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THE WELLPOSEDNESS OF FBSDES

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Abstract. In this paper we investigate the wellposedness of a class of Forward-Backward SDEs. Compared to the existing methods in the literature, our result has the following features: (i) arbitrary time duration; (ii) random coefficients; (iii) (possibly) degenerate forward diffusion; and (iv) no monotonicity condition. As a trade off, we impose some assumptions on the derivatives of the coefficients. A comparison theorem is also proved under the same conditions. This work is motivated by studying numerical methods for FBSDEs.

1. Introduction.

In this paper we study the wellposedness of the following Forward Backward SDE (FBSDE):

$$\begin{cases} X_t = x + \int_0^t b(\omega, s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(\omega, s, X_s, Y_s, Z_s)dW_s; \\ Y_t = g(\omega, X_T) + \int_t^T f(\omega, s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s; \end{cases} \tag{1}$$

where W is a standard Brownian Motion. The first seminal work on BSDE theory is [15]. Since then BSDEs and FBSDEs have been extensively studied and their applications have been found in many areas, including finance and stochastic control. In particular, [16] studies decoupled FBSDEs where b and σ do not involve Y or Z. We refer the readers to [6] and [7] for more details on BSDE theories, and [13] for FBSDEs .

There are mainly three approaches for the wellposedness of FBSDEs in the literature, each of which has its constraints. The first one is to use the fixed point theorem. This method works very well for BSDEs (and decoupled FBSDEs), but for FBSDEs one has to assume that T is small enough (see, e.g. [1]) or to assume some monotonicity conditions (see, e.g. [17]). The second one is the four step scheme [13], which allows T to be arbitrary large, but requires the coefficients to be deterministic and σ to be nondegenerate. To be precise, the authors assume that σ is independent of z and relate the FBSDE (1) to the following PDE

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(t,x,u)u_{xx} + b(t,x,u,\sigma(t,x,u)u_x)u_x + f(t,x,u,\sigma(t,x,u)u_x) = 0; \\ u(T,x) = g(x); \end{cases}$$

in the sense that

$$Y_t = u(t, X_t); \quad Z_t = \sigma(t, X_t, u(t, X_t))u_x(t, X_t).$$

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[5] also follows this line. The third one is the method of continuation, see e.g. [10], [18] and [20]. This method allows T to be large and the coefficients to be random, however, it requires some monotonicity conditions. For example, [10] assumes that, for some constant $\beta > 0$ and for any $\theta_i = (x_i, y_i, z_i), i = 1, 2$,

$$
\begin{aligned}
&[b(t,\theta_1) - b(t,\theta_2)]\Delta y + [\sigma(t,\theta_1) - \sigma(t,\theta_2)]\Delta z - [f(t,\theta_1) - f(t,\theta_2)]\Delta x \\
&\qquad \geq \beta[|\Delta x|^2 + |\Delta y|^2 + |\Delta z|^2]; \\
&[g(x_1) - g(x_2)]\Delta x \leq -\beta|\Delta x|^2.
\end{aligned}
$$

We would also like to mention that there are some works on linear FBSDEs (e.g. [21] and [22]) and some works on FBSDEs with nonsmooth coefficients (e.g. [2], [8], [9] and [14]).

Despite all these efforts, the FBSDE theories are far from complete. In this paper we provide a different approach. This work is motivated by our study of numerical methods for some FBSDEs, in which we need the wellposedness of a linear FBSDE with random coefficients. It turns out that none of the existing methods works in our case. We think our new result is interesting in its own right and is potentially useful in more applications, so we decide to publish it separately.

Our main idea is to obtain some uniform estimates of the solution to the FBSDE over small time interval, and then prove by induction that the time interval can be extended piece by piece while still keeping that estimate. Such an idea was also used by Delarue [5]. While [5] relies heavily on PDE arguments (so its coefficients have to be deterministic), we use purely probabilistic arguments. As a trade off, we need to impose a key compatibility condition (3).

Another main result of this paper is a comparison theorem. Unlike the BSDE case, in general comparison theorem does not hold true for FBSDEs. There are some positive results along this line (see, e.g. [3] and [19]). We prove the comparison result under the same conditions as for the wellposedness result.

The rest of the paper is organized as follows. In next section we state the main theorems. In §3 we study the small time duration case, in particular we obtain the key estimate in Lemma 2. In §4 we prove the wellposedness result and in §5 we prove the comparison theorem.

2. **Main Theorems.**

Assume $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}_0 \subset \mathcal{F}$ and W is a Brownian motion independent of $\mathcal{F}_0$. Let $\mathbf{F} \stackrel{\triangle}{=} \{\mathcal{F}_t\}_{0\leq t\leq T}$ be the filtration generated by W and $\mathcal{F}_0$, augmented by the null sets as usual. We study the following FBSDE:

$$
\begin{cases}
X_t = X_0 + \displaystyle\int_0^t b(\omega, s, \Theta_s)ds + \int_0^t \sigma(\omega, s, X_s, Y_s)dW_s; \\
Y_t = g(\omega, X_T) + \displaystyle\int_t^T f(\omega, s, \Theta_s)ds - \int_t^T Z_s dW_s.
\end{cases} \tag{2}
$$

where $\Theta \stackrel{\triangle}{=} (X, Y, Z)$, $X_0 \in \mathcal{F}_0$ and b, σ, f, g are progressively measurable. Moreover, for any $\theta \stackrel{\triangle}{=} (x, y, z)$, b, σ, f are $\mathbf{F}$-adapted and $g(\cdot, x) \in \mathcal{F}_T$. For technical reasons in this paper we assume all processes are 1-dimensional, and for notational simplicity we will always omit the variable ω. Some more general results are proved in an accompanying paper [23].

Our main result is the following theorem.

Theorem 1. *Assume that b, σ, f, g are differentiable with respect to x, y, z with uniformly bounded derivatives; and that*

$$\sigma_y b_z = 0; \quad b_y + \sigma_x b_z + \sigma_y f_z = 0. \tag{3}$$

Assume further that

$$I_0^2 \stackrel{\triangle}{=} E\Big\{|X_0|^2 + |g(0)|^2 + \int_0^T [|b(t,0,0,0)|^2 + |\sigma(t,0,0)|^2 + |f(t,0,0,0)|^2]dt\Big\} < \infty.$$

Then FBSDE (2) has a unique solution Θ such that

$$\|\Theta\|^2 \stackrel{\triangle}{=} E\Big\{\sup_{0\le t\le T}[|X_t|^2 + |Y_t|^2] + \int_0^T |Z_t|^2 dt\Big\} \le C I_0^2. \tag{4}$$

Remark 1. (3) can be replaced by the following stronger condition

$$\|\sigma_y\|_\infty \|b_z\|_\infty = 0; \quad \|b_y\|_\infty + \|\sigma_x\|_\infty \|b_z\|_\infty + \|\sigma_y\|_\infty \|f_z\|_\infty = 0. \tag{5}$$

Remark 2. The following three classes of FBSDEs satisfy condition (5) (and (3)):

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases}$$

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s, Z_s)ds + \int_0^t \sigma(s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \end{cases}$$

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s)ds - \int_t^T Z_s dW_s. \end{cases}$$

Also, instead of differentiability, it suffices to assume uniform Lipschitz continuity in these cases.

Another main result of this paper is the following comparison theorem.

Theorem 2. *Consider the following two FBSDEs:*

$$\begin{cases} X_t^i = X_0 + \int_0^t b(s, \Theta_s^i)ds + \int_0^t \sigma(s, X_s^i, Y_s^i)dW_s; \\ Y_t^i = g^i(X_T^i) + \int_t^T f^i(s, \Theta_s^i)ds - \int_t^T Z_s^i dW_s; \end{cases} \qquad i = 0, 1.$$

Assume

(i) $(b, \sigma, f^i, g^i), i = 0, 1$ satisfy all the conditions in Theorem 1;

(ii) for any (ω, t, θ), $g^0(x) \le g^1(x)$ and $f^0(t,\theta) \le f^1(t,\theta)$.

Then $Y_0^0 \le Y_0^1$.

3. **Small Time Duration.** It is well known that (2) has a unique solution when T is small. For any function φ, let L_φ denote its Lipschitz constant in (x, y, z).

Lemma 1. *Assume $L_b, L_\sigma, L_f \le K$ and $L_g \le K_0$. Then there exist δ_0, C_0 and C_1, depending on K and K_0, such that for $T \le \delta_0$, if $I_0^2 < \infty$, then (2) has a unique solution and it holds that*

$$\|\Theta\| \le C_0 I_0.$$

Moreover, the following estimate holds true:

$$\begin{aligned} & E\Big\{\sup_{0\le t\le T}[|X_t|^4 + |Y_t|^4] + (\int_0^T |Z_t|^2 dt)^2\Big\} \\ \le\ & C_1 E\Big\{|X_0|^4 + |g(0)|^4 + \int_0^T [|b(t,0,0,0)|^4 + |\sigma(t,0,0)|^4 + |f(t,0,0,0)|^4]dt\Big\}; \end{aligned}$$

given the right side at above is finite.

Proof. The existence and uniqueness as well as the first estimate are due to Antonelli [1]. The second estimate is standard (see, e.g. [5]). □

The following result is the key step of this paper.

Lemma 2. *Consider the following linear FBSDE:*

$$\left\{\begin{aligned} X_t &= 1 + \int_0^t [\alpha_s^1 X_s + \beta_s^1 Y_s + \gamma_s^1 Z_s]ds + \int_0^t [\alpha_s^2 X_s + \beta_s^2 Y_s]dW_s; \\ Y_t &= GX_T + \int_t^T [\alpha_s^3 X_s + \beta_s^3 Y_s + \gamma_s^3 Z_s]ds - \int_t^T Z_s dW_s. \end{aligned}\right. \tag{6}$$

Assume $|\alpha_t^i|, |\beta_t^i|, |\gamma_t^i| \le K, i = 1,2,3$ and $|G| \le K_0$. Let δ_0 be as in Lemma 1. Assume further that

$$\beta_t^2\gamma_t^1 = 0; \quad \beta_t^1 + \alpha_t^2\gamma_t^1 + \beta_t^2\gamma_t^3 = 0. \tag{7}$$

Then for $T \le \delta_0$, (6) has a unique solution (X, Y, Z) such that $|Y_t| \le \bar{K}_0|X_t|$, where

$$\bar{K}_0 \stackrel{\triangle}{=} [K_0 + 1]e^{(2K+K^2)T} - 1. \tag{8}$$

In particular, $|Y_0| \le \bar{K}_0$.

Proof. First by Lemma 1, (6) has a unique solution. For any $t \in [0, T)$ and any $\xi \in L^\infty(\mathcal{F}_t)$, denote $\hat{\Theta}_s \stackrel{\triangle}{=} \Theta_s \xi, s \in [t, T]$. Then $\hat{\Theta}$ satisfies the following FBSDE

$$\left\{\begin{aligned} \hat{X}_s &= X_t\xi + \int_t^s [\alpha_r^1 \hat{X}_r + \beta_r^1 \hat{Y}_r + \gamma_r^1 \hat{Z}_r]dr + \int_t^s [\alpha_r^2 \hat{X}_r + \beta_r^2 \hat{Y}_r]dW_r; \\ \hat{Y}_s &= G\hat{X}_T + \int_s^T [\alpha_r^3 \hat{X}_r + \beta_r^3 \hat{Y}_r + \gamma_r^3 \hat{Z}_r]dr - \int_s^T \hat{Z}_r dW_r. \end{aligned}\right.$$

By Lemma 1 again, we have

$$E\{|Y_t\xi|^2\} = E\{|\hat{Y}_t|^2\} \le C_0^2 E\{|X_t\xi|^2\}.$$

Since ξ is arbitrary, we have $|Y_t| \le C_0|X_t|, P$-a.s., $\forall t$. Moreover, both X and Y are continuous, thus

$$|Y_t| \le C_0|X_t|, \forall t, P - a.s.$$

Denote

$$\tau \stackrel{\triangle}{=} \inf\{t > 0 : X_t = 0\} \wedge T; \quad \tau_n \stackrel{\triangle}{=} \inf\{t > 0 : X_t = \frac{1}{n}\} \wedge T.$$

Then $\tau_n \uparrow \tau$ and $X_t > 0$ for $t \in [0, \tau)$. Denote

$$\tilde{Y}_t = Y_t[X_t]^{-1}; \quad \tilde{Z}_t \stackrel{\triangle}{=} Z_t[X_t]^{-1} - \tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]; \quad t \in [0, \tau).$$

Then $|\tilde{Y}_t| \le C_0$ and

$$\begin{aligned}
d\tilde{Y}_t &= [X_t]^{-1}dY_t - Y_t[X_t]^{-2}dX_t - [X_t]^{-2}d < X, Y >_t + Y_t[X_t]^{-3}d < X >_t \\
&= Z_t[X_t]^{-1}dW_t - [\alpha_t^3 + \beta_t^3\tilde{Y}_t + \gamma_t^3 Z_t[X_t]^{-1}]dt \\
&\quad -\tilde{Y}_t[\alpha_t^1 + \beta_t^1\tilde{Y}_t + \gamma_t^1 Z_t[X_t]^{-1}]dt \\
&\quad -\tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]dW_t - [\alpha_t^2 + \beta_t^2\tilde{Y}_t]Z_t[X_t]^{-1}dt + \tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]^2 dt \\
&= \tilde{Z}_t dW_t - \Big[\gamma_t^3 + \gamma_t^1\tilde{Y}_t + \alpha_t^2 + \beta_t^2\tilde{Y}_t\Big] Z_t[X_t]^{-1}dt \\
&\quad -\Big[\alpha_t^3 + \beta_t^3\tilde{Y}_t + \tilde{Y}_t[\alpha_t^1 + \beta_t^1\tilde{Y}_t] - \tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]^2\Big]dt \\
&= \tilde{Z}_t dW_t - \Big[\gamma_t^3 + \gamma_t^1\tilde{Y}_t + \alpha_t^2 + \beta_t^2\tilde{Y}_t\Big]\tilde{Z}_t dt \\
&\quad -\Big[\gamma_t^3 + \gamma_t^1\tilde{Y}_t + \alpha_t^2 + \beta_t^2\tilde{Y}_t\Big]\tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]dt \\
&\quad -\Big[\alpha_t^3 + \beta_t^3\tilde{Y}_t + \tilde{Y}_t[\alpha_t^1 + \beta_t^1\tilde{Y}_t] - \tilde{Y}_t[\alpha_t^2 + \beta_t^2\tilde{Y}_t]^2\Big]dt \\
&= \tilde{Z}_t dW_t - \Big[\gamma_t^3 + \gamma_t^1\tilde{Y}_t + \alpha_t^2 + \beta_t^2\tilde{Y}_t\Big]\tilde{Z}_t dt \\
&\quad -\Big[\beta_t^2\gamma_t^1\tilde{Y}_t^3 + [\beta^1 + \alpha^2\gamma^1 + \beta^2\gamma^3]\tilde{Y}_t^2 + [\alpha_t^1 + \beta_t^3 + \alpha_t^2\gamma_t^3]\tilde{Y}_t + \alpha_t^3\Big]dt \\
&= \tilde{Z}_t dW_t - \Big[[\gamma_t^3 + \alpha_t^2] + [\gamma_t^1 + \beta_t^2]\tilde{Y}_t\Big]\tilde{Z}_t dt - \Big[[\alpha_t^1 + \beta_t^3 + \alpha_t^2\gamma_t^3]\tilde{Y}_t + \alpha_t^3\Big]dt,
\end{aligned}$$

thanks to (7).

For each n, we have

$$\tilde{Y}_0 = \tilde{Y}_{\tau_n} - \int_0^{\tau_n}\tilde{Z}_s dW_t + \int_0^{\tau_n}\Big[\Big[[\gamma_t^3+\alpha_t^2]+[\gamma_t^1+\beta_t^2]\tilde{Y}_t\Big]\tilde{Z}_t + \Big[[\alpha_t^1+\beta_t^3+\alpha_t^2\gamma_t^3]\tilde{Y}_t+\alpha_t^3\Big]\Big]dt.$$

Denote

$$M_t = 1 + \int_0^t M_s\Big[[\gamma_s^3 + \alpha_s^2] + [\gamma_s^1 + \beta_s^2]\tilde{Y}_s\Big]1_{\{\tau > s\}}dW_s;$$

$$\Gamma_t = 1 + \int_0^t \Gamma_s[\alpha_s^1 + \beta_s^3 + \alpha_s^2\gamma_s^3]1_{\{\tau > s\}}ds.$$

Then

$$d(\Gamma_t M_t \tilde{Y}_t) = (\cdots)dW_t - \Gamma_t M_t \alpha_t^3 1_{\{\tau > t\}}dt,$$

and thus,

$$\tilde{Y}_0 = E\Big\{\Gamma_{\tau_n}M_{\tau_n}\tilde{Y}_{\tau_n} + \int_0^{\tau_n}\Gamma_t M_t \alpha_t^3 dt\Big\}. \tag{9}$$

Since $|\tilde{Y}_t| \le C_0$, M is a martingale and $|\Gamma_t| \le e^{(2K+K^2)t}$. Moreover, if $\tau = T$, then $|Y_\tau| = |Y_T| = |GX_T| = |GX_\tau| \le K_0|X_\tau|$. If $\tau < T$, then $X_\tau = 0$, and thus $|Y_\tau| \le C_0|X_\tau| = 0$. Therefore, in both cases it holds that $|Y_\tau| \le K_0|X_\tau|$. By standard arguments, we have

$$\begin{aligned}
&|Y_{\tau_n}|^2 + E_{\tau_n}\Big\{\int_{\tau_n}^{\tau}|Z_t|^2 dt\Big\} \\
=&E_{\tau_n}\Big\{|Y_\tau|^2 + 2\int_{\tau_n}^{\tau} Y_t[\alpha_t^3 X_t + \beta_t^3 Y_t + \gamma_t^3 Z_t]dt\Big\} \\
\le&E_{\tau_n}\Big\{K_0^2|X_\tau|^2 + C\int_{\tau_n}^{\tau}[|X_t|^2 + |Y_t|^2]dt + \frac{1}{2}\int_{\tau_n}^{\tau}|Z_t|^2 dt\Big\}.
\end{aligned}$$

Similarly,

$$E_{\tau_n}\{|X_\tau|^2\} \le E_{\tau_n}\Big\{|X_{\tau_n}|^2 + C\int_{\tau_n}^{\tau}[|X_t|^2+|Y_t|^2]dt + \frac{1}{2K_0^2}\int_{\tau_n}^{\tau}|Z_t|^2dt\Big\}.$$

Thus

$$|Y_{\tau_n}|^2 \le E_{\tau_n}\Big\{K_0^2|X_{\tau_n}|^2 + C\int_{\tau_n}^{\tau}[|X_t|^2+|Y_t|^2]dt\Big\}.$$

Note that $|X_{\tau_n}| \ge \frac{1}{n}$, then

$$|\tilde{Y}_{\tau_n}| \le K_0 + CE_{\tau_n}^{\frac{1}{2}}\Big\{\int_{\tau_n}^{\tau}[|\bar{X}_t|^2+|\bar{Y}_t|^2]dt\Big\} \le K_0 + CE_{\tau_n}^{\frac{1}{2}}\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^2+|\bar{Y}_t|^2][\tau-\tau_n]\Big\},$$

where

$$\bar{X}_t \triangleq X_t[X_{\tau_n}]^{-1}; \quad \bar{Y}_t \triangleq Y_t[X_{\tau_n}]^{-1}.$$

Now by (9),

$$\begin{aligned}
|\tilde{Y}_0| &\le K\int_0^T e^{(2K+K^2)t}dt \\
&\quad + E\Big\{e^{(2K+K^2)T}M_{\tau_n}\Big[K_0 + CE_{\tau_n}^{\frac{1}{2}}\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^2+|\bar{Y}_t|^2][\tau-\tau_n]\Big\}\Big]\Big\} \\
&\le e^{(2K+K^2)T} - 1 + K_0e^{(2K+K^2)T} \\
&\quad + CE\Big\{M_{\tau_n}E_{\tau_n}^{\frac{1}{2}}\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^2+|\bar{Y}_t|^2][\tau-\tau_n]\Big\}\Big\} \\
&\le \bar{K}_0 + CE^{\frac{1}{2}}\{|M_{\tau_n}|^2\}E^{\frac{1}{2}}\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^2+|\bar{Y}_t|^2][\tau-\tau_n]\Big\} \\
&\le \bar{K}_0 + CE^{\frac{1}{4}}\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^4+|\bar{Y}_t|^4]\Big\}E^{\frac{1}{4}}\{|\tau-\tau_n|^2\}.
\end{aligned}$$

Note that $(\bar{X},\bar{Y})$ satisfies the following FBSDE:

$$\left\{\begin{array}{lcl}
\bar{X}_t &=& 1+\int_0^t[\alpha_s^1 1_{\{\tau_n<s\}}\bar{X}_s + \beta_s^1 1_{\{\tau_n<s\}}\bar{Y}_s + \gamma_s^1 1_{\{\tau_n<s\}}\bar{Z}_s]ds \\
&& +\int_0^t[\alpha_s^2 1_{\{\tau_n<s\}}\bar{X}_s + \beta_s^2 1_{\{\tau_n<s\}}\bar{Y}_s]dW_s; \\
\bar{Y}_t &=& G\bar{X}_T + \int_t^T[\alpha_s^3 1_{\{\tau_n<s\}}\bar{X}_s + \beta_s^3 1_{\{\tau_n<s\}}\bar{Y}_s + \gamma_s^3 1_{\{\tau_n<s\}}\bar{Z}_s]ds - \int_t^T \bar{Z}_s dW_s.
\end{array}\right.$$

By Lemma 1,

$$E\Big\{\sup_{\tau_n\le t\le\tau}[|\bar{X}_t|^4+|\bar{Y}_t|^4]\Big\} \le E\Big\{\sup_{0\le t\le T}[|\bar{X}_t|^4+|\bar{Y}_t|^4]\Big\} \le C_1.$$

Thus

$$|\tilde{Y}_0| \le \bar{K}_0 + CE^{\frac{1}{4}}\{|\tau-\tau_n|^2\}.$$

Let $n\to\infty$, we get $|\tilde{Y}_0| \le \bar{K}_0$. That is, $|Y_0| \le \bar{K}_0|X_0| = \bar{K}_0$. Similarly, $|Y_t| \le \bar{K}_0|X_t|$ for all $t\in[0,T]$. □

The following result is important.

Corollary 1. *Assume that all the conditions in Lemma 1 hold true; and that (3) holds true. Let $\Theta^i, i=0,1,$ be the solution to FBSDEs:*

$$\left\{\begin{array}{l}
X_t^i = x_i + \int_0^t b(s,\Theta_s^i)ds + \int_0^t \sigma(s,X_s^i,Y_s^i)dW_s; \\
Y_t^i = g(X_T^i) + \int_t^T f(s,\Theta_s^i)ds - \int_t^T Z_s^i dW_s.
\end{array}\right.$$

Then $|Y_0^1 - Y_0^0| \le \bar{K}_0|x_1-x_0|$, where $\bar{K}_0$ is defined in (8).

Proof. For $0 \le \lambda \le 1$, let $\Theta^\lambda \stackrel{\triangle}{=} (X^\lambda, Y^\lambda, Z^\lambda)$ and $\nabla\Theta^\lambda \stackrel{\triangle}{=} (\nabla X^\lambda, \nabla Y^\lambda, \nabla Z^\lambda)$ be the solutions to FBSDEs:

$$\begin{cases} X_t^\lambda = x_0 + \lambda(x_1 - x_0) + \displaystyle\int_0^t b(s, \Theta_s^\lambda)ds + \int_0^t \sigma(s, X_s^\lambda, Y_s^\lambda)dW_s; \\ Y_t^\lambda = g(X_T^\lambda) + \displaystyle\int_t^T f(s, \Theta_s^\lambda)ds - \int_t^T Z_s^\lambda dW_s. \end{cases}$$

and

$$\begin{cases} \nabla X_t^\lambda &= 1 + \int_0^t [b_x(s,\Theta_s^\lambda)\nabla X_s^\lambda + b_y(s,\Theta_s^\lambda)\nabla Y_s^\lambda + b_z(s,\Theta_s^\lambda)\nabla Z_s^\lambda]ds \\ & \quad + \int_0^t [\sigma_x(s,\Theta_s^\lambda)\nabla X_s^\lambda + \sigma_y(s,\Theta_s^\lambda)\nabla Y_s^\lambda]dW_s; \\ \nabla Y_t^\lambda &= g'(X_T^\lambda)\nabla X_T^\lambda + \int_t^T [f_x(s,\Theta_s^\lambda)\nabla X_s^\lambda + f_y(s,\Theta_s^\lambda)\nabla Y_s^\lambda \\ & \quad + f_z(s,\Theta_s^\lambda)\nabla Z_s^\lambda]ds - \displaystyle\int_t^T \nabla Z_s^\lambda dW_s; \end{cases} \tag{10}$$

respectively. One can easily prove that

$$\Theta_t^1 - \Theta_t^0 = \int_0^1 \frac{d}{d\lambda}\Theta_t^\lambda d\lambda = [x_1 - x_0]\int_0^1 \nabla\Theta_t^\lambda d\lambda.$$

In particular,

$$Y_0^1 - Y_0^0 = [x_1 - x_0]\int_0^1 \nabla Y_0^\lambda d\lambda.$$

Note that (3) implies (7) for FBSDE (10). Then by Lemma 2 we have $|\nabla Y_0^\lambda| \le \bar K_0$, and thus

$$|Y_0^1 - Y_0^0| \le |x_1 - x_0|\int_0^1 |\nabla Y_0^\lambda| d\lambda \le \bar K_0 |x_1 - x_0|,$$

proving the lemma. □

4. **Proof of Theorem 1.** We now consider arbitrary large T. Let K and K_0 be as in Lemma 1, and $\bar K_0$ be defined by (8). Let δ_0 be a constant as in Lemma 1, but corresponding to $(K, \bar K_0)$ instead of (K, K_0). Assume $(n-1)\delta_0 < T \le n\delta_0$ for some integer n. Denote $T_i \stackrel{\triangle}{=} \frac{iT}{n}, i = 0, \cdots, n$. Define a mapping $F_n : \Omega \times \mathbb{R} \to \mathbb{R}$ by $F_n(\omega, x) \stackrel{\triangle}{=} g(\omega, x)$. Now for $t \in [T_{n-1}, T_n]$, consider the following FBSDE:

$$\begin{cases} X_t^n = x + \displaystyle\int_{T_{n-1}}^t b(s, \Theta_s^n)ds + \int_{T_{n-1}}^t \sigma(s, X_s^n, Y_s^n)dW_s; \\ Y_t^n = F_n(X_{T_n}^n) + \displaystyle\int_t^{T_n} f(s, \Theta_s^n)ds - \int_t^{T_n} Z_s^n dW_s. \end{cases}$$

Note that $L_{F_n} \le K_0 \le \bar K_0$, by Lemma 1 the above FBSDE has a unique solution for any x. Define $F_{n-1}(x) \stackrel{\triangle}{=} Y_{T_{n-1}}^n$. Then for fixed x, $F_{n-1}(x) \in \mathcal{F}_{T_{n-1}}$. Moreover, by Corollary 1 we have

$$L_{F_{n-1}} \le K_1 \stackrel{\triangle}{=} [K_0 + 1]e^{(2K+K^2)(T_n - T_{n-1})} - 1 \le \bar K_0.$$

Next we consider the following FBSDE over $[T_{n-2}, T_{n-1}]$:

$$\begin{cases} X_t^{n-1} = x + \displaystyle\int_{T_{n-2}}^t b(s, \Theta_s^{n-1})ds + \int_{T_{n-1}}^t \sigma(s, X_s^{n-1}, Y_s^{n-1})dW_s; \\ Y_t^{n-1} = F_{n-1}(X_{T_{n-1}}^{n-1}) + \displaystyle\int_t^{T_{n-1}} f(s, \Theta_s^{n-1})ds - \int_t^{T_{n-1}} Z_s^{n-1} dW_s. \end{cases}$$

Similarly we may define $F_{n-2}(x)$ such that

$$\begin{aligned} L_{F_{n-2}} \le K_2 &\stackrel{\triangle}{=} [K_1+1]e^{(2K+K^2)(T_{n-1}-T_{n-2})}-1 \\ &= [K_0+1]e^{(2K+K^2)(T_n-T_{n-2})}-1 \le \bar{K}_0. \end{aligned}$$

Repeat the arguments for $i = n, \cdots, 1$, we may define F_i such that

$$L_{F_i} \le K_{n-i} \stackrel{\triangle}{=} [K_0+1]e^{(2K+K^2)(T_n-T_i)}-1 \le \bar{K}_0.$$

Now for any $X_0 \in L^2(\mathcal{F}_0)$, we construct a solution for (2) as follows. For $i = 1, 2, \cdots, n$,

$$\left\{\begin{array}{ll} X_t = X_{T_{i-1}} + \int_{T_{i-1}}^t b(s,\Theta_s)ds + \int_{T_{i-1}}^t \sigma(s,X_s,Y_s)dW_s; & \\ & t \in [T_{i-1}, T_i]. \\ Y_t = F_i(X_{T_i}) + \int_t^{T_i} f(s,\Theta_s)ds - \int_t^{T_i} Z_s dW_s; & \end{array}\right.$$

Obviously this provides a solution to (2). From the construction and the uniqueness of each step, we know this solution is unique.

We next prove (4). Denote

$$I_t^2 \stackrel{\triangle}{=} |b(t,0,0,0)|^2 + |\sigma(t,0,0)|^2 + |f(t,0,0,0)|^2.$$

By Lemma 1 and the definition of F_i, we have

$$E\{|F_{i-1}(0)|^2\} \le C_0 E\Big\{|F_i(0)|^2 + \int_{T_{i-1}}^{T_i} I_t^2 dt\Big\}.$$

By induction one can easily prove that

$$\max_{0\le i\le n} E\{|F_i(0)|^2\} \le C_0^n E\Big\{|g(0)|^2 + \int_0^T I_t^2 dt\Big\} = CE\Big\{|g(0)|^2 + \int_0^T I_t^2 dt\Big\}.$$

We note that $n \le \frac{T}{\delta_0} + 1$ is a fixed constant depending only on K, K_0 and T, then so is C. Now for $t \in [T_0, T_1]$, by Lemma 1,

$$\begin{aligned} E\Big\{|X_t|^2 + |Y_t|^2\Big\} &\le CE\Big\{|X_0|^2 + |F_1(0)|^2 + \int_{T_0}^{T_1} I_t^2 dt\Big\} \\ &\le CE\Big\{|X_0|^2 + |g(0)|^2 + \int_0^T I_t^2 dt\Big\}. \end{aligned}$$

Then by induction one can prove

$$\sup_{0\le t\le T} E\Big\{|X_t|^2 + |Y_t|^2\Big\} \le CE\Big\{|X_0|^2 + |g(0)|^2 + \int_0^T I_t^2 dt\Big\}.$$

Now by Ito's formula,

$$\begin{aligned} E\Big\{|Y_0|^2 + \int_0^T |Z_t|^2 dt\Big\} &= E\Big\{|Y_T|^2 + 2\int_0^T Y_t f(t,\Theta_t)dt\Big\} \\ &\le E\Big\{|Y_T|^2 + C\int_0^T [|f(t,0,0,0)|^2 + |X_t|^2 \\ &\quad +|Y_t|^2]dt + \frac{1}{2}\int_0^T |Z_t|^2 dt\Big\}. \end{aligned}$$

Then

$$E\Big\{\int_0^T |Z_t|^2 dt\Big\} \le CE\Big\{|X_0|^2 + |g(0)|^2 + \int_0^T I_t^2 dt\Big\}.$$

Finally, (4) follows the Burkholder-Davis-Gundy Inequality. □

5. **Stability and Comparison Theorem.** We shall prove Theorem 2 in this section. First we establish the stability result for (2).

Theorem 3. *Assume* $(b^i, \sigma^i, f^i, g^i, X_0^i), i = 0, 1$, *satisfy all the conditions in Theorem 1. Let* Θ^i *be the corresponding solutions,* $\Delta\Theta \stackrel{\triangle}{=} \Theta^1 - \Theta^0$, $\Delta g \stackrel{\triangle}{=} g_1 - g_0$, *and define other terms similarly. Then*

$$\|\Delta\Theta\|^2 \le CE\Big\{|\Delta X_0|^2 + |\Delta g(X_T^1)|^2 + \int_0^T \Big[|\Delta b|^2 + |\Delta\sigma|^2 + |\Delta f|^2\Big](t, \Theta_t^1)dt\Big\}.$$

Proof. For $0 \le \lambda \le 1$, let Θ^λ and $\nabla\Theta^\lambda$ be the solutions to the following FBSDEs:

$$\begin{cases} X_t^\lambda &= X_0^0 + \lambda\Delta X_0 + \int_0^t [b^0(s, \Theta_s^\lambda) + \lambda\Delta b(s, \Theta_s^1)]ds \\ &\quad + \int_0^t [\sigma^0(s, \Theta_s^\lambda) + \lambda\Delta\sigma(s, \Theta_s^1)]dW_s; \\ Y_t^\lambda &= [g^0(X_T^\lambda) + \lambda\Delta g(X_T^1)] + \int_t^T [f^0(s, \Theta_s^\lambda) + \lambda\Delta f(s, \Theta_s^1)]ds - \int_t^T Z_s^\lambda dW_s. \end{cases}$$

and

$$\begin{cases} \nabla X_t^\lambda &= \Delta X_0 + \int_0^t \Big[\sigma_x^0(s, \Theta_s^\lambda)\nabla X_s^\lambda + \sigma_y^0(s, \Theta_s^\lambda)\nabla Y_s^\lambda + \Delta\sigma(s, \Theta_s^1)\Big]dW_s \\ &\quad + \int_0^t \Big[b_x^0(s, \Theta_s^\lambda)\nabla X_s^\lambda + b_y^0(s, \Theta_s^\lambda)\nabla Y_s^\lambda \\ &\quad + b_z^0(s, \Theta_s^\lambda)\nabla Z_s^\lambda + \Delta b(s, \Theta_s^1)\Big]ds; \\ \nabla Y_t^\lambda &= [g_x^0(X_T^\lambda)\nabla X_T^\lambda + \Delta g(X_T^1)] - \int_t^T \nabla Z_s^\lambda dW_s \\ &\quad + \int_t^T \Big[f_x^0(s, \Theta_s^\lambda)\nabla X_s^\lambda + f_y^0(s, \Theta_s^\lambda)\nabla Y_s^\lambda \\ &\quad + f_z^0(s, \Theta_s^\lambda)\nabla Z_s^\lambda + \Delta f(s, \Theta_s^1)\Big]ds; \end{cases}$$

respectively. By the uniqueness of solutions, we know that the two definitions of (Θ^0, Θ^1) are consistent. Also, one can show that

$$\Delta\Theta_t = \int_0^1 \frac{d}{d\lambda}\Theta_t^\lambda d\lambda = \int_0^1 \nabla\Theta_t^\lambda d\lambda. \tag{11}$$

Since (b^0, σ^0, f^0) satisfies (3), by Lemma 2 we have

$$\|\nabla\Theta^\lambda\|^2 \le CE\Big\{|\Delta X_0|^2 + |\Delta g(X_T^1)|^2 + \int_0^T \Big[|\Delta b|^2 + |\Delta\sigma|^2 + |\Delta f|^2\Big](t, \Theta_t^1)dt\Big\},$$

which obviously proves the theorem. □

Corollary 2. *Assume* $(b^n, \sigma^n, f^n, g^n, X_0^n), n = 0, 1, \cdots$ *satisfy all the conditions in Theorem 1 uniformly;* $X_0^n \to X_0^0$ *in* L^2*; for* $\varphi = b, \sigma, f, g$ *and for any* (t, θ), $\varphi^n(t, \theta) \to \varphi^0(t, \theta)$ *as* $n \to \infty$*; and*

$$E\Big\{|X_0^n - X_0|^2 + |g^n - g^0|^2(0) + \int_0^T [|b^n - b^0|^2 + |\sigma^n - \sigma^0|^2 + |f^n - f^0|^2](t, 0, 0, 0)dt\Big\} \to 0.$$

Let Θ^n *denote the corresponding solutions. Then*

$$\|\Theta^n - \Theta^0\| \to 0.$$

Proof. By Theorem 3,

$$\begin{aligned} \|\Theta^n - \Theta^0\|^2 &\le CE\Big\{|X_0^n - X_0^0|^2 + |g^n - g^0|^2(X_T^0) \\ &\quad + \int_0^T [|b^n - b|^2 + |\sigma^n - \sigma|^2 + |f^n - f^0|^2](t, \Theta_t^0)dt\Big\}. \end{aligned}$$

Let $n \to \infty$ and apply the Dominated Convergence Theorem we prove the result. $\square$

Next lemma is the linear version of Theorem 2.

Lemma 3. *Assume $|\alpha^i|, |\beta^i|, |\gamma^i| \le K$, $|G| \le K_0$, and (3) holds true. Assume further that $\xi \ge 0$ and $\eta \ge 0$. Let (X, Y, Z) be the solution to the following linear FBSDE:*

$$\left\{\begin{array}{l} X_t = \int_0^t [\alpha_s^1 X_s + \beta_s^1 Y_s + \gamma_s^1 Z_s] ds + \int_0^t [\alpha_s^2 X_s + \beta_s^2 Y_s] dW_s; \\ Y_t = G X_T + \xi + \int_t^T [\alpha_s^3 X_s + \beta_s^3 Y_s + \gamma_s^3 Z_s + \eta_s] ds - \int_t^T Z_s dW_s. \end{array}\right.$$

Then $Y_0 \ge 0$.

Proof. We prove the result in several steps.

Step 1. Assume $G = 0, \eta = 0$. If $Y_0 < 0$, let

$$\tau \stackrel{\triangle}{=} \inf\{t : Y_t = 0\} \wedge T.$$

Since $Y_T = \xi \ge 0$, we have $Y_\tau = 0$. Denote

$$\begin{array}{l} \bar{\alpha}_t^i \stackrel{\triangle}{=} \alpha_t^i 1_{\{\tau > t\}}; \bar{\beta}_t^i \stackrel{\triangle}{=} \beta_t^i 1_{\{\tau > t\}}; \bar{\gamma}_t^i \stackrel{\triangle}{=} \gamma_t^i 1_{\{\tau > t\}}; \\ \bar{X}_t \stackrel{\triangle}{=} X_{\tau \wedge t}; \quad \bar{Y}_t \stackrel{\triangle}{=} Y_{\tau \wedge t}; \quad \bar{Z}_t \stackrel{\triangle}{=} Z_t 1_{\{\tau > t\}}. \end{array}$$

Then

$$\left\{\begin{array}{l} \bar{X}_t = \int_0^t [\bar{\alpha}_s^1 \bar{X}_s + \bar{\beta}_s^1 \bar{Y}_s + \bar{\gamma}_s^1 \bar{Z}_s] ds + \int_0^t [\bar{\alpha}_s^2 \bar{X}_s + \bar{\beta}_s^2 \bar{Y}_s] dW_s; \\ \bar{Y}_t = \int_t^T [\bar{\alpha}_s^3 \bar{X}_s + \bar{\beta}_s^3 \bar{Y}_s + \bar{\gamma}_s^3 \bar{Z}_s] ds - \int_t^T \bar{Z}_s dW_s. \end{array}\right.$$

By uniqueness $\bar{Y}_t = 0$. Then $Y_0 = \bar{Y}_0 = 0$, contradiction. Thus $Y_0 \ge 0$.

Step 2. Assume $\eta = 0$ and $|g| \le C$ where $G = E\{G\} + \int_0^T g_t dW_t$. Denote

$$G_t \stackrel{\triangle}{=} E\{G\} + \int_0^t g_s dW_s; \quad \tilde{Y}_t \stackrel{\triangle}{=} Y_t - G_t X_t; \quad \tilde{Z}_t \stackrel{\triangle}{=} Z_t - G_t[\alpha_t^2 X_t + \beta_t^2 Y_t] - g_t X_t.$$

Then

$$\begin{array}{rcl} dX_t &=& \Big[\alpha_t^1 X_t + \beta_t^1 [\tilde{Y}_t + G_t X_t] + \gamma_s^1 [\tilde{Z}_t + G_t \alpha_t^2 X_t + G_t \beta_t^2 [\tilde{Y}_t + G_t X_t] + g_t X_t]\Big] dt \\ && + \Big[\alpha_t^2 X_t + \beta_t^2 [\tilde{Y}_t + G_t X_t]\Big] dW_t \\ &=& \Big[\tilde{\alpha}_t^1 X_t + \tilde{\beta}_t^1 \tilde{Y}_t + \tilde{\gamma}_t^1 \tilde{Z}_t\Big] dt + \Big[\tilde{\alpha}_t^2 X_t + \tilde{\beta}_t^2 \tilde{Y}_t\Big] dW_t; \end{array}$$

and

$$\begin{array}{rcl} d\tilde{Y}_t &=& -[\alpha_t^3 X_t + \beta_t^3 Y_t + \gamma_t^3 Z_t] dt + Z_t dW_t - g_t[\alpha_t^2 X_t + \beta_t^2 Y_t] dt \\ && - G_t[\alpha_t^1 X_t + \beta_t^1 Y_t + \gamma_t^1 Z_t] dt - G_t[\alpha_t^2 X_t + \beta_t^2 Y_t] dW_t - g_t X_t dW_t \\ &=& \tilde{Z}_t dW_t - \Big[[\alpha_t^3 + g_t \alpha_t^2 + G_t \alpha_t^1] X_t + [\beta_t^3 + g_t \beta_t^2 + G_t \beta_t^1][\tilde{Y}_t + G_t X_t] \\ && + [\gamma_t^3 + G_t \gamma_t^1][\tilde{Z}_t + [g_t + G_t \alpha_t^2] X_t + G_t \beta_t^2 [\tilde{Y}_t + G_t X_t]]\Big] dt \\ &=& -[\tilde{\alpha}_t^3 X_t + \tilde{\beta}_t^3 \tilde{Y}_t + \tilde{\gamma}_t^3 \tilde{Z}_t] dt + \tilde{Z}_t dW_t, \end{array}$$

where

$$
\begin{cases}
\tilde{\alpha}^1_t \triangleq \alpha^1_t + G_t\beta^1_t + G_t\alpha^2_t\gamma^1_t + |G_t|^2\beta^2_t\gamma^1_t + g_t\gamma^1_t;\\
\tilde{\beta}^1_t \triangleq \beta^1_t + G_t\beta^2_t\gamma^1_t = \beta^1_t;\\
\tilde{\gamma}^1_t \triangleq \gamma^1_t;\\
\tilde{\alpha}^2_t \triangleq \alpha^2 + G_t\beta^2_t;\\
\tilde{\beta}^2_t \triangleq \beta^2_t;\\
\tilde{\alpha}^3_t \triangleq \alpha^3_t + g_t\alpha^2_t + G_t\alpha^1_t + [\beta^3_t + g_t\beta^2_t + G_t\beta^1_t]G_t\\
\qquad +[\gamma^3_t + G_t\gamma^1_t][g_t + G_t\alpha^2_t + |G_t|^2\beta^2_t];\\
\tilde{\beta}^3_t \triangleq \beta^3_t + g_t\beta^2_t + G_t\beta^1_t + G_t\beta^2_t\gamma^3_t + |G_t|^2\beta^2_t\gamma^1_t;\\
\tilde{\gamma}^3_t \triangleq \gamma^3_t + G_t\gamma^1_t.
\end{cases}
$$

One can easily check that $\tilde{\alpha}^i, \tilde{\beta}^i, \tilde{\gamma}^i$ are bounded and still satisfy (3). Note that $\tilde{Y}_T = \xi \geq 0$. By Step 1 we know $Y_0 = \tilde{Y}_0 \geq 0$.

Step 3. Assume $\eta = 0$. One can find G_n such that $|G_n| \leq K$, $G_n \to G$ a.s., and G_n satisfies the condition in Step 2. Let (X^n, Y^n, Z^n) denote the solution corresponding to G_n. By Step 2 we have $Y^n_0 \geq 0$. Then by Corollary 2 we get $Y_0 = \lim\limits_{n\to\infty} Y^n_0 \geq 0$.

Step 4. Assume $\xi = 0, \frac{1}{m} \leq \eta \leq m$ and $T \leq \delta$ where $\delta > 0$ is a small constant depending only on K, K_0 and m. By otherwise applying the Girsanov Theorem, without loss of generality we assume $\gamma^3_t = 0$. By standard arguments (see, e.g. [1]), for and $\varepsilon > 0$ we have

$$
\begin{aligned}
&\sup_{0\leq t\leq T} E\Big\{|X_t|^2 + |Y_t|^2\Big\} + E\Big\{\int_0^T |Z_t|^2 dt\Big\}\\
\leq &C\varepsilon^{-1} E\Big\{\int_0^T [|X_t|^2 + |Y_t|^2]dt\Big\} + \frac{\varepsilon}{2} E\Big\{\int_0^T |\eta_t|^2 dt\Big\}\\
\leq &C\varepsilon^{-1} T \sup_{0\leq t\leq T} E\Big\{|X_t|^2 + |Y_t|^2\Big\} + \frac{\varepsilon}{2} m^2 T.
\end{aligned}
$$

We choose $\delta = \frac{\varepsilon}{2C}$ and will specify ε later. Then for $T \leq \delta$, we have

$$
\sup_{0\leq t\leq T} E\Big\{|X_t|^2 + |Y_t|^2\Big\} + E\Big\{\int_0^T |Z_t|^2 dt\Big\} \leq m^2 \varepsilon T.
$$

Moreover,

$$
\begin{aligned}
E\{|X_T|^2\} &\leq CE\Big\{|\int_0^T [\alpha^1_t X_t + \beta^1_t Y_t + \gamma^1_t Z_t]dt|^2 + |\int_0^T [\alpha^2_t X_t + \beta^2_t Y_t]dW_t|^2\Big\}\\
&\leq CE\Big\{T\int_0^T [|X_t|^2 + |Y_t|^2 + |Z_t|^2]dt + \int_0^T [|X_t|^2 + |Y_t|^2]dt\Big\}\\
&\leq Cm^2\varepsilon T^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|E\Big\{GX_T + \int_0^T [\alpha^3_t X_t + \beta^3_t Y_t]dt\Big\}|\\
\leq &CE^{\frac{1}{2}}\{|X_T|^2\} + CT \sup_{0\leq t\leq T} E^{\frac{1}{2}}\{|X_t|^2 + |Y_t|^2\} \leq Cm\sqrt{\varepsilon}T.
\end{aligned}
$$

Therefore,

$$\begin{aligned} Y_0 &= E\Big\{GX_T+\int_0^T[\alpha_t^3X_t+\beta_t^3Y_t+\eta_t]dt\Big\}\\ &\geq m^{-1}T-|E\Big\{GX_T+\int_0^T[\alpha_t^3X_t+\beta_t^3Y_t]dt\Big\}|\\ &\geq m^{-1}T-Cm\sqrt{\varepsilon}T. \end{aligned}$$

Now choose $\varepsilon=C^{-2}m^{-4}$, we get $Y_0\geq 0$.

Step 5. Assume $\frac{1}{m}\leq\eta\leq m$ and $T\leq\delta$ where δ is the same as in Step 4. Denote

$$\begin{cases} X_t^1=\displaystyle\int_0^t[\alpha_s^1X_s^1+\beta_s^1Y_s^1+\gamma_s^1Z_s^1]ds+\int_0^t[\alpha_s^2X_s^1+\beta_s^2Y_s^1]dW_s;\\ Y_t^1=GX_T^1+\xi+\displaystyle\int_t^T[\alpha_s^3X_s^1+\beta_s^3Y_s^1+\gamma_s^3Z_s^1]ds-\int_t^TZ_s^1dW_s; \end{cases}$$

and

$$\begin{cases} X_t^2=\displaystyle\int_0^t[\alpha_s^1X_s^2+\beta_s^1Y_s^2+\gamma_s^1Z_s^2]ds+\int_0^t[\alpha_s^2X_s^2+\beta_s^2Y_s^2]dW_s;\\ Y_t^2=GX_T^2+\displaystyle\int_t^T[\alpha_s^3X_s^2+\beta_s^3Y_s^2+\gamma_s^3Z_s^2+\eta_s]ds-\int_t^TZ_s^2dW_s. \end{cases}$$

By Step 3, $Y_0^1\geq 0$, and by Step 4, $Y_0^2\geq 0$. Therefore, $Y_0=Y_0^1+Y_0^2\geq 0$.

Step 6. Assume $\frac{1}{m}\leq\eta\leq m$. Let δ be as in Step 4 but corresponding to $(K,\bar K_0,m)$ instead of (K,K_0,m), and assume $(n-1)\delta<T\leq n\delta$. Denote $T_i\stackrel{\triangle}{=}\frac{iT}{n}$. Denote $G_n\stackrel{\triangle}{=}G,\xi_n\stackrel{\triangle}{=}\xi$. For $t\in[T_{n-1},T_n]$, let

$$\begin{cases} X_t^{n,1} &= 1+\int_{T_{n-1}}^t[\alpha_s^1X_s^{n,1}+\beta_s^1Y_s^{n,1}+\gamma_s^1Z_s^{n,1}]ds\\ &\quad+\int_{T_{n-1}}^t[\alpha_s^2X_s^{n,1}+\beta_s^2Y_s^{n,1}]dW_s;\\ Y_t^{n,1} &= G_nX_{T_n}^{n,1}+\int_t^{T_n}[\alpha_s^3X_s^{n,1}+\beta_s^3Y_s^{n,1}+\gamma_s^3Z_s^{n,1}]ds-\int_t^{T_n}Z_s^{n,1}dW_s; \end{cases}$$

and

$$\begin{cases} X_t^{n,2} &= \int_{T_{n-1}}^t[\alpha_s^1X_s^{n,2}+\beta_s^1Y_s^{n,2}+\gamma_s^1Z_s^{n,2}]ds+\int_0^t[\alpha_s^2X_s^{n,2}+\beta_s^2Y_s^{n,2}]dW_s;\\ Y_t^{n,2} &= G_nX_T^n+\xi_n+\int_t^{T_n}[\alpha_s^3X_s^{n,2}+\beta_s^3Y_s^{n,2}+\gamma_s^3Z_s^{n,2}+\eta_s]ds\\ &\quad-\int_t^{T_n}Z_s^{n,2}dW_s. \end{cases}$$

Denote $G_{n-1}\stackrel{\triangle}{=}Y_{T_{n-1}}^{n,1},\xi_{n-1}\stackrel{\triangle}{=}Y_{T_{n-1}}^{n,2}$. By the proof of Theorem 1 we know $|G_{n-1}|\leq K_1\leq\bar K_0$. By Step 5, $\xi_{n-1}\geq 0$. We note that, for $t\in[0,T_{n-1}]$, (X,Y,Z) satisfies

$$\begin{cases} X_t=\displaystyle\int_0^t[\alpha_s^1X_s+\beta_s^1Y_s+\gamma_s^1Z_s]ds+\int_0^t[\alpha_s^2X_s+\beta_s^2Y_s]dW_s;\\ Y_t=G_{n-1}X_{T_{n-1}}+\xi_{n-1}+\displaystyle\int_t^{T_{n-1}}[\alpha_s^3X_s+\beta_s^3Y_s+\gamma_s^3Z_s+\eta_s]ds-\int_t^{T_1}Z_sdW_s. \end{cases}$$

Repeat the arguments we may define G_1 and $\xi_1\geq 0$, and it holds that

$$\begin{cases} X_t=\displaystyle\int_0^t[\alpha_s^1X_s+\beta_s^1Y_s+\gamma_s^1Z_s]ds+\int_0^t[\alpha_s^2X_s+\beta_s^2Y_s]dW_s;\\ Y_t=G_1X_{T_1}+\xi_1+\displaystyle\int_t^{T_1}[\alpha_s^3X_s+\beta_s^3Y_s+\gamma_s^3Z_s+\eta_s]ds-\int_t^{T_1}Z_sdW_s. \end{cases}$$

By Step 5 again, $Y_0\geq 0$.

Step 7. In general case, denote $\eta^m \stackrel{\triangle}{=} (\eta \wedge m) \vee \frac{1}{m}$ and let (X^m, Y^m, Z^m) denote the solution corresponding to η^m. By Step 6, $Y_0^m \geq 0$. Then by Corollary 2, $Y_0 = \lim_{m\to\infty} Y_0^m \geq 0$. □

Remark 3. In general one cannot expect $Y_t \geq 0$.

Proof. Consider the following FBSDE:

$$\begin{cases} X_t = \int_0^t Y_s dW_s; \\ Y_t = X_T + \int_t^T ds - \int_t^T Z_s dW_s. \end{cases}$$

Assume $Y_T \geq 0$. By the BSDE we have $X_T = Y_T \geq 0$. Since $E\{X_T\} = 0$, we get $X_T = 0$ a.s. Then on one hand, by the FSDE we have $Y_t = 0$. On the other hand, by the BSDE we get $Y_t = T - t$. Contradiction! □

Proof of Theorem 2: Let Θ^λ and $\nabla\Theta^\lambda$ be as in the proof of Theorem 3. Then

$$\Delta X_0 = 0, \quad \Delta b = 0, \quad \Delta\sigma = 0, \Delta f \geq 0, \quad \Delta g \geq 0.$$

By Lemma 3, $\nabla Y_0^\lambda \geq 0$. The result follows (11) now. □

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