mathbb{R}^N$, and define

$$\mathbb{V}(t,\omega):=\bigcap_{\varepsilon>0}\mathbb{V}_\varepsilon(t,\omega)\ \text{ where }\ \mathbb{V}_\varepsilon(t,\omega):=\big\{y\in O_\varepsilon\big(J(t,\omega;\alpha^\varepsilon)\big):\alpha^\varepsilon\in NE_\varepsilon(t,\omega)\big\}.$$

Clearly $\mathbb{V}_0(t,\omega)\subset\mathbb{V}(t,\omega)$. Moreover, we have the following simple but important properties.

Proposition 4. *Let Assumption 1 hold.*

i. *$\mathbb{V}_\varepsilon(t,\omega)$ is bounded and open;*

ii. *For any $\varepsilon'<\varepsilon$, the closure $\text{cl}(\mathbb{V}_{\varepsilon'}(t,\omega))\subset\mathbb{V}_\varepsilon(t,\omega)$;*

iii. *$\mathbb{V}(t,\omega)$ is compact. Moreover, $\mathbb{V}(t,\omega)\neq\emptyset$ whenever $NE_\varepsilon(t,\omega)\neq\emptyset$ for all $\varepsilon>0$.*

Proof.

i. This result is obvious.

ii. One can easily see that $\text{cl}(\mathbb{V}_{\varepsilon'}(t,\omega))\subset\{y\in O_{\varepsilon-\varepsilon'}(\tilde y):\tilde y\in\mathbb{V}_{\varepsilon'}(t,\omega)\}\subset\mathbb{V}_\varepsilon(t,\omega)$.

iii. Because $\mathbb{V}_\varepsilon(t,\omega)$ is bounded, the $\text{cl}(\mathbb{V}_\varepsilon(t,\omega))$ is compact. By (ii), we see that $\mathbb{V}(t,\omega)=\cap_{\varepsilon>0}\text{cl}(\mathbb{V}_\varepsilon(t,\omega))$ is also compact. Moreover, again because each $\text{cl}(\mathbb{V}_\varepsilon(t,\omega))$ is compact, we see that $\mathbb{V}(t,\omega)\neq\emptyset$ whenever $\text{cl}(\mathbb{V}_\varepsilon(t,\omega))\neq\emptyset$ for all $\varepsilon>0$. Q.E.D.

Remark 11.

i. It is obvious that $\text{cl}(\mathbb{V}_0(t,\omega))\subset\mathbb{V}(t,\omega)$; however, the inclusion could be strict. Note that $\mathbb{V}_0(t,\omega)\neq\emptyset$ if and only if the game has a true equilibrium, whereas $\mathbb{V}(t,\omega)\neq\emptyset$ can occur even if no equilibrium exists. Such a relaxation could be useful for more general games where a true equilibrium may not exist; see for example, Frei and dos Reis [17], Buckdahn et al. [6], and Lin [26] for some results in this direction (the latter two use strategies instead of closed-loop controls, though).

ii. When we view a stochastic control problem as a game with one player and denote its (standard) value function as $v(t,\omega)$, then we always have $\mathbb{V}(t,\omega)=\{v(t,\omega)\}$, but $\mathbb{V}_0(t,\omega)$ could be empty. Similarly for a two-person, zero-sum game, the standard value function corresponds to $\mathbb{V}$, not $\mathbb{V}_0$.

For the rest of the properties, we impose the following regularities.

Assumption 2.

i. *b,f are uniformly continuous in (t,ω) under $\mathbf{d}$, and ξ is uniformly continuous in ω under $\|\cdot\|$, with a common modulus of continuity function ρ_0.*

ii. *b,f are uniformly continuous in a.*

We then have the regularity and stability of $\mathbb{V}$ in the spirit of Feinstein [15]. However, we note that Feinstein [15] considers the set of equilibriums, whereas we consider the set of values. Given $D_n\subset\mathbb{R}^N$, we define the set valued limits as in Aubin and Frankowska [2]:

$$\underline{\lim}_{n\to\infty}D_n=\Big\{y\in\mathbb{R}^N:\lim_{n\to\infty}\inf_{y_n\in D_n}|y-y_n|=0\Big\}$$

$$\overline{\lim}_{n\to\infty}D_n=\Big\{y\in\mathbb{R}^N:\underline{\lim}_{n\to\infty}\inf_{y_n\in D_n}|y-y_n|=0\Big\}.$$

That is, the limit inferior (superior) denotes the set of $y\in\mathbb{R}^N$ such that there exists $y_n\in D_n$ (resp. subsequence), satisfying $\lim_{n\to\infty}y_n=y$.

Theorem 3. *Let Assumptions 1 and 2 (i) hold.*

i. *For any $\varepsilon_1<\varepsilon_2$, there exists $\delta>0$ such that*

$$\mathbb{V}_{\varepsilon_1}(\tilde t,\tilde\omega)\subset\mathbb{V}_{\varepsilon_2}(t,\omega)\quad \textit{for all } (t,\omega),(\tilde t,\tilde\omega) \textit{ satisfying } \mathbf{d}\Big((t,\omega),(\tilde t,\tilde\omega)\Big)\le\delta. \tag{3}$$

ii. *If $\mathbf{d}((t_n,\omega^n),(t,\omega))\to0$, then $\mathbb{V}(t,\omega)=\bigcap_{\varepsilon>0}[\underline{\lim}_{n\to\infty}\mathbb{V}_\varepsilon(t_n,\omega^n)]=\bigcap_{\varepsilon>0}[\overline{\lim}_{n\to\infty}\mathbb{V}_\varepsilon(t_n,\omega^n)]$.*

iii. *Assume that* (b^n, f^n, ξ^n) *satisfies Assumption 2 uniformly, and define* $\mathbb{V}^n_\varepsilon(t,\omega)$ *in the obvious way. If* $(b^n, f^n, \xi^n) \to (b, f, \xi)$ *uniformly, then*

$$\mathbb{V}(t,\omega) = \bigcap_{\varepsilon>0}\Big[\varliminf_{n\to\infty} \mathbb{V}^n_\varepsilon(t,\omega)\Big] = \bigcap_{\varepsilon>0}\Big[\varlimsup_{n\to\infty} \mathbb{V}^n_\varepsilon(t,\omega)\Big].$$

Proof.

i. We first claim that there exists a modulus of continuity function ρ such that

$$|J(t,\omega,\alpha) - J(\tilde{t},\tilde{\omega},\alpha)| \le \rho\Big(\mathbf{d}\Big((t,\omega),(\tilde{t},\tilde{\omega})\Big)\Big), \quad \forall (t,\omega),(\tilde{t},\tilde{\omega}),\ \forall \alpha. \tag{4}$$

Then, let $\mathbf{d}((t,\omega),(\tilde{t},\tilde{\omega})) \le \delta$ and $y \in O_{\varepsilon_1}(J(\tilde{t},\tilde{\omega},\alpha^{\varepsilon_1})) \subset \mathbb{V}_{\varepsilon_1}(\tilde{t},\tilde{\omega})$, where $\alpha^{\varepsilon_1} \in NE_{\varepsilon_1}(\tilde{t},\tilde{\omega})$. For any i and α^i, by 4 we have

$$\begin{aligned} J_i(t,\omega,\alpha^{\varepsilon_1}) &\le J_i(\tilde{t},\tilde{\omega},\alpha^{\varepsilon_1}) + \rho(\delta) \le J_i(\tilde{t},\tilde{\omega},\alpha^{\varepsilon_1,-i},\alpha^i) + \varepsilon_1 + \rho(\delta) \\ &\le J_i(t,\omega,\alpha^{\varepsilon_1,-i},\alpha^i) + \varepsilon_1 + 2\rho(\delta). \end{aligned}$$

Choose $\delta > 0$ small enough such that $2\rho(\delta) \le \varepsilon_2 - \varepsilon_1$, we see that $\alpha^{\varepsilon_1} \in NE_{\varepsilon_2}(t,\omega)$. Moreover, by (4) again we have

$$|y - J_i(t,\omega,\alpha^{\varepsilon_1})| \le |y - J_i(\tilde{t},\tilde{\omega},\alpha^{\varepsilon_1})| + \rho(\delta) < \varepsilon_1 + \rho(\delta) \le \varepsilon_2.$$

So $y \in \mathbb{V}_{\varepsilon_2}(t,\omega)$, and hence 3 holds.

We next prove (4). By (2) we have

$$J_i(t,\omega,\alpha) = \mathbb{E}^{\mathbb{P}_0}\left[M^{t,\omega,\alpha}_{T-t}\left[\xi^{t,\omega}_i(B.) + \int_0^{T-t} f^{t,\omega}_i\Big(s,B.,\alpha^i_s\Big)\,ds\right]\right]. \tag{5}$$

Similarly, we have the representation for $J_i(\tilde{t},\tilde{\omega},\alpha)$. Denote

$$OSC_\delta(B) := \sup_{|s-\tilde{s}|\le\delta} |B_s - B_{\tilde{s}}|, \quad \rho'(\delta) := \mathbb{E}\Big[\rho_0^2(\delta + OSC_\delta(B))\Big].$$

Assume without loss of generality that $t \le \tilde{t}$. Then,

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^{T-t} f^{t,\omega}_i\Big(s,B.,\alpha^i_s\Big)\,ds - \int_0^{T-\tilde{t}} f^{\tilde{t},\tilde{\omega}}_i\Big(s,B.,\alpha^i_s\Big)ds\right|^2\right] \\ &\le C\mathbb{E}\left[\int_0^{T-\tilde{t}} |f^{t,\omega}_i\Big(s,B.,\alpha^i_s\Big) - f^{\tilde{t},\tilde{\omega}}_i\Big(s,B.,\alpha^i_s\Big)|^2 ds + |\int_{T-\tilde{t}}^{T-t} f^{t,\omega}_i\Big(s,B.,\alpha^i_s\Big)\,ds|^2\right] \\ &\le C\mathbb{E}\left[\int_0^{T-\tilde{t}} \rho_0^2\Big(\mathbf{d}\Big((t+s,\omega\otimes_t B.),\Big(\tilde{t}+s,\tilde{\omega}\otimes_{\tilde{t}} B.\Big)\Big)\Big)\,ds\right] + C\delta^2 \\ &\le C\mathbb{E}\left[\int_0^{T-\tilde{t}} \rho_0^2\Big(\mathbf{d}\Big((t,\omega),(\tilde{t},\tilde{\omega})\Big) + OSC_{\tilde{t}-t}(B)\Big)\,ds\right] + C\delta^2 \\ &\le C\mathbb{E}\left[\int_0^{T-\tilde{t}} \rho_0^2(\delta + OSC_\delta(B))\,ds\right] + C\delta^2 \le C\rho'(\delta) + C\delta^2. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}_0}\Big[|\xi^{t,\omega}_i(B.) - \xi^{\tilde{t},\tilde{\omega}}_i(B.)|^2\Big] \le \rho'(\delta); \\ &\mathbb{E}^{\mathbb{P}_0}\left[\left|\int_0^{T-t} b^{t,\omega}(s,B.,\alpha_s)\,dB_s - \int_0^{T-\tilde{t}} b^{\tilde{t},\tilde{\omega}}(s,B.,\alpha_s)dB_s\right|^2\right] \\ &\quad \le C\mathbb{E}^{\mathbb{P}_0}\left[\int_0^{T-\tilde{t}} |b^{t,\omega} - b^{\tilde{t},\tilde{\omega}}|^2(s,B.,\alpha_s)\,ds + \int_{T-\tilde{t}}^{T-t} |b^{t,\omega}(s,B.,\alpha_s)|^2 ds\right] \le C\rho'(\delta) + C\delta; \\ &\mathbb{E}^{\mathbb{P}_0}\left[\left|\int_0^{T-t} |b^{t,\omega}(s,B.,\alpha_s)|^2 ds - \int_0^{T-\tilde{t}} |b^{\tilde{t},\tilde{\omega}}(s,B.,\alpha_s)|^2 ds\right|^2\right] \\ &\quad \le C\mathbb{E}^{\mathbb{P}_0}\left[\int_0^{T-\tilde{t}} \Big||b^{t,\omega}|^2 - |b^{\tilde{t},\tilde{\omega}}|^2\Big|^2(s,B.,\alpha_s)\,ds + \left(\int_{T-\tilde{t}}^{T-t} |b^{t,\omega}(s,B.,\alpha_s)|^2 ds\right)^2\right] \le C\rho'(\delta) + C\delta^2. \end{aligned}$$

We note that, because b is bounded, for any $p \geq 1$,

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\left[(M_T^\alpha)^p + (M_T^\alpha)^{-p}\right] \leq C_p < \infty. \tag{6}$$

Moreover, note that $|e^x - e\tilde{x}| \leq [e^x + e\tilde{x}]|x-\tilde{x}|$. Then

$$\mathbb{E}^{\mathbb{P}_0}\Big[M_{T-t}^{t,\omega,\alpha}\Big[|\xi_i^{t,\omega}(B.) - \xi_i^{\tilde{t},\tilde{\omega}}(B.)|\Big]\Big] \leq C\bigg(\mathbb{E}^{\mathbb{P}_0}\Big[|\xi_i^{t,\omega}(B.) - \xi_i^{\tilde{t},\tilde{\omega}}(B.)|^2\Big]\bigg)^{\frac{1}{2}} \leq C\sqrt{\rho'(\delta)};$$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\left[M_{T-t}^{t,\omega,\alpha}\,|\int_0^{T-t} f_i^{t,\omega}\left(s,B.,\alpha_s^i\right)ds - \int_0^{T-\tilde{t}} f_i^{\tilde{t},\tilde{\omega}}\left(s,B.,\alpha_s^i\right)ds|\right]\\
&\quad\leq C\left(\mathbb{E}^{\mathbb{P}_0}\left[|\int_0^{T-t} f_i^{t,\omega}\left(s,B.,\alpha_s^i\right)ds - \int_0^{T-\tilde{t}} f_i^{\tilde{t},\tilde{\omega}}\left(s,B.,\alpha_s^i\right)ds|^2\right]\right)^{\frac{1}{2}}\\
&\quad\leq C\sqrt{\rho'(\delta)+\delta^2};
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\Big[|M_{T-t}^{t,\omega,\alpha} - M_{T-\tilde{t}}^{\tilde{t},\tilde{\omega},\alpha}|\Big] \leq \mathbb{E}^{\mathbb{P}_0}\Big[\Big[M_{T-t}^{t,\omega,\alpha} + M_{T-\tilde{t}}^{\tilde{t},\tilde{\omega},\alpha}\Big]\times\\
&\quad\bigg[|\int_0^{T-t} b^{t,\omega}(s,B.,\alpha_s)dB_s - \int_0^{T-\tilde{t}} b^{\tilde{t},\tilde{\omega}}(s,B.,\alpha_s)dB_s|\\
&\quad+\frac{1}{2}|\int_0^{T-t}|b^{t,\omega}(s,B.,\alpha_s)|^2ds - \int_0^{T-\tilde{t}}|b^{\tilde{t},\tilde{\omega}}(s,B.,\alpha_s)|^2ds|\bigg]\bigg]\\
&\quad\leq C\sqrt{\rho'(\delta)+\delta};
\end{aligned}$$

$$\begin{aligned}
&|J(t,\omega,\alpha) - J(\tilde{t},\tilde{\omega},\alpha)| \leq \mathbb{E}^{\mathbb{P}_0}\Big[M_{T-t}^{t,\omega,\alpha}|\xi_i^{t,\omega}(B.) - \xi_i^{\tilde{t},\tilde{\omega}}(B.)|\\
&\quad+M_{T-t}^{t,\omega,\alpha}|\int_0^{T-t} f_i^{t,\omega}\left(s,B.,\alpha_s^i\right)ds - \int_0^{T-\tilde{t}} f_i^{\tilde{t},\tilde{\omega}}\left(s,B.,\alpha_s^i\right)ds|\\
&\quad+|M_{T-t}^{t,\omega,\alpha} - M_{T-\tilde{t}}^{\tilde{t},\tilde{\omega},\alpha}||\xi_i^{\tilde{t},\tilde{\omega}}(B.) + \int_0^{T-\tilde{t}} f_i^{\tilde{t},\tilde{\omega}}\left(s,B.,\alpha_s^i\right)ds|\Big]\\
&\quad\leq C\sqrt{\rho'(\delta)+\delta} =: \rho(\delta).
\end{aligned}$$

Clearly ρ' and hence, ρ are modulus of continuity functions, we thus obtain (4).

ii. Denote $\delta_n := \mathbf{d}((t_n,\omega^n),(t,\omega)) \to 0$. For any $\varepsilon_1 < \varepsilon_2$, by (3) and its proof we have

$$\mathbb{V}_{\varepsilon_1}(t,\omega) \subset \mathbb{V}_{\varepsilon_2}(t_n,\omega^n),\;\; \mathbb{V}_{\varepsilon_1}(t_n,\omega^n) \subset \mathbb{V}_{\varepsilon_2}(t,\omega), \quad \text{whenever } 2\rho(\delta_n) \leq \varepsilon_2 - \varepsilon_1. \tag{7}$$

Now fix ε_2 and set $\rho(\delta_n) \leq \frac{\varepsilon_2}{4}$; we see that (7) holds for all $\varepsilon_1 \leq \frac{\varepsilon_2}{2}$. This implies immediately that $\mathbb{V}(t,\omega) \subset \mathbb{V}_{\varepsilon_2}(t_n,\omega^n)$ and $\bigcap_{\varepsilon_1>0}[\overline{\lim}_{n\to\infty}\mathbb{V}_{\varepsilon_1}(t,\omega^n)] \subset \mathbb{V}_{\varepsilon_2}(t,\omega)$. Now send $\varepsilon_2 \to 0$, and we have $\bigcap_{\varepsilon_1>0}[\overline{\lim}_{n\to\infty}\mathbb{V}_{\varepsilon_1}(t,\omega^n)] \subset \mathbb{V}(t,\omega) \subset \cap_{\varepsilon>0}[\underline{\lim}_{n\to\infty}\mathbb{V}_\varepsilon(t,\omega^n)]$. Because the limit inferior is always contained in the limit superior, hence they are all equal.

iii. Let J^n be defined by (5), but corresponding to (b^n, f^n, ξ^n). It is clear that $c_n := \sup_{t,\omega,\alpha}|[J^n - J](t,\omega,\alpha)| \to 0$. Then the result follows similar arguments to (ii). Q.E.D.

To study the measurability of the mapping $(t,\omega) \mapsto \mathbb{V}(t,\omega)$, we introduce

$$\hat{\mathbb{V}}_\varepsilon(t,\omega) := \bigcup_{\varepsilon'<\varepsilon}\mathbb{V}_{\varepsilon'}(t,\omega).$$

It is clear that

$$\hat{\mathbb{V}}_\varepsilon(t,\omega) \subset \mathbb{V}_\varepsilon(t,\omega) \subset \hat{\mathbb{V}}_{\tilde{\varepsilon}}(t,\omega), \quad \forall\, \varepsilon < \tilde{\varepsilon}, \quad \text{hence} \;\; \mathbb{V}(t,\omega) = \bigcap_{\varepsilon>0}\hat{\mathbb{V}}_\varepsilon(t,\omega).$$

We then have the following result, which will be quite useful for the DPP below.

Theorem 4. *Let Assumptions 1 and 2 (i) hold. For any $\varepsilon > 0$, any $\mathbb{F}$-stopping time τ, and any $\eta \in \mathbb{L}^0(\mathcal{F}_\tau)$, the events $\{\omega \in \Omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(\tau(\omega), \omega)\}$ and $\{\omega \in \Omega : \eta(\omega) \in \mathbb{V}(\tau(\omega), \omega)\}$ are $\mathcal{F}_\tau$-measurable.*

Proof. First, note that $\{\omega : \eta(\omega) \in \mathbb{V}(\tau(\omega), \omega)\} = \bigcap_{n\geq 1}\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{1/n}(\tau(\omega), \omega)\}$; then the measurability for $\mathbb{V}$ clearly follows from the measurability for $\hat{\mathbb{V}}_{\frac{1}{n}}$. We now prove the claimed measurability for $\hat{\mathbb{V}}_\varepsilon$ in three steps.

Step 1. We first show that, for any t and any compact set $K \subset\subset \mathbb{R}^N$, the event $\{\omega \in \Omega : K \subset \hat{\mathbb{V}}_\varepsilon(t, \omega)\}$ is open (in terms of ω under $\|\cdot\|$) and thus is obviously $\mathcal{F}_t$-measurable. Indeed, fix ω such that $K \subset \hat{\mathbb{V}}_\varepsilon(t, \omega) = \bigcup_{\varepsilon'<\varepsilon} \mathbb{V}_{\varepsilon'}(t, \omega)$. Because $K \subset \mathbb{R}^N$ is compact and $\mathbb{V}_{\varepsilon'}(t, \omega) \subset \mathbb{R}^N$ is open and increasing in ε', there exists $\varepsilon_1 < \varepsilon$ such that $K \subset \mathbb{V}_{\varepsilon_1}(t, \omega)$. Now by (3) we see that there exists $\delta > 0$ such that $K \subset \mathbb{V}_{\varepsilon_1+\varepsilon/2}(t, \tilde{\omega}) \subset \hat{\mathbb{V}}_\varepsilon(t, \tilde{\omega})$ whenever $\|\tilde{\omega}_{t\wedge\cdot} - \omega_{t\wedge\cdot}\| \leq \delta$.

Step 2. We next show the result when $\tau \equiv t$ is a constant. Note that the set of closed balls in $\mathbb{R}^N$ with rational centers and rational radii is countable, numerated as $\{K_i\}_{i\geq 1}$. Because $\hat{\mathbb{V}}_\varepsilon(t, \omega)$ is open, for $\eta \in \mathbb{L}^0(\mathcal{F}_t)$, one can easily verify that

$$\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(t, \omega)\right\} = \bigcup_{i\geq 1}(E_i \cap \{\omega : \eta(\omega) \in K_i\}), \text{where } E_i := \left\{\omega : K_i \subset \hat{\mathbb{V}}_\varepsilon(t, \omega)\right\}.$$

Clearly, $\{\eta \in K_i\}$ is $\mathcal{F}_t$-measurable, and by Step 1 the events E_i are also $\mathcal{F}_t$-measurable, and then so is the event $\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(t, \omega)\}$.

Step 3. We now consider stopping times τ. If τ is discrete, namely, taking only finitely many values, $t_1, \cdots, t_n$, then

$$\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(\tau(\omega), \omega)\right\} = \bigcup_{i=1}^{n}\left(\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(t_i, \omega)\right\} \cap \{\tau = t_i\}\right),$$

By Step 2, $\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(t_i, \omega)\} \in \mathcal{F}_{t_i}$ for each i, and then the above clearly implies $\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(\tau(\omega), \omega)\} \in \mathcal{F}_\tau$.

Now, for general τ, there exist stopping times $\tau_n \downarrow \tau$ such that each τ_n is discrete and $0 \leq \tau_n - \tau \leq 2^{-n}T$. Choose an arbitrary sequence $\varepsilon_m \uparrow \varepsilon$. By (3), for any m we have

$$\begin{aligned}\{\omega : \eta(\omega) \in \mathbb{V}_{\varepsilon_{m-1}}(\tau(\omega), \omega)\} &\subset \underline{\lim}_{n\to\infty}\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega), \omega)\right\}\\ &\subset \overline{\lim}_{n\to\infty}\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega), \omega)\right\} \subset \{\omega : \eta(\omega) \in \mathbb{V}_{\varepsilon_{m+1}}(\tau(\omega), \omega)\}.\end{aligned}$$

Send $m \to \infty$ and note that the first and the last terms above have the same limit, and then the middle two terms have to converge to the same limit, namely

$$\lim_{m,n\to\infty}\left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega), \omega)\right\} = \left\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(\tau(\omega), \omega)\right\}. \tag{8}$$

We already have $\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega), \omega)\} \in \mathcal{F}_{\tau_n}$. Because $\mathbb{F}$ is right continuous and $\tau_n \downarrow \tau$, then $\lim_{n\to\infty}\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega), \omega)\} \in \mathcal{F}_\tau$, and thus $\{\omega : \eta(\omega) \in \hat{\mathbb{V}}_\varepsilon(\tau(\omega), \omega)\} \in \mathcal{F}_\tau$. Q.E.D.

3.3. Dynamic Programming Principle

Given an $\mathbb{F}$-stopping time τ and $\eta \in \mathbb{L}^\infty(\mathcal{F}_\tau; \mathbb{R}^N)$, one may consider the game on $[0, \tau]$ with terminal condition η. In particular,

$$J_i(\tau, \eta; t, \omega, \alpha) := \mathbb{E}^{\mathbb{P}^{t,\omega,\alpha}}\left[\eta_i^{t,\omega} + \int_0^{\tau^{t,\omega}-t} f_i^{t,\omega}\left(s, B_\cdot, \alpha_s^i\right) ds\right], \quad t \leq \tau(\omega), \; i = 1, \cdots, N, \tag{9}$$

and we can define equilibrium and ε-equilibrium at $(\tau, \eta; t, \omega)$ in the obvious sense. We now state our main result of this section, extending Theorem 1 to the continuous time model.

Theorem 5. *Let Assumptions 1 and 2 hold. For any (t,ω) and any $\mathbb{F}$-stopping time τ with $\tau(\omega) > t$, we have*

$$\begin{aligned}\mathbb{V}(t,\omega) = \bigcap_{\varepsilon>0}\Big\{y \in O_\varepsilon(J(\tau,\eta;t,\omega,\alpha^\varepsilon)) : \text{ for all } \eta \in \mathbb{L}^\infty(\mathcal{F}_\tau;\mathbb{R}^{\mathrm{N}}) \text{ and } \alpha^\varepsilon \in \mathcal{A} \\ \text{such that } \alpha^\varepsilon \in NE_\varepsilon(\tau,\eta;\mathrm{t},\omega) \text{ and } \mathbb{P}_0\Big(\eta^{\mathrm{t},\omega} \notin \hat{\mathbb{V}}_\varepsilon\Big(\tau^{\mathrm{t},\omega},\mathrm{B}^{\mathrm{t},\omega}_\cdot\Big)\Big) \le \varepsilon\Big\}.\end{aligned} \tag{10}$$

To prove the theorem, we first need a lemma.

Lemma 1.
i. *Let τ be an $\mathbb{F}$-stopping time and $\eta \in \mathbb{L}^\infty(\mathcal{F}_\tau;\mathbb{R}^N)$. For any $\delta > 0$, there exists a discrete $\mathbb{F}$-stopping time τ_δ with $0 \le \tau_\delta - \tau \le \delta$ and an $\eta_\delta \in \mathbb{L}^\infty(\mathcal{F}_{\tau_\delta};\mathbb{R}^N)$ with the same bound as η such that*

$$\mathbb{E}^{\mathbb{P}_0}[|\eta_\delta - \eta|] \le \delta, \quad \text{and } \eta_\delta \text{ is uniformly continuous in } \omega. \tag{11}$$

ii. *For any $\alpha \in \mathcal{A}$ and $\delta > 0$, there exists a discrete $\alpha^\delta = \sum_{i=0}^{n-1} \alpha^\delta_{t_i}\mathbf{1}_{[t_i,t_i+1)} \in \mathcal{A}$ such that*

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T\Big[|\alpha^\delta_t - \alpha_t| \wedge 1\Big]dt\right] \le \delta, \text{ and each } \alpha^\delta_{\mathrm{t_i}} \text{ is uniformly continuous in } \omega.$$

Proof. i. The case $\tau \equiv t$ follows the same approximations in Theorem 2.5.2, Steps 1–4 of Zhang [40], and in this case we actually have $\tau_\delta \equiv t$ as well. We now prove (i) for general stopping time τ. First, clearly there exists discrete τ_δ such that $0 \le \tau_\delta - \tau \le \delta$. Assume that τ_δ takes values $t_1, \cdots, t_n$. Because the space $(\Omega, \|\cdot\|)$ is Polish and thus $\mathbb{P}_0$ is tight, see for example, Billingsley [5]; then, for each i there exists a compact set[1] $K_i \subset\subset \{\tau_\delta = t_i\}$ such that $K_i \in \mathcal{F}_{t_i}$ and $\mathbb{P}_0(\{\tau_\delta = t_i\}\backslash K_i) < \delta/3C_0n$, where C_0 is the bound of η. Then, one may easily construct uniformly continuous functions $I_i \in \mathbb{L}^0(\mathcal{F}_{t_i};[0,1])$ such that $\mathbb{E}^{\mathbb{P}_0}[|I_i - \mathbf{1}_{K_i}|] \le \delta/3C_0n$. Next, note that $\eta \in \mathbb{L}^\infty(\mathcal{F}_{\tau_\delta};\mathbb{R}^N)$, and then $\eta\mathbf{1}_{\{\tau_\delta = t_i\}}$ is $\mathcal{F}_{t_i}$-measurable. Apply (11) for the deterministic time case; there exists $\eta_i \in \mathbb{L}^\infty(\mathcal{F}_{t_i};\mathbb{R}^N)$ with the same bound as η such that

$$\mathbb{E}^{\mathbb{P}_0}[|\eta_i - \eta\mathbf{1}_{\{\tau_\delta = t_i\}}|] \le \frac{\delta}{3n}, \quad \text{and } \eta_i \text{ is uniformly continuous in } \omega.$$

Denote $\eta_\delta := \Sigma_{i=1}^n \eta_i I_i$. Then one can easily verify that η_δ is $\mathcal{F}_{\tau_\delta}$ measurable, uniformly continuous, and

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_0}[|\eta_\delta - \eta|] \le \sum_{i=1}^n \mathbb{E}^{\mathbb{P}_0}[|\eta_i I_i - \eta\mathbf{1}_{\{\tau_\delta = t_i\}}|] \\ &\le \sum_{i=1}^n \mathbb{E}^{\mathbb{P}_0}[|\eta_i[I_i - \mathbf{1}_{K_i}]| + |\eta_i[\mathbf{1}_{K_i} - \mathbf{1}_{\{\tau_\delta = t_i\}}]| + |\eta_i - \eta\mathbf{1}_{\{\tau_\delta = t_i\}}|\mathbf{1}_{\{\tau_\delta = t_i\}}] \\ &\le \sum_{i=1}^n \left[C_0\frac{\delta}{3C_0n} + C_0\frac{\delta}{3C_0n} + \frac{\delta}{3n}\right] = \delta.\end{aligned}$$

This proves (11) for the general stopping time τ.

ii. First, denote $\alpha^R_t := \alpha_t\mathbf{1}_{\{|\alpha_t| \le R\}}$. Then, $\lim_{R\to\infty}\mathbb{E}\left[\int_0^T[|\alpha^R_t - \alpha_t| \wedge 1]\,dt\right] = 0$. By otherwise choosing an α^R, without loss of generality we assume that α is bounded. Next, for each n, denote $t_i := iT/n$ and $i = 0, \cdots, n$. Denote $\alpha^n_t := 0$, $t \in [t_0, t_1]$, $\alpha^n_t := \frac{n}{T}\int_{t_{i-1}}^{t_i} \alpha_s ds$, $t \in (t_i, t_{i+1}]$, and $i = 1, \cdots, n-1$. Then, $\mathbb{E}\left[\int_0^T |\alpha^n_t - \alpha_t|\,dt\right] \le \delta/2$ for n large. Now, fix such an n. For each $\alpha^n_{t_i}$, by (i) we may construct uniformly continuous $\alpha^\delta_{t_i}$ such that $\mathbb{E}[|\alpha^\delta_{t_i} - \alpha^n_{t_i}|] \le \delta/2$. Then, clearly α^δ satisfies all the claimed properties. Q.E.D.

Proof of Theorem 5. For notational simplicity, we assume that $t = 0$; then, (10) becomes

$$\mathbb{V}(0,0) = \tilde{V}(0,0) := \bigcap_{\varepsilon>0} \tilde{V}_\varepsilon(0,0) \quad \text{where}$$
$$\tilde{V}_\varepsilon(0,0) := \Big\{y \in O_\varepsilon\left(J(\tau,\eta;0,0,\alpha^\varepsilon)\right): \quad \text{for all } \eta \in \mathbb{L}^\infty(\mathcal{F}_\tau;\mathbb{R}^N) \text{ and } \alpha^\varepsilon \in \mathcal{A} \tag{12}$$
$$\text{such that } \alpha^\varepsilon \in NE_\varepsilon(\tau,\eta;0,0) \text{ and } \mathbb{P}_0\Big(\eta \notin \hat{\mathbb{V}}_\varepsilon(\tau,B.)\Big) \le \varepsilon\Big\}.$$

Step 1. We first prove the $\subset$ part. Fix an arbitrary $y \in \mathbb{V}(0,0)$. To show $y \in \tilde{V}(0,0)$, we fix an arbitrary $\varepsilon > 0$. Let $\delta > 0$ be a small number that will be specified later.

Because $y \in \mathbb{V}_\delta(0,0)$, there exists $\tilde{\alpha}^\delta \in NE_\delta(0,0)$ such that $|y - J(0,0,\tilde{\alpha}^\delta)| \le \delta$. For any $\delta_1 > 0$, apply Lemma 1 (ii) on $\tilde{\alpha}^\delta$; there exists $\alpha^\delta = \Sigma_{i=0}^{n-1}\alpha^\delta_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \in \mathcal{A}$ such that $\alpha^\delta_{t_i}$ is uniformly continuous in ω and $\mathbb{E}^{\mathbb{P}_0}[\int_0^T[|\tilde{\alpha}^\delta_t - \alpha^\delta_t| \wedge 1]dt] \le \delta_1$. By Assumption 2 (ii) and (5), for δ_1 small enough (depending on δ), we see that

$$\alpha^\delta \in NE_{2\delta}(0,0) \quad \text{and} \quad |y - J\left(0,0,\alpha^\delta\right)| \le 2\delta. \tag{13}$$

Define

$$\eta^\delta(\omega) := J\left(\tau(\omega),\omega,(\alpha^\delta)^{\tau(\omega),\omega}\right). \tag{14}$$

By (4) and Assumption 2 (ii) again, it is clear that η^δ is $\mathcal{F}_\tau$-measurable. Note that, for any $\alpha \in \mathcal{A}$, $J(\tau,\eta^\delta;0,0,\alpha) = J(0,0,\tilde{\alpha})$, where $\tilde{\alpha} := \alpha\mathbf{1}_{[0,\tau]} + \alpha^\delta\mathbf{1}_{(\tau,T]}$. Then, (13) implies $\alpha^\delta \in NE_{2\delta}(\tau,\eta^\delta,0,0)$ and $|y - J(\tau,\eta^\delta;0,0,\alpha^\delta)| \le 2\delta$. We shall always set $2\delta \le \varepsilon$. Moreover, set $\varepsilon_m \uparrow \varepsilon$ and $\tau_n \downarrow \tau$ as in Theorem 4, Step 3. We claim that, for any m,

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{n\to\infty}}\,\mathbb{P}_0\Big(\Big\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega),\omega)\Big\}\Big) = 0. \tag{15}$$

Then, by (8) and noting that $\hat{\mathbb{V}}_\varepsilon$ is increasing in ε, we can easily see that

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\,\mathbb{P}_0(\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_\varepsilon(\tau(\omega),\omega)\}) &= \overline{\lim_{\delta\to 0}}\,\lim_{m,n\to\infty}\mathbb{P}_0(\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_{\varepsilon_m}(\tau_n(\omega),\omega)\})\\ &\le \overline{\lim_{\delta\to 0}}\,\overline{\lim_{n\to\infty}}\,\mathbb{P}_0(\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_{\varepsilon_1}(\tau_n(\omega),\omega)\}) = 0.\end{aligned}$$

This verifies all the requirements in (12) and thus $y \in \tilde{\mathbb{V}}_\varepsilon(0,0)$.

We now prove (15) for $m = 1$. Because α^δ is uniformly continuous in ω, by (9) and Assumption 2 (ii), similar to (4), we have

$$\lim_{n\to\infty}\mathbb{E}\left[|\eta^\delta_n - \eta^\delta|\right] = 0, \quad \text{where} \quad \eta^\delta_n(\omega) := J\left(\tau_n(\omega),\omega,(\alpha^\delta)^{\tau_n(\omega),\omega}\right). \tag{16}$$

Note that

$$\begin{aligned}\Big\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_{\varepsilon_1}(\tau_n(\omega),\omega)\Big\} &\subset \{\omega : \eta^\delta(\omega) \notin \mathbb{V}_{\varepsilon_2}(\tau_n(\omega),\omega)\}\\ &\subset E^\delta_n \cup \{\omega : |\eta^\delta(\omega) - \eta^\delta_n(\omega)| > \varepsilon_2\},\end{aligned}$$

where, assuming that τ_n takes values t_i, $i = 0,\cdots,2^n$,

$$E^\delta_n := \bigcup_{i=0}^{2^n} E_i, \quad E_i := \{\tau_n = t_i\} \cap \Big\{\omega : (\alpha^\delta)^{t_i,\omega} \notin NE_{\varepsilon_2}(t_i,\omega)\Big\}. \tag{17}$$

Then,

$$\mathbb{P}_0(\{\omega : \eta^\delta(\omega) \notin \hat{\mathbb{V}}_{\varepsilon_1}(\tau_n(\omega),\omega)\}) \le \mathbb{P}_0(E^\delta_n) + \frac{1}{\varepsilon_2}\mathbb{E}^{\mathbb{P}_0}\Big[|\eta^\delta_n - \eta^\delta(\omega)|\Big].$$

By (16), it suffices to show that

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{n\to\infty}}\,\mathbb{P}_0\left(E^\delta_n\right) = 0. \tag{18}$$

Now, fix δ, n. Let $\delta' > 0$ be another small number that will be specified later. Note that Ω is separable, and we may have a decomposition $E_i = \cup_{j\ge 1} E_{i,j}$ on $\mathcal{F}_{t_i}$ such that, for some fixed $\omega^{i,j} \in E_{i,j}$, $\sup_{0\le s\le t_i} |\omega_s - \omega_s^{i,j}| \le \delta'$ for all $\omega \in E_{i,j}$. Now, for each (i,j), because $(\alpha^\delta)^{t_i,\omega^{i,j}} \notin NE_{\varepsilon_2}(t_i, \omega^{i,j})$, there exists $k = 1, \cdots, N$ and $\alpha^{i,j,k} \in \mathcal{A}$ such that

$$\left(\eta_n^\delta\right)_k(\omega^{i,j}) = J_k\left(t_i, \omega^{i,j}, (\alpha^\delta)^{t_i,\omega^{i,j}}\right) > J_k\left(t_i, \omega^{i,j}, (\alpha^{\delta,-k})^{t_i,\omega^{i,j}}, \alpha^{i,j,k}\right) + \varepsilon_2. \tag{19}$$

Again, by (9) and Assumption 2 (ii), and because α^δ is uniformly continuous in ω, then $\omega \longmapsto J_k(t_i, \omega, (\alpha^\delta)^{t_i,\omega})$ and $\omega \longmapsto J_k(t_i, \omega, (\alpha^{\delta,-k})^{t_i,\omega}, \alpha^{i,j,k})$ are uniformly continuous. Thus, for δ' small enough,

$$J_k\left(t_i, \omega, (\alpha^\delta)^{t_i,\omega}\right) > J_k\left(t_i, \omega, (\alpha^{\delta,-k})^{t_i,\omega}, \alpha^{i,j,k}\right) + \frac{\varepsilon_2}{2}, \quad \forall \omega \in E_{i,j}. \tag{20}$$

Denote $E_{i,j}^k := \{\omega \in E_{i,j} : 19 \text{ holds}\}$. Then, $E_n^\delta = \bigcup_{k=1}^N E^k$, where $E^k := \bigcup_{i=0}^{2^n} \bigcup_{j\ge 1} E_{ij}^k$. One can easily construct $\alpha^k \in \mathcal{A}_k$ such that $(\alpha^k)_t^{t_i,\omega} = \alpha_t^{i,j,k}$ for $(t,\omega) \in [t_i, T] \times E_{i,j}^k$, and $\alpha_t^k = (\alpha^\delta)_t^k$ for all other (t,ω). Then, by (20) we have

$$\begin{aligned} &J_k\left(\tau_n(\omega), \omega, (\alpha^\delta)^{\tau_n(\omega),\omega}\right) > J_k\left(\tau_n(\omega), \omega, (\alpha^{\delta,-k})^{\tau_n(\omega),\omega}, (\alpha^k)^{\tau_n(\omega),\omega}\right) + \frac{\varepsilon_2}{2}, \quad \forall \omega \in E^k; \\ &J_k\left(\tau_n, \omega, (\alpha^\delta)^{\tau_n(\omega),\omega}\right) = J_k\left(\tau_n(\omega), \omega, (\alpha^{\delta,-k})^{\tau_n(\omega),\omega}, (\alpha^k)^{\tau_n(\omega),\omega}\right), \quad \forall \omega \notin E^k. \end{aligned}$$

Note that $\alpha_t^k = (\alpha^\delta)_t^k$ for $t \le \tau_n$. Then, because $\alpha^\delta \in NE_{2\delta}(0,0)$,

$$\begin{aligned} 2\delta &\ge J_k\left(0,0,\alpha^\delta\right) - J_k\left(0,0,\alpha^{\delta,-k},\alpha^k\right) \\ &= \mathbb{E}^{\mathbb{P}^{0,0,\alpha^\delta}}\left[J_k\left(\tau_n(\omega), \omega, (\alpha^\delta)^{\tau_n(\omega),\omega}\right) - J_k\left(\tau_n(\omega), \omega, (\alpha^{\delta,-k})^{\tau_n(\omega),\omega}, (\alpha^k)^{\tau_n(\omega),\omega}\right)\right] \\ &\ge \frac{\varepsilon_2}{2} \mathbb{P}^{0,0,\alpha^\delta}(E^k) = \frac{\varepsilon_2}{2} \mathbb{E}^{\mathbb{P}_0}\left[M_{\tau_n}^{0,0,\alpha^\delta} \mathbf{1}_{E^k}\right]. \end{aligned}$$

Thus, by (6),

$$\mathbb{P}_0(E^k) = \mathbb{E}^{\mathbb{P}_0}\left[\left(M_{\tau_n}^{0,0,\alpha^\delta}\right)^{-\frac{1}{2}} \left(M_{\tau_n}^{0,0,\alpha^\delta}\right)^{\frac{1}{2}} \mathbf{1}_{E^k}\right] \le C\left(\mathbb{E}^{\mathbb{P}_0}\left[M_{\tau_n}^{0,0,\alpha^\delta} \mathbf{1}_{E^k}\right]\right)^{\frac{1}{2}} \le C\sqrt{\frac{\delta}{\varepsilon_2}}.$$

Then, $\mathbb{P}_0(E_n^\delta) \le CN\sqrt{\delta/\varepsilon_2}$. This implies (18) and hence, (15) immediately.

Step 2. To see the opposite inclusion, we fix $y \in \tilde{\mathbb{V}}(0,0)$ and $\varepsilon > 0$. Let $\delta > 0$ be a small number that will be specified later. Because $y \in \tilde{V}_\delta(0,0)$, let η, α^δ be the corresponding terms in (12) corresponding to δ. Moreover, set $\delta_n \downarrow 0$ and let (τ_n, η_n) be the approximations of (τ, η) as in Lemma 1 (i) with error δ_n. Note that, for any $k = 1, \cdots, N$ and any $\alpha^k \in \mathcal{A}_k$,

$$\begin{aligned} &\left|J_k(\tau_n, \eta_n; 0, 0, \alpha^{\delta,-k}, \alpha^k) - J_k(\tau, \eta; 0, 0, \alpha^{\delta,-k}, \alpha^k)\right| \\ &= \left|\mathbb{E}^{\mathbb{P}_0}\left[M_{\tau_n}^{\alpha^{\delta,-k},\alpha^k}\left[\left[(\eta_n)_k - \eta_k\right] + \int_\tau^{\tau_n} f_k\left(s, B, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right]\right]\right| \le C\left(\mathbb{E}^{\mathbb{P}_0}\left[|\eta_n - \eta|\right]\right)^{\frac{1}{2}} + C2^{-n} \le \delta, \end{aligned} \tag{21}$$

when n is large enough. Thus

$$\begin{aligned} &J_k\left(\tau_n, \eta_n; 0, 0, \alpha^\delta\right) - J_k\left(\tau_n, \eta_n; 0, 0, \alpha^{\delta,-k}, \alpha^k\right) \\ &\le J_k\left(\tau, \eta; 0, 0, \alpha^\delta\right) - J_k\left(\tau, \eta; 0, 0, \alpha^{\delta,-k}, \alpha^k\right) + 2\delta \le 3\delta, \quad \forall \alpha^k \in \mathcal{A}_k. \end{aligned} \tag{22}$$

That is, $\alpha^\delta \in NE_{3\delta}(\tau_n, \eta_n; 0, 0)$ and $y \in O_{2\delta}(J(\tau_n, \eta_n; 0, 0, \alpha^\delta))$ for n large enough.

Next, by (8) and noting that $\hat{\mathbb{V}}_\delta$ is increasing in δ, we have

$$\overline{\lim_{n\to\infty}}\, \mathbb{P}_0\left(\eta \notin \hat{\mathbb{V}}_\delta(\tau_n, B.)\right) \le \mathbb{P}_0\left(\eta \notin \hat{\mathbb{V}}_\delta(\tau, B.)\right) \le \delta.$$

Note that $\{\eta \in \hat{\mathbb{V}}_\delta(\tau_n, B)\} \cap \{|\eta_n - \eta| \le \delta\} \subset \{\eta_n \in \hat{\mathbb{V}}_{2\delta}(\tau_n, B)\}$; then,

$$\overline{\lim_{n\to\infty}}\, \mathbb{P}_0\Big(\eta_n \notin \hat{\mathbb{V}}_{2\delta}(\tau_n, B.)\Big) \le \overline{\lim_{n\to\infty}}\left[\mathbb{P}_0\Big(\eta \notin \hat{\mathbb{V}}_\delta(\tau_n, B.)\Big) + \frac{1}{\delta}\mathbb{E}^{\mathbb{P}_0}\big[|\eta_n - \eta|\big]\right] \le \delta.$$

Thus, for n large enough,

$$\mathbb{P}_0\left(\left(E_n^\delta\right)^c\right) \le 2\delta, \quad \text{where} \quad E_n^\delta := \left\{\eta_n \in \hat{\mathbb{V}}_{2\delta}(\tau_n, B.)\right\}. \tag{23}$$

Now fix δ, n. Denote $E_i := E_n^\delta \cap \{\tau_n = t_i\}$. Let $\delta' > 0$ be another small number that will be specified later. Similar to Step 1, we have decomposition $E_i = \cup_{j\ge 1} E_{i,j}$ on $\mathcal{F}_{t_i}$ such that, for some fixed $\omega^{i,j} \in E_{i,j}$, $\sup_{0\le s\le t_i} |\omega_s - \omega_s^{i,j}| \le \delta'$ for all $\omega \in E_{i,j}$. Now, for each (i,j), because $\eta_n(\omega^{i,j}) \in \hat{\mathbb{V}}_{2\delta}(t_i, \omega^{i,j}) \subset \mathbb{V}_{2\delta}(t_i, \omega^{i,j})$, there exists $\alpha^{i,j} \in \mathcal{A}$ such that

$$\alpha^{i,j} \in NE_{2\delta}\,(t_i, \omega^{i,j})), \quad |\eta_n(\omega^{i,j}) - J(t_i, \omega^{i,j}, \alpha^{i,j})| \le 2\delta.$$

Because η_n and $J(t_i, \omega, \alpha^{i,j})$ are uniformly continuous in ω, for δ' small enough we have

$$|\eta_n(\omega) - \eta_n(\omega^{i,j})| \le \delta, \quad \sup_\alpha |J\,(t_i, \omega, \alpha) - J\,(t_i, \omega^{i,j}, \alpha)| \le \delta, \quad \forall\, \omega \in E_{i,j}.$$

Denote

$$\tilde{\eta}_n(\omega) := \sum_{i,j} \mathbf{1}_{E_{i,j}}(\omega) J\,\left(t_i, \omega, \alpha^{i,j}\right) + \mathbf{1}_{\left(E_n^\delta\right)^c}(\omega) J\,\left(\tau_n(\omega), \omega, (\alpha^\delta)^{\tau_n(\omega),\omega}\right). \tag{24}$$

Then,

$$\begin{aligned}
|\tilde{\eta}_n - \eta_n| &\le \sum_{I,j} \mathbf{1}_{E_{i,j}}(\omega) \mid J(t_i, \omega, \alpha^{i,j}) - \eta_n(\omega) \mid + C\mathbf{1}_{\left(E_n^\delta\right)^c} \\
&\le \sum_{i,j} \mathbf{1}_{E_{i,j}}(\omega) \Big[\Big|J\,\left(t_i, \omega, \alpha^{i,j}\right) - J\,\left(t_i, \omega^{i,j}, \alpha^{i,j}\right)\Big| + \Big|J\,\left(t_i, \omega^{i,j}, \alpha^{i,j}\right) - \eta_n(\omega^{i,j})\Big| \\
&\quad + \big|\eta_n(\omega^{i,j}) - \eta_n(\omega)\big|\Big] + C\mathbf{1}_{\left(E_n^\delta\right)^c} \le 4\delta + C\mathbf{1}_{\left(E_n^\delta\right)^c}.
\end{aligned}$$

Similar to (21) and (22), by (23), one can easily show that

$$\alpha^\delta \in NE_{C\sqrt{\delta}}(\tau_n, \tilde{\eta}_n; 0, 0) \quad \text{and} \quad y \in O_{C\sqrt{\delta}}\left(J\left(\tau_n, \tilde{\eta}_n; 0, 0, \alpha^\delta\right)\right). \tag{25}$$

We now define

$$\alpha_t^{\delta,n} := \alpha_t^\delta \mathbf{1}_{[0,\tau_n]}\,(t) + \mathbf{1}_{(\tau_n, T]}(t)\left[\sum_{i,j} \mathbf{1}_{E_{i,j}} \alpha_{t-\tau_n}^{i,j} + \alpha_t^\delta \mathbf{1}_{\left(E_n^\delta\right)^c}\right]. \tag{26}$$

Then, $\tilde{\eta}_n(\omega) = J(\tau_n(\omega), \omega, (\alpha^{\delta,n})^{\tau_n(\omega),\omega})$ for all $\omega \in \Omega$. For $k = 1, \cdots, N$ and for $\alpha^k \in \mathcal{A}_k$,

$$\begin{aligned}
J_k\,\left(0, 0, \alpha^{\delta,n}\right) &= J_k\,\left(\tau_n, \tilde{\eta}_n; 0, 0, \alpha^\delta\right) \le J_k\,\left(\tau_n, \tilde{\eta}_n; 0, 0, \alpha^{\delta,-k}, \alpha^k\right) + C\sqrt{\delta} \\
&= \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[\tilde{\eta}_n(\omega) + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&\le \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[\sum_{i,j} \mathbf{1}_{E_{i,j}} J_k\,\left(t_i, \omega, \alpha^{i,j}\right) + C\mathbf{1}_{\left(E_n^\delta\right)^c} + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&\le \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[\sum_{i,j} \mathbf{1}_{E_{i,j}} J_k\,\left(t_i, \omega^{i,j}, \alpha^{i,j}\right) + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&\le \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[\sum_{i,j} \mathbf{1}_{E_{i,j}} J_k\left(t_i, \omega^{i,j}, \alpha^{i,j,-k}, (\alpha^k)^{t_i,\omega}\right) + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&\le \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[\sum_{i,j} \mathbf{1}_{E_{i,j}} J_k\left(t_i, \omega, \alpha^{i,j,-k}, (\alpha^k)^{t_i,\omega}\right) + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&\le \mathbb{E}^{\mathbb{P}^{\alpha^{\delta,-k},\alpha^k}}\left[J_k\left(\tau_n(\omega), \omega, (\alpha^{\delta,n,-k}, \alpha^k)^{\tau_n(\omega),\omega}\right) + \int_0^{\tau_n} f_k\,\left(s, \omega, \alpha_s^{\delta,-k}, \alpha_s^k\right) ds\right] + C\sqrt{\delta} \\
&= J_k\,\left(0, 0, \alpha^{\delta,n,-k}, \alpha^k\right) + C\sqrt{\delta}.
\end{aligned}$$

That is, $\alpha^{\delta,n} \in NE_{C\sqrt{\delta}}(0,0)$, and $y \in O_{C\sqrt{\delta}}(J(\tau_n,\tilde{\eta}_n;0,0,\alpha^\delta)) = O_{C\sqrt{\delta}}(J(0,0,\alpha^{\delta,n}))$. Then, $y \in \mathbb{V}_{C\sqrt{\delta}}(0,0)$, and thus $y \in \mathbb{V}_\varepsilon(0,0)$ when $C\sqrt{\delta} \le \varepsilon$. Q.E.D.

Remark 12. In the state-dependent setting, namely

$$b = b(t,\omega_t,a), \quad f = f(t,\omega_t,a), \quad \xi = g(\omega_T), \tag{27}$$

as in Section 2.4, we can show that $\mathbb{V}(t,\omega) = \mathbb{V}(t,\omega_t)$ is also state dependent, but the DPP still involves path-dependent η and α^ε.

3.4. A Duality Result

In this subsection, we provide an alternative characterization for the set value $\mathbb{V}(t,\omega)$. The idea is similar to the level set or nodal set approach; see, for example, Barles et al. [3], Ma and Yong [27], and Karnam et al. [25]. In particular, this method could be efficient for numerical purposes.

We first note that, for any (t,ω) and $\alpha \in \mathcal{A}$, $J(t,\omega,\alpha) = Y_0^{t,\omega,\alpha}$, where $(Y^{t,\omega,\alpha}, Z^{t,\omega,\alpha})$ is the solution to the following (linear) BSDE on $[0,T-t]$:

$$\begin{gathered} Y_s^{t,\omega,\alpha,i} = \xi_i^{t,\omega}(B) + \int_s^{T-t} f_i^{t,\omega}\left(r,B,\alpha_r,Z_r^{t,\omega,\alpha,i}\right) dr - \int_s^{T-t} Z_r^{t,\omega,\alpha,i} dB_r, \\ \text{where} \quad f_i(t,\omega,a,z_i) := f_i(t,\omega,a_i) + b(t,\omega,a)z_i. \end{gathered} \tag{28}$$

For each i and $a^{-i} = (a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_N)$, denote

$$\underline{f}_i(t,\omega,a^{-i},z_i) := \inf_{a_i \in A_i} f_i(t,\omega,a^{-i},a_i,z_i).$$

Because b is bounded, $\underline{f}_i$ is uniformly Lipschitz continuous in z_i. Introduce the following multidimensional BSDE: $i = 1,\cdots,N$,

$$\underline{Y}_s^{t,\omega,\alpha,i} = \xi_i^{t,\omega}(B) + \int_s^{T-t} \underline{f}_i^{t,\omega}\left(r,B,\alpha_r^{-i},\underline{Z}_r^{t,\omega,\alpha,i}\right) dr - \int_s^{T-t} \underline{Z}_r^{t,\omega,\alpha,i} dB_r. \tag{29}$$

It is clear that (see, e.g., El-Karoui and Hamadene [13]) $\alpha^* \in NE(t,\omega)$ if and only if

$$\underline{f}_i^{t,\omega}\left(r,B,\alpha_r^{*,-i},\underline{Z}_r^{t,\omega,\alpha^*,i}\right) = f_i^{t,\omega}\left(r,B,\alpha_r^*,\underline{Z}_r^{t,\omega,\alpha^*,i}\right), \ a.s., 0 \le r \le T-t, \ i = 1,\cdots,N. \tag{30}$$

Our main idea of the duality approach is to rewrite (29) as a forward diffusion by viewing the component Z as a control. To be precise, fix $(t,\omega,y) \in [0,T] \times \Omega \times \mathbb{R}^N$. For any $\alpha \in \mathcal{A}$ and $Z = (Z^1,\cdots,Z^N)$, denote

$$Y_s^{t,\omega,y,\alpha,Z,i} := y_i - \int_0^s \underline{f}_i^{t,\omega}\left(r,B,\alpha_r^{-i},Z_r^i\right) dr + \int_0^s Z_r^i dB_r. \tag{31}$$

We then introduce an auxiliary control problem:

$$\begin{gathered} W(t,\omega,y) := \inf_{\alpha \in \mathcal{A},\, Z \in \mathbb{L}^2(\mathbb{F},\mathbb{P}_0)} \sum_{i=1}^N \mathbb{E}^{\mathbb{P}_0}\Bigg[|\xi_i^{t,\omega}(B) - Y_{T-t}^{t,\omega,y,\alpha,Z,i}|^2 \\ + \int_0^{T-t} \left[\Delta f_i^{t,\omega}\left(s,B,\alpha_s,Z_s^i\right)\right]^{\frac{3}{2}} ds\Bigg], \\ \text{where} \quad \Delta f_i(t,\omega,a,z_i) := f_i(t,\omega,a,z_i) - \underline{f}_i(t,\omega,a^{-i},z_i). \end{gathered} \tag{32}$$

Here, the power 3/2 (between 1 and 2) for the f-term is for some technical reasons on which we will elaborate later. By (29) and (30), it is obvious that $W(t,\omega,y) = 0$ for all $y \in \mathbb{V}_0(t,\omega)$.

Our main result of this subsection is that the set value agrees with the nodal set of W.

Theorem 6. *Let Assumptions 1 and 2 hold. Then, for any* (t,ω),

$$\mathbb{V}(t,\omega) = \mathbb{N}(t,\omega) := \left\{y \in \mathbb{R}^N : W(t,\omega,y) = 0\right\}.$$

Proof. Without loss of generality, we assume that $(t,\omega) = (0,0)$, and for notational simplicity we may omit $(0,0)$ when there is no confusion, for example $J(\alpha) := J(0,0,\alpha)$.

i. We first show that $\mathbb{N}(0,0) \subset \mathbb{V}(0,0)$. Fix $y \in \mathbb{N}(0,0)$. For any $\varepsilon > 0$, there exist α^ε and Z^ε such that, denoting $Y^\varepsilon := Y^{y,\alpha^\varepsilon,Z^\varepsilon}$,

$$\mathbb{E}^{\mathbb{P}_0}\left[|\xi_i - Y_T^{\varepsilon,i}|^2 + \int_0^T \left[\Delta f_i\left(s,B,\alpha_s^\varepsilon,Z_s^{\varepsilon,i}\right)\right]^{\frac{3}{2}} ds\right] \le \varepsilon^2, \ i \ge 1. \tag{33}$$

Let $(\tilde{Y}^{\varepsilon}, \tilde{Z}^{\varepsilon})$ solve the following BSDE:

$$\tilde{Y}_t^{\varepsilon,i} = \xi_i(B) + \int_t^T f_i\left(s, B, \alpha_s^{\varepsilon}, \tilde{Z}_s^{\varepsilon,i}\right) ds - \int_t^T \tilde{Z}_s^{\varepsilon,i} dB_s.$$

Note that

$$Y_t^{\varepsilon,i} = Y_T^{\varepsilon,i} + \int_t^T \underline{f}_i\left(s, B, \alpha_s^{\varepsilon,-i}, Z_s^{\varepsilon,i}\right) ds - \int_t^T Z_s^{\varepsilon,i} dB_s. \tag{34}$$

Then, denoting $\Delta Y^i := \tilde{Y}^{\varepsilon,i} - Y^{\varepsilon,i}$ and $\Delta Z^i := \tilde{Z}^{\varepsilon,i} - Z^{\varepsilon,i}$, we have

$$\Delta Y_t^i = \xi_i(B) - Y_T^{\varepsilon,i} + \int_t^T \Delta f_i\left(s, B, \alpha_s^{\varepsilon}, Z_s^{\varepsilon,i}\right) ds + \int_t^T b\left(s, B, \alpha_s^{\varepsilon}\right) \Delta Z_s^i ds - \int_t^T \Delta Z_s^i dB_r.$$

Thus, recalling (2) for M,

$$\Delta Y_0^i = \mathbb{E}^{\mathbb{P}_0}\left[M_T^{\alpha^{\varepsilon}}\left[\xi_i(B) - Y_T^{\varepsilon,i} + \int_0^T \Delta f_i\left(s, B, \alpha_s^{\varepsilon}, Z_s^{\varepsilon,i}\right) ds\right]\right].$$

By (6) and (33) (in particular, noting the power 3/2 for the f-term is greater than 1), it is clear that $|\tilde{Y}_0^{\varepsilon,i} - Y_0^{\varepsilon,i}| \le C\varepsilon$. Moreover, let $(\hat{Y}^{\varepsilon}, \hat{Z}^{\varepsilon})$ solve the following BSDE:

$$\hat{Y}_s^{\varepsilon,i} = \xi_i(B) + \int_s^T \underline{f}_i\left(s, B, \alpha_s^{\varepsilon,-i}, \hat{Z}_s^{\varepsilon,i}\right) dr - \int_s^T \hat{Z}_r^{\varepsilon,i} dB_r. \tag{35}$$

Comparing (34) and (35), it follows from (33) again that $|\tilde{Y}_0^{\varepsilon,i} - \hat{Y}_0^{\varepsilon,i}| \le C\varepsilon$, and thus $|\hat{Y}_0^{\varepsilon,i} - Y_0^{\varepsilon,i}| \le C\varepsilon$.

On the other hand, for any α^i, applying the comparison principle on BSDEs (28) and (35), we see that $J_i(\alpha^{\varepsilon,-i}, \alpha^i) \ge \hat{Y}_0^{\varepsilon,i}$. Then,

$$J_i(\alpha^{\varepsilon}) = Y_0^{\varepsilon,i} \le \hat{Y}_0^{\varepsilon,i} + C\varepsilon \le J_i(\alpha^{\varepsilon,-i}, \alpha^i) + C\varepsilon,$$

and thus $\alpha^{\varepsilon} \in NE_{C\varepsilon}(0,0)$. Recall $J(\alpha^{\varepsilon}) = Y_0^{\varepsilon} = y$; then, $y \in \mathbb{V}_{C\varepsilon}(0,0)$. Because ε is arbitrary, we obtain $y \in \mathbb{V}(0,0)$.

ii. We next show that $\mathbb{V}(0,0) \subset \mathbb{N}(0,0)$. Fix $y \in \mathbb{V}(0,0)$. For any $\varepsilon > 0$, there exists $\alpha^{\varepsilon} \in NE_{\varepsilon}(0,0)$ such that $|y - J(\alpha^{\varepsilon})| \le \varepsilon$. Recall that $J(\alpha^{\varepsilon}) = Y_0^{\alpha^{\varepsilon}}$, where $(Y^{\alpha^{\varepsilon}}, Z^{\alpha^{\varepsilon}})$ is defined by (28). Let $(\hat{Y}^{\varepsilon}, \hat{Z}^{\varepsilon})$ be defined by (35). For each i, there exists α^i such that

$$f_i\left(r, B, \alpha_r^{\varepsilon,-i}, \alpha_r^i, \hat{Z}_r^{\varepsilon,i}\right) \le \underline{f}_i\left(r, B, \alpha^{\varepsilon,-i}, \hat{Z}_r^{\varepsilon,i}\right) + \varepsilon. \tag{36}$$

Let $(\check{Y}^{\varepsilon,i}, \check{Z}^{\varepsilon,i})$ solve the following BSDE:

$$\check{Y}_s^{\varepsilon,i} = \xi_i(B) + \int_s^T f_i\left(r, B, \alpha_r^{\varepsilon,-i}, \alpha_r^i, \check{Z}_r^{\varepsilon,i}\right) dr - \int_s^T \check{Z}_r^{\varepsilon,i} dB_r. \tag{37}$$

Compare BSDEs (35) and (37), it follows from (36) that $\check{Y}_0^{\varepsilon,i} \le \hat{Y}_0^{\varepsilon,i} + C\varepsilon$. Moreover, because $\alpha^{\varepsilon} \in NE_{\varepsilon}(0,0)$, then $Y_0^{\alpha^{\varepsilon},i} \le \check{Y}_0^{\varepsilon,i} + \varepsilon \le \hat{Y}_0^{\varepsilon,i} + C\varepsilon$. By the comparison principle of BSDEs, we know that $Y_0^{\alpha^{\varepsilon},i} \ge \hat{Y}_0^{\varepsilon,i}$. Thus $|Y_0^{\alpha^{\varepsilon},i} - \hat{Y}_0^{\varepsilon,i}| \le C\varepsilon$. This, together with $|y - Y_0^{\alpha^{\varepsilon}}| \le \varepsilon$, implies that $|y - \hat{Y}_0^{\varepsilon}| \le C\varepsilon$.

Finally, note that

$$Y_T^{y,\alpha^{\varepsilon},\hat{Z}^{\varepsilon},i} - \xi_i(B) = Y_T^{y,\alpha^{\varepsilon},\hat{Z}^{\varepsilon},i} - Y_T^{\hat{Y}_0^{\varepsilon},\alpha^{\varepsilon},\hat{Z}^{\varepsilon},i} = y_i - \hat{Y}_0^{\varepsilon,i}. \tag{38}$$

Moreover, note that $\underline{f}_i$ is uniformly Lipschitz in z. Then, denoting $\Delta Z^i := Z^{\alpha^{\varepsilon},i} - \hat{Z}^{\varepsilon,i}$,

$$\begin{aligned}
C\varepsilon &\ge Y_0^{\alpha^{\varepsilon},i} - \hat{Y}_0^{\varepsilon,i} \\
&= \int_0^T \left[f_i\left(s, B, \alpha_s^{\varepsilon}, Z_s^{\alpha^{\varepsilon},i}\right) - \underline{f}_i\left(s, B, \alpha_s^{\varepsilon,-i}, \hat{Z}_s^{\varepsilon,i}\right)\right] ds - \int_0^T \Delta Z_s^i dB_s \\
&= \int_0^T \Delta f_i\left(s, B, \alpha_s^{\varepsilon}, \hat{Z}_s^{\varepsilon,i}\right) ds + \int_0^T b\left(s, B, \alpha_s^{\varepsilon}\right) \Delta Z_s^i ds - \int_0^T \Delta Z_s^i dB_s.
\end{aligned}$$

This implies that

$$\mathbb{E}^{\mathbb{P}_0}\left[M_T^{\alpha^\varepsilon}\int_0^T \Delta f_i\left(s,B,\alpha_s^\varepsilon,\hat{Z}_s^{\varepsilon,i}\right)ds\right] \le C\varepsilon. \tag{39}$$

Because ξ and f are bounded, by standard BSDE estimates we have $\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T |\hat{Z}_s^{\varepsilon,i}|^2 ds\right] \le C$. Note further that

$$0 \le \Delta f_i\,(t,\omega,a,z) \le C\,[1+|z|].$$

One can easily derive from (6) and (39) that (thanks to the fact that $3/2 < 2$)

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T \left[\Delta f_i\left(s,\,B,\,\alpha_s^\varepsilon,\,\hat{Z}_s^{\varepsilon,\,i}\right)\right]^{\frac{3}{2}} ds\right]\\
&\le C\mathbb{E}^{\mathbb{P}_0}\left[\left(M_T^{\alpha^\varepsilon}\right)^{-\frac{1}{4}}\left(M_T^{\alpha^\varepsilon}\right)^{\frac{1}{4}}\int_0^T \left[\Delta f_i\left(s,\,B,\,\alpha_s^\varepsilon,\,\hat{Z}_s^{\varepsilon,\,i}\right)\right]^{\frac{1}{4}} ds \int_0^T \left[1+|\hat{Z}_s^{\varepsilon,\,i}|^{\frac{5}{4}}\right]\right]\\
&\le C\left(\mathbb{E}^{\mathbb{P}_0}\left[\left(M_T^{\alpha^\varepsilon}\right)^{-2}\right]\right)^{\frac{1}{8}}\left(\mathbb{E}^{\mathbb{P}_0}\left[M_T^{\alpha^\varepsilon}\int_0^T \Delta f_i\left(s,\,B,\,\alpha_s^\varepsilon,\,\hat{Z}_s^{\varepsilon,i}\right)ds\right]\right)^{\frac{1}{4}}\left(\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T\left[1+|\hat{Z}_s^{\varepsilon,i}|2\right]\right]\right)^{\frac{5}{8}}\\
&\le C\varepsilon^{\frac{1}{4}}.
\end{aligned}$$

This, together with (38), implies that

$$\mathbb{E}^{\mathbb{P}_0}\left[|\xi_i(B) - Y_T^{y,\,\alpha^\varepsilon,\,\hat{Z}^\varepsilon,\,i}|^2 + \int_0^T \left[\Delta f_i\left(s,\,B,\,\alpha_s^\varepsilon,\,\hat{Z}_s^{\varepsilon,\,i}\right)\right]^{\frac{3}{2}} ds\right] \le C\varepsilon^{\frac{1}{4}}.$$

Then, by (32) we have $W(0,0,y) \le CN\varepsilon^{\frac{1}{4}}$. Because ε is arbitrary, we get $W(0,0,y) = 0$, that is, $y \in \mathbb{N}(0,0)$. Q.E.D.

Note that (32) is a standard path-dependent control problem. Following Section 11.3.3 in Zhang [40], we have the following result, whose proof is omitted.

Proposition 5. *Let Assumptions 1 and 2 hold. Then, $W \in C([0,T \times \Omega \times \mathbb{R}^N)$ is a viscosity solution of the following path-dependent PDE:*

$$\begin{aligned}
\partial_t W + \inf_{a\in A,\, z\in\mathbb{R}^{N\times d}}&\left[\frac{1}{2}\mathrm{tr}\left(\partial^2_{\omega\omega}W\right) + \frac{1}{2}\mathrm{tr}\left(z^\top \partial^2_{yy}Wz\right) + \mathrm{tr}\left(z^\top\partial_{y\omega}W\right)\right.\\
&\left.+\sum_{i=1}^N\left[\left[\Delta f_i\;(t,\omega,a,z_i)\right]^{\frac{3}{2}} - \underline{f}_i\;(t,\omega,a^{-i},z_i)\;\partial_{y_i}W\right]\right] = 0;\\
&W(T,\omega,y) = |\xi(\omega) - y|^2.
\end{aligned} \tag{40}$$

Remark 13.

i. The path derivatives $\partial_\omega W, \partial^2_{\omega\omega}W$ are introduced by Dupire [10], and we refer to section 9.4 in Zhang [40] for more details. Note that this path-dependent PDE is always degenerate, and the control is unbounded, so the uniqueness of viscosity solution is not completely covered by Ekren et al. [11, 12] and Ren et al. [35]. This problem is in general challenging and is left for future research.

ii. In the state-dependent case as in Remark 12, $W = W(t,x,y)$ also becomes state dependent, and the path-dependent PDE 40 reduces to a standard HJB equation:

$$\begin{aligned}
\partial_t W + \inf_{a\in A,\, z\in\mathbb{R}^{N\times d}}\left[\frac{1}{2}\mathrm{tr}\left(\partial^2_{xx}W\right) + \frac{1}{2}\mathrm{tr}\left(z^\top\partial^2_{yy}Wz\right) + \mathrm{tr}\left(z^\top\partial_{yx}W\right) + \sum_{i=1}^N\left[\left[\Delta f_i(t,x,a,z_i)\right]^{\frac{3}{2}} - \underline{f}_i(t,x,a^{-i},z_i)\partial_{y_i}W\right]\right] = 0;\\
W(T,x,y) = |g(x) - y|^2.
\end{aligned}$$

This PDE is also degenerate and with unbounded controls, though.

iii. In light of Theorem 6, PPDE 40, especially PDE (41) in the state-dependent case, is quite useful for numerical computation of the set value $\mathbb{V}(t,\omega)$.

Remark 14. Roughly speaking (modulus the existence of optimal controls in (32)), y is in the nodal set $N(t,\omega)$ if and only if there exists α, Z such that $Y^{t,\omega,y,\alpha,Z}$ in (31) hits the target $\xi^{t,\omega}(B)$ at $T-t$. This is in the spirit of Cardaliaguet et al. [9]. However, we note that Cardaliaguet et al. [9] use strategy versus controls, whereas we use closed-loop controls for all players.

Remark 15. In this remark, we make a further connection between the game and BSDEs.

i. In the literature, one may indeed use (30) to find equilibriums, especially in the state-dependent setting (27); see, for example, Hamadene et al. [23], Hamadene and Mu [21, 22], and Espinosa and Youzi [14]. To be precise, assume there exists a measurable function $\varphi : [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d)^N \to A$ such that, for $i = 1, \cdots, N$,

$$\underline{f}_i\left(t,x,\varphi^{-i}(t,x,z),z^i\right) = f_i\left(t,x,\varphi\,(t,x,z),z^i\right), \tag{41}$$

and the following BSDEs have a strong solution (setting $(t,x) = (0,0)$ for simplicity):

$$\underline{Y}^i_s = g_i(B_T) + \int_s^T \underline{f}_i\left(r,B_r,\varphi^{-i}(r,B_r,\underline{Z}_r),\underline{Z}^i_r\right)dr - \int_s^T \underline{Z}^i_r dB_r, \tag{42}$$

and then $\alpha^*_t := \varphi(t,B_t,\underline{Z}_t)$ is a Nash equilibrium at $(0,0)$. However, we should note that the function φ, assuming its existence, may not be continuous, and thus the well-posedness of (42) may not be easy. Even worse, in order to obtain the whole set $\mathbb{V}_0(0,0)$, as we noted before, we need to consider path-dependent $\varphi : [0,T] \times \Omega \times (\mathbb{R}^d)^N \to A$, which will make the well-posedness of (42) even harder. Nevertheless, by (30), it is true that the set $\mathbb{V}_0$ can be constructed by first finding all path-dependent functions φ satisfying (41) and then finding all strong solutions of the multidimensional BSDE (42), where both (41) and (42) should be extended to the path-dependent setting.

ii. One may replace the linear BSDE (28) with nonlinear BSDEs:

$$Y^{t,\omega,\alpha,i}_s = \xi^{t,\omega}_i(B) + \int_s^{T-t} f^{t,\omega}_i\left(r,B,\alpha_r,Y^{t,\omega,\alpha,i}_r,Z^{t,\omega,\alpha,i}_r\right)dr - \int_s^{T-t} Z^{t,\omega,\alpha,i}_r dB_r,$$

where $f_i : [0,T] \times \Omega \times A \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is nonlinear in (y,z). Still, define $J(t,\omega,\alpha) := Y^{t,\omega,\alpha}_0$, and then one can show without significant difficulties that all the results in this section hold true after obvious modifications.

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Endnote

[1] More rigorously, we should first get an $\mathcal{F}^B_{t_i}$-measurable set $E_i \subset \{\tau_\delta = t_i\}$ with $\mathbb{P}_0(\{\tau_\delta = t_i\}\backslash E_i) = 0$ and then apply (5) to obtain the desired $K_i \subset\subset E_i$.

References

[1] Abreu D, Pearce D, Stacchetti E (1990) Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica* 58(5):1041–1063.

[2] Aubin JP, Frankowska H (2009) *Set-Valued Analysis* (Springer Science & Business Media, Berlin).

[3] Barles G, Soner H, Souganidis P (1993) Front propagation and phase field theory. *SIAM J. Control Optim.* 31(2):439–469.

[4] Bensoussan A, Frehse J (2000) Stochastic games for *n* players. *J. Optim. Theory Appl.* 105:543–565.

[5] Billingsley P (1999) *Convergence of Probability Measures*, 2nd ed. (Wiley, New York).

[6] Buckdahn R, Cardaliaguet P, Rainer C (2004) Nash equilibrium payoffs for nonzero-sum stochastic differential games. *SIAM J. Control Optim.* 43(2):624–642.

[7] Buckdahn R, Li J, Quincampoix M (2014) Value in mixed strategies for zero-sum stochastic differential games without isaacs condition. *Ann. Probab.* 42(4):1724–1768.

[8] Cardaliaguet P, Plaskacz S (2003) Existence and uniqueness of a nash equilibrium feedback for a simple nonzero-sum differential game. *Internat. J. Game Theory.* 32:33–71.

[9] Cardaliaguet P, Quincampoix M, Saint-Pierre P (1999) *Set-Valued Numerical Analysis for Optimal Control and Differential Games* (Birkhäuser, Boston), 177–247.

[10] Dupire B (2019) Functional itô calculus. *Quant. Finance* 19(5):721–729.

[11] Ekren I, Touzi N, Zhang J (2016) Viscosity solutions of fully nonlinear parabolic path dependent pdes: Part i. *Ann. Probab.* 44(2):1212–1253.

[12] Ekren I, Touzi N, Zhang J (2016) Viscosity solutions of fully nonlinear parabolic path dependent pdes: Part ii. *Ann. Probab.* 44(4):2507–2553.

[13] El-Karoui N, Hamadene S (2003) Bsdes and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Process. Appl.* 107(1):145–169.

[14] Espinosa GE, Touzi N (2015) Optimal investment under relative performance concerns. *Math. Finance* 25(2):221–257.

[15] Feinstein Z (2020) Continuity and sensitivity analysis of parameterized nash games. Preprint, submitted July 10, https://arxiv.org/pdf/200704388.pdf.
[16] Feinstein Z, Rudloff B (2019) Time consistency for scalar multivariate risk measures Preprint, submitted January 8, https://arxiv.org/pdf/1810.04978.pdf.
[17] Frei C, dos Reis G (2011) A financial market with interacting investors: Does an equilibrium exist? *Math. Financial Econom.* 4:161–182.
[18] Friedman A (1972) Stochastic differential games. *J. Differential Equations* 11:79–108.
[19] Hamadene S (1999) Nonzero sum linear-quadratic stochastic differential games and backward-forward equations. *Stoch. Anal. Appl.* 17: 117–130.
[20] Hamadene S, Mannucci P (2019) Regularity of Nash payoffs of Markovian nonzero-sum stochastic differential games. *Stochastics* 91(5): 695–715.
[21] Hamadene S, Mu R (2014) Bangbang-type nash equilibrium point for markovian nonzero-sum stochastic differential game. *C. R. Math. Acad. Sci. Paris* 352(9):699–706.
[22] Hamadene S, Mu R (2015) Existence of nash equilibrium points for markovian non-zero-sum stochastic differential games with unbounded coefficients. *Stochastics* 87(1):85–111.
[23] Hamadene S, Lepeltier JP, Peng S (1997) BSDEs with continuous coefficients and stochastic differential games. El Karoui N, Mazliak L, eds. *Pittman Research Notes in Mathematics Series* (Longman, UK) 115–128.
[24] Ho K, Rosen AM (2017) *Partial Identification in Applied Research: Benefits and Challenges*, Economic Society of Monographs, vol. 2 (Cambridge University Press, Cambridge, UK), 307–359.
[25] Karnam C, Ma J, Zhang J (2017) Dynamic approaches for some time inconsistent problems. *Ann. Appl. Probab.* 27(6):3435–3477.
[26] Lin Q (2012) A BSDE approach to nash equilibrium payoffs for stochastic differential games with nonlinear cost functionals. *Stochastic Process. Appl.* 122(1):357–385.
[27] Ma J, Yong J (1995) Solvability of forward-backward SDEs and the nodal set of hamilton-jacobi-bellman equations. *Chin. Ann. Math. Ser. B.* 16:279–298.
[28] Mannucci P (2004) Nonzero-sum stochastic differential games with discontinuous feedback. *SIAM J. Control Optim.* 43(4):1222–1233.
[29] Mannucci P (2014) Nash points for nonzero-sum stochastic differential games with separate hamiltonians. *Dyn. Games Appl.* 4:329–344.
[30] Mertens J, Sorin S, Zamir S (2015) *Repeated Games (Econometric Society Monographs).* (Cambridge University Press, Cambridge, UK).
[31] Olsder GJ (2001) On open- and closed-loop bang-bang control in nonzero-sum differential games. *SIAM J. Control Optim.* 40(4):1087–1106.
[32] Pham T, Zhang J (2004) Two person zero–sum game in weak formulation and path dependent bellman-isaacs equation. *SIAM J. Control Optim.* 52(4):2090–2121.
[33] Possamai D, Touzi N, Zhang J (2020) Zero-sum path-dependent stochastic differential games in weak formulation. *Ann. Appl. Probab.* 30(4):1415–1457.
[34] Rainer C (2007) Two different approaches to nonzero-sum stochastic differential games. *Appl. Math. Optim.* 56:131–144.
[35] Ren Z, Touzi N, Zhang J (2017) Comparison of viscosity solutions of fully nonlinear degenerate parabolic path-dependent PDEs. *SIAM J. Math. Anal.* 49(5):4093–4116.
[36] Sannikov Y (2007) Games with imperfectly observable actions in continuous time. *Econometrica* 75(5):1285–1329.
[37] Sun J, Yong J (2019) Linear-quadratic stochastic two-person nonzero-sum differential games: open-loop and closed-loop nash equilibria. *Stochastic Process. Appl.* 129(2):381–418.
[38] Uchida K (1978) On existence of a nash equilibrium point in n-person nonzero sum stochastic differential games. *SIAM J. Control Optim.* 16(1):142–149.
[39] Wu Z (2005) Forward-backward stochastic differential equations, linear quadratic stochastic optimal control and nonzero sum differential games. *J. Systems Sci. Complexity* 18(2):179–192.
[40] Zhang J (2017) *Backward Stochastic Differential Equations — From Linear to Fully Nonlinear Theory* (Springer, New York).