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Pathwise Itô calculus for rough paths and rough PDEs with path dependent coefficients

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Abstract

This paper introduces path derivatives, in the spirit of Dupire's functional Itô calculus, for controlled rough paths in rough path theory with possibly non-geometric rough paths. We next study rough PDEs with coefficients depending on the rough path itself, which corresponds to stochastic PDEs with random coefficients. Such coefficients are less regular in the time variable, which is not covered in the existing literature. The results are useful for studying viscosity solutions of stochastic PDEs.

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1. Introduction

Firstly initiated by Lyons [33], rough path theory has been studied extensively and its applications have been found in many areas, including the recent application on KPZ equations by Hairer [24]. We refer to Lyons [34], Friz and Hairer [9], Friz and Victoir [21], and the references therein for the general theory and its applications.

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On the other hand, the functional Itô calculus, initiated by Dupire [13] and further developed by Cont and Fournie [9], has received very strong attention in recent years. In particular, it has proven to be a very convenient language for the theory of path dependent PDEs, see Peng and Wang [37], Ekren, Keller, Touzi and Zhang [14], and Ekren, Touzi and Zhang [15,16]. We also refer to Buckdahn, Ma and Zhang [6], Cosso and Russo [10], Leao, Ohashi and Simas [27], and Oberhauser [36] for some recent related works on functional Itô calculus.

The first goal of this paper is to develop the pathwise Itô calculus, in the spirit of Dupire's functional Itô calculus, in the rough path framework with possibly non-geometric rough paths. Based on the quadratic compensator of rough paths, which plays the role of quadratic variation in semimartingale theory, we introduce path derivatives for controlled rough paths of Gubinelli [22]. Our first order spatial path derivative is the same as Gubinelli's derivative, and the time derivative is closely related to a second order Taylor expansion of the controlled rough paths. This allows us to study the structure of a fairly general class of controlled rough paths, and more importantly, to treat rough path integration and rough ODEs/PDEs in the same manner as standard Itô calculus. In particular,

- the pathwise Taylor expansion and the pathwise Itô formula become equivalent;
- as observed by Buckdahn, Ma and Zhang [6] in a Brownian motion setting, the pathwise Itô–Ventzell formula is equivalent to the chain rule of our path derivatives, which is crucial for studying rough PDEs and stochastic PDEs;
- We may study rough ODEs/PDEs whose "drift term" is driven by the quadratic compensator, instead of dt. See (1.1) and (1.3). This is natural in semimartingale theory when the driving martingale is not a Brownian motion.

We shall remark though, while we believe such presentation of path derivatives in the rough path framework is new, many related ideas have already been discussed in the literature. Besides [18] and the reference therein, we also refer to the recent work Perkowski and Prömel [38] for some related studies.

We next study the following rough differential equations in the form:

$$d\theta_t = g(t,\theta_t)d\boldsymbol{\omega}_t + f(t,\theta_t)d\langle\boldsymbol{\omega}\rangle_t, \tag{1.1}$$

where $\boldsymbol{\omega}$ is a Hölder-α continuous rough path and $\langle\boldsymbol{\omega}\rangle$ is its quadratic compensator. We remark that, as mentioned in previous paragraph, we use Young's integration $f(t,\theta_t)d\langle\boldsymbol{\omega}\rangle_t$ rather than Lebesgue integration $f(t,\theta_t)dt$ in the "drift" term above, and they become the same when $\boldsymbol{\omega}$ is induced by a sample path of Brownian motion with Itô integration. Our study of above RDE is mainly motivated from the following stochastic differential equations with random coefficients:

$$dX_t = g(t,\omega,X_t)dB_t + f(t,\omega,X_t)dt, \tag{1.2}$$

where B is a Brownian motion in the canonical probability space $(\Omega,\mathcal{F},\mathbb{P})$, dB is Itô integration, and g, f are adapted, namely depend on the history of the path: $\{\omega_s\}_{0\le s\le t}$. In the literature, typically the coefficients g and f in (1.1) do not depend on t, or at least is Hölder-$(1-\alpha)$ continuous in t, see Lejay and Victoir [28]. However, since a Brownian motion sample path ω is only Hölder-$(\frac{1}{2}-\varepsilon)$ continuous, by setting $\alpha = \frac{1}{2}-\varepsilon$, for (1.2) it is not reasonable to assume the mapping $t \mapsto g(\cdot,\omega,x)$ is Hölder-$(1-\alpha)$ continuous as required by [28]. Consequently, we are not able to apply the existing results in the rough path literature to study SDE (1.2) with random coefficients. We shall provide various estimates for rough path integrations, which follow more or less standard arguments, and then establish the well-posedness of RDE (1.1) under minimum regularity conditions on the coefficients. To be precise, we require only that $g(\cdot,x)$, $f(\cdot,x)$, and

$\partial_\omega g(\cdot, x)$ are Hölder-β continuous for some $\beta \in (1-2\alpha, \alpha]$, where $\partial_\omega g$ is the spatial path derivative corresponding to Gubnelli's derivative. This can be easily satisfied for the coefficients of (1.2) when $\frac{1}{3} < \alpha < \frac{1}{2}$. We note that the recent works Gubinelli, Tindel and Torrecilla [23] and Lyons and Yang [35] have also studied rough integration for more general integrands.

As a direct consequence of the above well-posedness result of RDE (1.1), we obtain the pathwise solution of SDE (1.2) with random coefficients. Moreover, by restricting the canonical space Ω slightly and by using the pathwise stochastic integration, we construct the second order process $\underline{\omega}$ via ω itself. Then the pathwise solution exists for all $\omega \in \Omega$, without the exceptional $\mathbb{P}$-null set, and the solution $X(\omega)$ is continuous in ω under the rough path topology.

We would also like to mention that, for linear RDEs, we introduce a decoupling strategy and provide a semi-explicit solution, by using the local solution of certain Riccati-type RDEs. The result seems new even for standard linear SDEs in the multidimensional setting.

Finally, we extend the theory to the following rough PDEs with less regular coefficients:

$$du(t,x) = \big[\sigma(t,x)\partial_x u + g(t,x,u)\big]d\boldsymbol{\omega}_t + f(t,x,u,\partial_x u,\partial^2_{xx}u)d\langle\boldsymbol{\omega}\rangle_t, \tag{1.3}$$

again motivated from pathwise analysis for stochastic PDEs with random coefficients:

$$du(t,\omega,x) = \big[\sigma(t,\omega,x)\partial_x u + g(t,\omega,x,u)\big]dB_t + f(t,\omega,x,u,\partial_x u,\partial^2_{xx}u)dt. \tag{1.4}$$

As standard in the literature, see e.g. Kunita [26] for Stochastic PDEs and [18] for Rough PDEs, the main tool is the (pathwise) characteristics. We construct the pathwise characteristics via RDEs against a backward rough path. We remark that the backward rough path we construct is also a rough path. Our result here will be crucial for the study of viscosity solutions of SPDEs in Buckdahn, Keller, Ma and Zhang [2].

The rest of the paper is organized as follows. In Section 2 we introduce the basics of our pathwise Itô calculus, in particular the path derivatives of controlled rough paths. In Section 3 we study functions of controlled rough paths and their path derivatives. We shall provide related estimates and prove the chain rule of path derivatives, which is equivalent to the pathwise Itô–Ventzell formula. In Section 4 we study the well-posedness results of rough differential equations. In particular, for linear RDEs we introduce a decoupling strategy which enables us to construct semi-explicit global solution. In Section 5 we apply the RDE results to SDEs with random coefficients. Finally in Section 6 we extend the results to rough PDEs and stochastic PDEs.

Below we collect some notations used throughout the paper:

- $T > 0$ is a fixed time; and $\mathbb{T} := [0,T]$, $\mathbb{T}^2 := \{(s,t) : 0 \le s < t \le T\}$.
- d is the fixed dimension for rough paths, and $\mathbb{S}^d$ the space of $d \times d$ symmetric matrices.
- E (and $\tilde{E}$) is a generic Euclidean space, and $|E|$ is the dimension of E, namely $E = \mathbb{R}^{|E|}$.
- By default E^n is viewed as a column vector. However, for a function $g : y \in E \to \tilde{E}$, we take the convention that the first order derivative $\partial_y g \in \tilde{E}^{1\times|E|}$ is viewed as a row vector, and the second order derivative $\partial^2_{yy} g := \partial_y[(\partial_y g)^*] \in \tilde{E}^{|E|\times|E|}$ is symmetric. Moreover, for $g : (x,y) \in E_1 \times E_2 \to \tilde{E}$, $\partial_{xy} g := \partial_x[(\partial_y g)^*] \in \tilde{E}^{|E_2|\times|E_1|}$ and $\partial_{yx} g := \partial_y[(\partial_x g)^*] \in \tilde{E}^{|E_1|\times E_2}$.
- $\varphi_{s,t} := \varphi_t - \varphi_s$ for any function $\varphi : \mathbb{T} \to E$ and any $(s,t) \in \mathbb{T}^2$.
- For $A \in E^{m\times n}$, $A^* \in E^{n\times m}$ is its transpose.
- For $x \in E^d$ and $y \in \mathbb{R}^d$, $x \cdot y \in E$ is their inner product.
- For $A \in E^{m\times n}$ and $\tilde{A} \in \mathbb{R}^{m\times n}$, $A : \tilde{A} := \mathrm{Trace}(A\tilde{A}^*) \in E$.

- For $A = [a_{i,j} : 1 \le i \le m,\ 1 \le j \le |E|] \in \tilde{E}^{m\times|E|}$ and $x = [x_{i,j},\ 1 \le i \le n,\ 1 \le j \le |E|] \in E^n = \mathbb{R}^{n\times|E|}$, $Ax \in \tilde{E}^{m\times n}$ is their tensor contraction, whose (i, j)-th component is $\sum_{k=1}^{|E|} a_{i,k}x_{j,k}$.
- For $A = [a_{i,j} : 1 \le i \le |E_1|,\ 1 \le j \le E_2] \in \tilde{E}^{|E_1|\times|E_2|}$ and $x = [x_{i,j},\ 1 \le i \le m,\ 1 \le j \le |E_1|] \in E_1^m = \mathbb{R}^{m\times|E_1|}$, $y = [y_{i,j},\ 1 \le i \le n,\ 1 \le j \le |E_2|] \in E_2^n = \mathbb{R}^{n\times|E_2|}$, $A\,[x, y] \in \tilde{E}^{m\times n}$ is their (double) tensor contraction, whose (i, j)-th component is $\sum_{k=1}^{|E_1|}\sum_{l=1}^{|E_2|} a_{k,l}x_{i,k}y_{j,l}$.

2. Rough path integration and path derivatives

In this section we present the basics of rough path theory as well as our pathwise Itô calculus.

2.1. Rough path and its quadratic compensator

Denote, for a constant $\alpha > 0$,

$$\begin{aligned}
\Omega_\alpha(E) &:= \Big\{\omega \in C(\mathbb{T}, E) : \|\omega\|_\alpha < \infty\Big\}, \quad \text{where } \|\omega\|_\alpha := \sup_{(s,t)\in\mathbb{T}^2} \frac{|\omega_{s,t}|}{|t-s|^\alpha};\\
\underline{\Omega}_\alpha(E) &:= \Big\{\underline{\omega} \in C(\mathbb{T}^2, E) : \|\underline{\omega}\|_\alpha < \infty\Big\}, \quad \text{where } \|\underline{\omega}\|_\alpha := \sup_{(s,t)\in\mathbb{T}^2} \frac{|\underline{\omega}_{s,t}|}{|t-s|^\alpha}.
\end{aligned} \tag{2.1}$$

It is clear that

$$\|\omega\|_\infty := \sup_{0\le t\le T} |\omega_t| \le |\omega_0| + T^\alpha\|\omega\|_\alpha, \quad \forall \omega \in \Omega_\alpha(E). \tag{2.2}$$

From now on, we shall fix two parameters:

$$\boldsymbol{\alpha} := (\alpha, \beta) \quad \text{where } \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right), \ \beta \in (1-2\alpha, \alpha]. \tag{2.3}$$

Our space of rough paths is:

$$\begin{aligned}
\boldsymbol{\Omega}_\alpha^0 := \Big\{&\boldsymbol{\omega} = (\omega, \underline{\omega}) \in \Omega_\alpha(\mathbb{R}^d) \times \underline{\Omega}_{2\alpha}(\mathbb{R}^{d\times d}) :\\
&\underline{\omega}_{s,t} - \underline{\omega}_{s,r} - \underline{\omega}_{r,t} = \omega_{s,r}\omega_{r,t}^* \ \forall 0 \le s < r < t \le T\Big\}
\end{aligned} \tag{2.4}$$

equipped with:

$$\|\boldsymbol{\omega}\|_\alpha := \|\omega\|_\alpha + \|\underline{\omega}\|_{2\alpha}. \tag{2.5}$$

The requirement in second line of (2.4) is called Chen's relation. We remark that in general $\|\lambda\boldsymbol{\omega}\|_\alpha \ne |\lambda|\|\boldsymbol{\omega}\|_\alpha$ for a constant λ.

We next introduce the quadratic compensator of $\boldsymbol{\omega}$:

$$\langle\boldsymbol{\omega}\rangle_t := \omega_{0,t}(\omega_{0,t})^* - \underline{\omega}_{0,t} - \underline{\omega}_{0,t}^* \in \mathbb{S}^d. \tag{2.6}$$

By (2.4), one can easily check that

$$\langle\boldsymbol{\omega}\rangle_{s,t} = \omega_{s,t}(\omega_{s,t})^* - \underline{\omega}_{s,t} - \underline{\omega}_{s,t}^* \quad \text{and thus } \langle\boldsymbol{\omega}\rangle \in \Omega_{2\alpha}(\mathbb{S}^d). \tag{2.7}$$

Remark 2.1. (i) Clearly $\langle\boldsymbol{\omega}\rangle = 0$ if and only if $\boldsymbol{\omega}$ is a geometric rough path. This process is intrinsic for non-geometric rough paths, and makes our study much more convenient.

(ii) The process $\langle\boldsymbol{\omega}\rangle$ is called the bracket process, denoted as $[\boldsymbol{\omega}]$, of the so-called reduced rough path in [18]. As we will see later,

- this process plays essentially the same role as the quadratic variation process in semi-martingale theory;
- $\omega_t^2 - \langle\boldsymbol{\omega}\rangle_t$ is always a rough path integration, which can be viewed as the counterpart of martingale. So in spirit $\langle\boldsymbol{\omega}\rangle_t$ plays the similar role for ω_t^2 as the compensator for a random measure.

For these reasons, in this paper we call $\langle\boldsymbol{\omega}\rangle$ the quadratic compensator of $\boldsymbol{\omega}$. However, we shall note that a typical rough path may not have finite quadratic variation. ■

The following result is straightforward and its proof is omitted.

Lemma 2.2. *For any $\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}} \in \boldsymbol{\Omega}_\alpha^0$, we have*

$$\|\langle\boldsymbol{\omega}\rangle\|_{2\alpha} \le \|\boldsymbol{\omega}\|_\alpha[2 + \|\boldsymbol{\omega}\|_\alpha]; \qquad \|\langle\boldsymbol{\omega}\rangle - \langle\tilde{\boldsymbol{\omega}}\rangle\|_{2\alpha} \le [\|\omega\|_\alpha + \|\tilde{\omega}\|_\alpha + 2]\|\boldsymbol{\omega} - \tilde{\boldsymbol{\omega}}\|_\alpha. \tag{2.8}$$

2.2. *Rough path integration*

To study rough path integration against $\boldsymbol{\omega}$, we first introduce the controlled rough paths of Gubinelli [22], which can be viewed as C^1-regularity of the paths against the rough path.

Definition 2.3. For each $\omega \in \Omega_\alpha(\mathbb{R}^d)$, the space $\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ of controlled rough paths consists of E-valued paths $\theta \in \Omega_\beta(E)$ such that there exists $\partial_\omega\theta \in \Omega_\beta(E^{1\times d})$ satisfying:

$$R^{\omega,\theta} \in \underline{\Omega}_{\alpha+\beta}(E) \quad \text{where } R^{\omega,\theta}_{s,t} := \theta_{s,t} - \partial_\omega\theta_s\omega_{s,t}\ \forall (s,t) \in \mathbb{T}^2.$$

We note that for notational simplicity we take the convention that $\partial_\omega\theta$ is a row vector.

Remark 2.4. (i) The path derivative $\partial_\omega\theta$ depends on ω, but not on $\underline{\omega}$.

(ii) In general $\partial_\omega\theta$ is not unique. However, when $\boldsymbol{\omega}$ is truly rough, namely $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$ as defined in (2.9), $\partial_\omega\theta$ is unique. See [18, Proposition 6.4]. For the ease of presentation, in this paper we shall assume $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$. However, most of our results still hold true when $\boldsymbol{\omega} \in \boldsymbol{\Omega}_\alpha^0$, provided that we specify a version of $\partial_\omega\theta$.

(iii) $\partial_\omega\theta$ is called the Gubinelli derivative in the rough path literature. As we will see in Section 5, when ω is a sample path of Brownian motion, it coincides with the path derivative introduced in [6]. So in this paper we also call it path derivative. ■

For the ease of presentation, from now on we restrict to $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$ so that $\partial_\omega\theta$ is unique:

$$\boldsymbol{\Omega_a} := \Big\{\boldsymbol{\omega} \in \boldsymbol{\Omega}_\alpha^0 : \text{ there exists a dense subset } A \subset [0,T) \text{ such that}$$
$$\overline{\lim_{t\downarrow s}} \frac{|v\cdot\omega_{s,t}|}{(t-s)^{\alpha+\beta}} = \infty \text{ for all } s \in A \text{ and } v \in \mathbb{R}^d \setminus \{0\}\Big\}. \tag{2.9}$$

For $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$, we equip the space $\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ with the semi-norms:

$$\begin{aligned} \|\theta\|_{\omega,\boldsymbol{\alpha}} &:= \|\partial_\omega\theta\|_\beta + \|R^{\omega,\theta}\|_{\alpha+\beta}, & d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\theta,\tilde{\theta}) &:= \|\partial_\omega\theta - \partial_{\tilde{\omega}}\tilde{\theta}\|_\beta + \|R^{\omega,\theta} - R^{\tilde{\omega},\tilde{\theta}}\|_{\alpha+\beta}, \\ \|\!|\theta|\!\|_{\omega,\boldsymbol{\alpha}} &:= \|\theta\|_{\omega,\boldsymbol{\alpha}} + |\partial_\omega\theta_0|, & \boldsymbol{d}^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\theta,\tilde{\theta}) &:= d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\theta,\tilde{\theta}) + |\partial_\omega\theta_0 - \partial_{\tilde{\omega}}\tilde{\theta}_0|. \end{aligned} \tag{2.10}$$

In particular, we note that

$$d_{\boldsymbol{\alpha}}^{\omega}(\theta,\tilde\theta) := d_{\boldsymbol{\alpha}}^{\omega,\omega}(\theta,\tilde\theta) = \|\theta-\tilde\theta\|_{\omega,\boldsymbol{\alpha}}, \qquad \boldsymbol{d}_{\boldsymbol{\alpha}}^{\omega}(\theta,\tilde\theta) := \boldsymbol{d}_{\boldsymbol{\alpha}}^{\omega,\omega}(\theta,\tilde\theta) = \|\!|\theta-\tilde\theta|\!\|_{\omega,\boldsymbol{\alpha}}. \tag{2.11}$$

By (2.2) one can easily check that

$$\begin{aligned} \Omega_{\alpha+\beta}(E) \subset \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E), &\quad \text{with } \partial_\omega\theta = 0 \text{ and } \|\theta\|_{\omega,\boldsymbol{\alpha}} = \|\theta\|_{\alpha+\beta},\ \forall\theta\in\Omega_{\alpha+\beta};\\ \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E) \subset \Omega_\alpha(E), &\quad \text{with } \|\theta\|_\alpha \le |\partial_\omega\theta_0|\|\omega\|_\alpha + T^\beta[1+\|\omega\|_\alpha]\|\theta\|_{\omega,\boldsymbol{\alpha}}\ \forall\theta\in\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E). \end{aligned} \tag{2.12}$$

We are now ready to define the rough path integration. For each $\boldsymbol{\omega}\in\boldsymbol{\Omega}_{\boldsymbol{a}}$, $\theta\in\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$, and each partition $\pi : 0 = t_0 < \cdots < t_n = T$, denote

$$\Theta^\pi_t := \sum_{i=0}^{n-1}\Big[\theta_{t_i\wedge t}\cdot\omega_{t_i\wedge t,t_{i+1}\wedge t} + \partial_\omega\theta_{t_i\wedge t} : \underline{\omega}_{t_i\wedge t,t_{i+1}\wedge t}\Big]. \tag{2.13}$$

Here, for $\theta = [\theta_1,\dots,\theta_d]^*$, we take the convention that $\partial_\omega\theta\in E^{d\times d}$ with ith row $\partial_\omega\theta_i$. Following Gubinelli [22], we may define the rough integral as the unique limit of Θ^π:

Lemma 2.5. *For each $\boldsymbol{\omega}\in\boldsymbol{\Omega}_{\boldsymbol{a}}$, $\theta\in\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$, the rough integral*

$$\int_0^t \theta_s\cdot d\boldsymbol{\omega}_s := \Theta_t := \lim_{|\pi|\to0}\Theta^\pi_t \in E \tag{2.14}$$

exists, and is independent of the choice of π. Moreover, $\Theta\in\mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ with $\partial_\omega\Theta = \theta^*$ and

$$\begin{aligned} &\Big|\Theta_{s,t} - \theta_s\cdot\omega_{s,t} - \partial_\omega\theta_s : \underline{\omega}_{s,t}\Big| \le C_{\boldsymbol{\alpha}}\|\boldsymbol{\omega}\|_\alpha\|\theta\|_{\omega,\boldsymbol{\alpha}}|t-s|^{2\alpha+\beta};\\ &\quad\|\Theta\|_{\omega,\boldsymbol{\alpha}} \le T^{\alpha-\beta}\|\boldsymbol{\omega}\|_\alpha|\partial_\omega\theta_0| + C_{\boldsymbol{\alpha}}T^\alpha[1+\|\boldsymbol{\omega}\|_\alpha]\|\theta\|_{\omega,\boldsymbol{\alpha}}, \end{aligned} \tag{2.15}$$

where the constant $C_{\boldsymbol{\alpha}}$ depends only on $\boldsymbol{\alpha}$ and the dimensions $|E|$ and d.

Proof. This result follows the same arguments in [18, Theorem 4.10], except that the second line of (2.15) appears slightly differently. To see that, by the first estimate we have

$$\|R^{\omega,\theta}\|_{\alpha+\beta} \le \|\partial_\omega\theta\|_\infty\|\boldsymbol{\omega}\|_\alpha T^{\alpha-\beta} + CT^\alpha\|\boldsymbol{\omega}\|_\alpha\|\theta\|_{\omega,\boldsymbol{\alpha}}.$$

Plug (2.2) with ω replaced by ∂_ω and α replaced by β into above and then use the inequality of (2.12). We obtain the second estimate of (2.15) immediately. ■

Moreover, we have the following stability result in terms of the rough integral, which improves [18, Theorem 4.16] slightly.

Lemma 2.6. *Let $(\boldsymbol{\omega},\theta,\Theta)$ be as in Lemma 2.5 and consider $(\tilde{\boldsymbol{\omega}},\tilde\theta,\tilde\Theta)$ similarly. Denote*

$$M := \|\theta\|_{\omega,\boldsymbol{\alpha}} + \|\tilde\theta\|_{\tilde\omega,\boldsymbol{\alpha}} + \|\boldsymbol{\omega}\|_\alpha + \|\tilde{\boldsymbol{\omega}}\|_\alpha, \quad \text{and} \quad \Delta\varphi := \tilde\varphi - \varphi, \quad \text{for } \varphi = \boldsymbol{\omega},\theta,\Theta.$$

Then, there exists a constant $C_{\boldsymbol{\alpha},M}$, depending on $\boldsymbol{\alpha}$, M, and $|E|$, d, such that

$$d_{\boldsymbol{\alpha}}^{\omega,\tilde\omega}(\Theta,\tilde\Theta) \le T^{\alpha-\beta}\Big[|\partial_\omega\tilde\theta_0|\|\Delta\boldsymbol{\omega}\|_\alpha + \|\boldsymbol{\omega}\|_\alpha|\Delta\partial_\omega\theta_0|\Big] + C_{\boldsymbol{\alpha},M}T^\alpha\Big[\|\Delta\boldsymbol{\omega}\|_\alpha + d_{\boldsymbol{\alpha}}^{\omega,\tilde\omega}(\theta,\tilde\theta)\Big].$$

Proof. First, similar to the first estimate in (2.15), or following the same arguments as in [18, Theorem 4.16], we have

$$\Big|[R^{\tilde\omega,\tilde\Theta}_{s,t} - \partial_\omega\tilde\theta_s : \underline{\tilde\omega}_{s,t}] - [R^{\omega,\Theta}_{s,t} - \partial_\omega\theta_s : \underline{\omega}_{s,t}]\Big| \le CT^\alpha\Big[\|\Delta\boldsymbol{\omega}\|_\alpha + d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta)\Big](t-s)^{\alpha+\beta}.$$

Note that, by (2.2),

$$\begin{aligned}|\partial_\omega\tilde\theta_s : \underline{\tilde\omega}_{s,t} - \partial_\omega\theta_s : \underline{\omega}_{s,t}| &\le \Big[\|\Delta\partial_\omega\theta\|_\infty\|\underline{\omega}\|_{2\alpha} + \|\partial_\omega\tilde\theta\|_\infty\|\Delta\underline{\omega}\|_{2\alpha}\Big](t-s)^{2\alpha}\\ &\le \Big[[|\Delta\partial_\omega\theta_0|\|\underline{\omega}\|_{2\alpha} + |\partial_\omega\tilde\theta_0|\|\Delta\underline{\omega}\|_{2\alpha}]\\ &\quad + CT^\beta[\|\Delta\partial_\omega\theta\|_\beta + \|\Delta\underline{\omega}\|_{2\alpha}]\Big](t-s)^{2\alpha}.\end{aligned}$$

Then we obtain the desired estimate for $\|R^{\tilde\omega,\tilde\Theta} - R^{\omega,\Theta}\|_{\alpha+\beta}$ immediately. Moreover,

$$\begin{aligned}|\Delta\partial_\omega\Theta_{s,t}| = |\Delta\theta_{s,t}| &= \Big|[\partial_\omega\tilde\theta_s\tilde\omega_{s,t} + R^{\tilde\omega,\tilde\theta}_{s,t}] - [\partial_\omega\theta_s\omega_{s,t} + R^{\omega,\theta}_{s,t}]\Big|\\ &\le \Big[\|\Delta\partial_\omega\theta\|_\infty\|\omega\|_\alpha + \|\partial_\omega\tilde\theta\|_\infty\|\Delta\omega\|_\alpha + T^\beta\|R^{\tilde\omega,\tilde\theta} - R^{\omega,\theta}\|_{\alpha+\beta}\Big](t-s)^\alpha.\end{aligned}$$

By (2.2) again we obtain the desired estimate for $\|\Delta\partial_\omega\Theta\|_\beta$, completing the proof. ■

We conclude this subsection with the Young's integration against $\langle\boldsymbol{\omega}\rangle$. Since $\langle\boldsymbol{\omega}\rangle \in \Omega_{2\alpha}(\mathbb{S}^d)$, by (2.3) the Young's integral $\theta_t : d\langle\boldsymbol{\omega}\rangle_t$ is well defined for all $\theta \in \Omega_\beta(E^{d\times d})$. We collect below some results concerning this integration. Since the proofs are standard and are much easier than Lemmas 2.5 and 2.6, we omit them.

Lemma 2.7. (i) *Let $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$, $\theta \in \Omega_\beta(E^{d\times d})$, $\Theta_t := \int_0^t \theta_s : d\langle\boldsymbol{\omega}\rangle_s$. Then $\Theta \in \Omega_{\alpha+\beta}(E)$ and*

$$\begin{aligned}&|\Theta_{s,t} - \theta_s : \langle\boldsymbol{\omega}\rangle_{s,t}| \le C\|\theta\|_\beta\|\langle\boldsymbol{\omega}\rangle\|_{2\alpha}(t-s)^{2\alpha+\beta},\\ &\|\Theta\|_{\alpha+\beta} \le \Big[T^{\alpha-\beta}|\theta_0| + CT^\alpha\|\theta\|_\beta\Big]\|\langle\boldsymbol{\omega}\rangle\|_{2\alpha}.\end{aligned} \tag{2.16}$$

(ii) *Let $(\tilde{\boldsymbol{\omega}}, \tilde\theta, \tilde\Theta)$ satisfy the same properties. Then, denoting $\Delta\varphi := \varphi - \tilde\varphi$ for $\varphi = \boldsymbol{\omega}, \theta, \Theta$,*

$$\begin{aligned}\|\Delta\Theta\|_{\alpha+\beta} &\le T^{\alpha-\beta}\|\langle\boldsymbol{\omega}\rangle\|_{2\alpha}|\Delta\theta_0|\\ &\quad + CT^\alpha\Big[\|\langle\boldsymbol{\omega}\rangle\|_{2\alpha}\|\Delta\theta\|_\beta + \|\tilde\theta\|_\beta\|\langle\boldsymbol{\omega}\rangle - \langle\tilde{\boldsymbol{\omega}}\rangle\|_{2\alpha}\Big].\end{aligned} \tag{2.17}$$

2.3. Path derivatives

We next introduce further path derivatives of θ. Our following definition is motivated from the path derivatives introduced in Ekren, Touzi and Zhang [15] and Buckdahn, Ma and Zhang [6], which in turn were motivated by the functional Itô calculus of Dupire [13].

Definition 2.8. For each $\boldsymbol{\omega} \in \boldsymbol{\Omega_a}$, the space $\mathcal{C}^2_{\boldsymbol{\omega},\boldsymbol{\alpha}}(E)$ consists of E-valued controlled rough paths $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ such that $\partial_\omega\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^{1\times d})$ and there exists symmetric $D^{\boldsymbol{\omega}}_t\theta \in \Omega_\beta(E^{d\times d})$ satisfying the following pathwise Itô formula:

$$\begin{aligned}&d\theta_t = \partial_\omega\theta_t d\boldsymbol{\omega}_t + \Big[D^{\boldsymbol{\omega}}_t\theta_t + \frac{1}{2}\partial^2_{\omega\omega}\theta_t\Big] : d\langle\boldsymbol{\omega}\rangle_t,\\ &\text{where } \partial^2_{\omega\omega}\theta_t := \partial_\omega[(\partial_\omega\theta_t)^*] \in E^{d\times d}.\end{aligned} \tag{2.18}$$

Remark 2.9. (i) In general $D_t^{\boldsymbol{\omega}}\theta$ may not be unique. Similar to (2.9), one can easily check that $D_t^{\boldsymbol{\omega}}\theta$ is unique if $\boldsymbol{\omega}$ is restricted to the following $\widehat{\boldsymbol{\Omega}}_{\boldsymbol{a}}$:

$$\widehat{\boldsymbol{\Omega}}_{\boldsymbol{a}} := \Big\{\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{a}} : \text{ there exists a dense subset } A \subset [0, T) \text{ such that}$$

$$\overline{\lim_{t\downarrow s}} \frac{|v : \langle\boldsymbol{\omega}\rangle_{s,t}|}{(t-s)^{2\alpha+\beta}} = \infty \text{ for all } s \in A \text{ and } v \in \mathbb{S}^d \setminus \{0\}\Big\}. \tag{2.19}$$

(ii) However, $\langle\boldsymbol{\omega}\rangle$ is more regular than $\boldsymbol{\omega}$, and thus (2.19) is much more difficult to satisfy than (2.9). For example, if $\boldsymbol{\omega}$ is a sample path of Brownian motion with Itô integration, then $\langle\boldsymbol{\omega}\rangle_t = tI_d$ as we will see in Section 5. In the case $d \geq 2$, by considering $v \in \mathbb{S}^d \setminus \{0\}$ with $\mathrm{Trace}(v) = 0$, we see that $\widehat{\boldsymbol{\Omega}}_{\boldsymbol{a}} = \emptyset$. In the case $d = 1$ however, we have $\widehat{\boldsymbol{\Omega}}_{\boldsymbol{a}} = \boldsymbol{\Omega}_{\boldsymbol{a}}$ because $v \neq 0$ and $2\alpha + \beta > 1$.

(iii) In many cases in this paper, θ already takes the form $d\theta_t = a_t \cdot d\boldsymbol{\omega}_t + b_t : d\langle\boldsymbol{\omega}\rangle_t$, then clearly $\partial_\omega\theta = a^*$ and we shall always set, thanks to the symmetry of $\langle\boldsymbol{\omega}\rangle$,

$$D_t^{\boldsymbol{\omega}}\theta := \frac{1}{2}\Big[\Big(b - \frac{1}{2}\partial_\omega a\Big) + \Big(b - \frac{1}{2}\partial_\omega a\Big)^*\Big]. \tag{2.20}$$

(iv) In the case that $\langle\boldsymbol{\omega}\rangle_t = t$, we will actually define $\partial_t^{\boldsymbol{\omega}}\theta := \mathrm{Trace}(D_t^{\boldsymbol{\omega}}\theta)$. Then we see that $\partial_t^{\boldsymbol{\omega}}\theta$ is unique (see Theorem 1, [20]). ■

Remark 2.10. (i) In general ∂_{ω^i} and ∂_{ω^j} do not commute, and $D_t^{\boldsymbol{\omega}}$ and ∂_ω are also not commutative. In particular, $\partial^2_{\omega\omega}\theta$ is not symmetric. However, since $\langle\boldsymbol{\omega}\rangle$ is symmetric, we see that (2.18) is equivalent to

$$d\theta_t = \partial_\omega\theta_t d\boldsymbol{\omega}_t + \Big[D_t^{\boldsymbol{\omega}}\theta_t + \frac{1}{4}[\partial^2_{\omega\omega}\theta_t + (\partial^2_{\omega\omega}\theta_t)^*]\Big] : d\langle\boldsymbol{\omega}\rangle_t. \tag{2.21}$$

(ii) One can easily check that the pathwise Itô formulae (2.18) and (2.21) are equivalent to the following pathwise Taylor expansion:

$$\begin{aligned}\theta_{s,t} &= \partial_\omega\theta_s\omega_{s,t} + \frac{1}{2}\partial^2_{\omega\omega}\theta_s : [\omega_{s,t}\omega^*_{s,t} + \underline{\omega}_{s,t} - \underline{\omega}^*_{s,t}]\\ &\quad + D_t^{\boldsymbol{\omega}}\theta_s : \langle\boldsymbol{\omega}\rangle_{s,t} + O((t-s)^{2\alpha+\beta}).\end{aligned} \tag{2.22}$$

In the case that $\partial^2_{\omega\omega}\theta$ is symmetric, which is always the case when $d = 1$, (2.22) becomes

$$\theta_{s,t} = \partial_\omega\theta_s\omega_{s,t} + \frac{1}{2}\partial^2_{\omega\omega}\theta_s : [\omega_{s,t}\omega^*_{s,t}] + D_t^{\boldsymbol{\omega}}\theta_s : \langle\boldsymbol{\omega}\rangle_{s,t} + O((t-s)^{2\alpha+\beta}). \tag{2.23}$$

We refer to [6] for related works in Brownian motion setting. ■

2.4. Backward rough integration

In this subsection we introduce the backward rough path, which is also a rough path and will play an important role in constructing the pathwise characteristics in Section 6. Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{a}}$ and $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$. For any $t_0 \in [0, T]$ and $0 \leq s \leq t \leq t_0$, define

$$\begin{aligned}&\overset{\leftarrow t_0}{\omega}_t := \omega_{t_0} - \omega_{t_0-t}, \quad \overset{\leftarrow t_0}{\underline{\omega}}_{s,t} := \omega_{t_0-t,t_0-s}\omega^*_{t_0-t,t_0-s} - \underline{\omega}_{t_0-t,t_0-s}, \quad \overset{\leftarrow t_0}{\boldsymbol{\omega}} := (\overset{\leftarrow t_0}{\omega}, \overset{\leftarrow t_0}{\underline{\omega}});\\ &\qquad \overset{\leftarrow t_0}{\theta}_t := \theta_{t_0-t}, \quad \overset{\leftarrow}{(\partial_\omega\theta)}{}^{t_0}_t := -\partial_\omega\theta_{t_0-t}.\end{aligned} \tag{2.24}$$

By restricting the processes on $[0, t_0]$ in obvious sense, we have

Lemma 2.11. *Let $\boldsymbol{\omega} \in \boldsymbol{\Omega_\alpha}$ and $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$. Then $\overset{\leftarrow t_0}{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^0_\alpha$, $\overset{\leftarrow t_0}{\theta} \in \mathcal{C}^1_{\overset{\leftarrow t_0}{\omega},\alpha}(E^d)$ with*

$$\partial_{\overset{\leftarrow t_0}{\boldsymbol{\omega}}} \overset{\leftarrow t_0}{\theta} = \overset{\leftarrow}{(\partial_\omega \theta)}^{t_0} \quad \textit{and} \quad \int_{t_0-t}^{t_0-s} \overset{\leftarrow t_0}{\theta}_r \cdot d\overset{\leftarrow t_0}{\boldsymbol{\omega}}_r = \int_s^t \theta_r \cdot d\boldsymbol{\omega}_r, \quad 0 \le s < t \le t_0. \tag{2.25}$$

Proof. In this proof we omit the superscript ^{t_0} and denote $t' := t_0 - t$, $s' := t_0 - s$, $r' := t_0 - r$, $\delta := t - s$. First, one can easily check that

$$\overset{\leftarrow}{\omega}_{s,t} = \omega_{t',s'}, \qquad \overset{\leftarrow}{\underline{\omega}}_{s,t} - \overset{\leftarrow}{\underline{\omega}}_{s,r} - \overset{\leftarrow}{\underline{\omega}}_{r,t} = \omega_{r',s'}\omega^*_{t',s'} = \overset{\leftarrow}{\omega}_{s,r}\overset{\leftarrow}{\omega}_{r,t}.$$

This implies that $\overset{\leftarrow}{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^0_\alpha$. Next,

$$\overset{\leftarrow}{\theta}_{s,t} = -\theta_{t',s'} = -\partial_\omega \theta_{t'}\omega_{t',s'} - R^{\omega,\theta}_{t',s'} = \overset{\leftarrow}{\partial_\omega \theta_s}\, \overset{\leftarrow}{\omega}_{s,t} + \partial_\omega \theta_{t',s'}\omega_{t',s'} - R^{\omega,\theta}_{t',s'}.$$

Then clearly $\overset{\leftarrow}{\partial_\omega \theta}$ is a Gubinelli derivative of $\overset{\leftarrow}{\theta}$ with respect to $\overset{\leftarrow}{\omega}$. Finally, the second equality of (2.25) is exactly the same as [18, Proposition 5.10]. ■

Remark 2.12. (i) Note that the $\overline{\lim}$ in (2.9) is taken from the right. Due to the time change, it is not clear that the backward rough path $\overset{\leftarrow t_0}{\boldsymbol{\omega}}$ will still be truly rough.

(ii) However, thanks to the additional regularity requirement of the path derivative, $\partial_{\overset{\leftarrow t_0}{\boldsymbol{\omega}}} \overset{\leftarrow t_0}{\theta}$ is still unique. Indeed, let η be an arbitrary path satisfying the desired properties of the path derivative $\partial_{\overset{\leftarrow t_0}{\boldsymbol{\omega}}} \overset{\leftarrow t_0}{\theta}$. Then, for $0 \le s < t \le t_0$,

$$\begin{aligned}\theta_{s,t} &= \overset{\leftarrow t_0}{\theta}_{t_0-t,t_0-s} = \eta_{t_0-t}\overset{\leftarrow}{\omega}_{t_0-t,t_0-s} + O(|t-s|^{\alpha+\beta})\\ &= \eta_{t_0-t}\omega_{s,t} + O(|t-s|^{\alpha+\beta}) = \eta_{t_0-s}\omega_{s,t} + O(|t-s|^{\alpha+\beta}).\end{aligned}$$

By the uniqueness of $\partial_\omega \theta$, we see that $\eta_{t_0-s} = \partial_\omega \theta_s$, and thus $\eta_s = \partial_\omega \theta_{t_0-s}$ is unique. ■

3. Functions of controlled rough paths

In this section we study functions $\varphi : \mathbb{T} \times \tilde{E} \to E$ and its related path derivatives. Similar to (2.18), we shall take the notational convention that

$$\partial_{yy}\varphi := \partial_y[(\partial_y \varphi)^*], \qquad \partial_{y\omega}\varphi := \partial_y[(\partial_\omega \varphi)^*], \qquad \partial_{\omega y}\varphi := \partial_\omega[(\partial_y \varphi)^*]. \tag{3.1}$$

Definition 3.1. (i) For $k \ge 0$, let $\mathcal{C}^k_{loc}(\tilde{E}, E)$ be the set of mappings $g : \mathbb{T} \times \tilde{E} \to E$ such that g is kth differentiable in y. Moreover, let $\mathcal{C}^k(\tilde{E}, E) \subset \mathcal{C}^k_{loc}(\tilde{E}, E)$ be such that

$$\|g\|_k := \sum_{i=0}^k \sup_{y \in \tilde{E}} \|\partial_y^{(i)} g(\cdot, y)\|_\infty < \infty. \tag{3.2}$$

(ii) For $k \ge 0$, let $\mathcal{C}^k_{\beta,loc}(\tilde{E}, E) \subset \mathcal{C}^k_{loc}(\tilde{E}, E)$ be such that, for $i = 0, \dots, k$, $\partial_y^{(i)} g$ is Hölder-β continuous in t, and the mapping $y \mapsto \partial_y^{(i)} g(\cdot, y)$ is continuous under $\|\cdot\|_\beta$. Moreover, let $\mathcal{C}^k_\beta(\tilde{E}, E) \subset \mathcal{C}^k_{\beta,loc}(\tilde{E}, E)$ be such that

$$\|g\|_{k,\beta} := \sum_{i=0}^k \sup_{y \in \tilde{E}} \|\partial_y^{(i)} g(\cdot, y)\|_\beta < \infty. \tag{3.3}$$

(iii) Let $\mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E) \subset \mathcal{C}^2_{loc}(\tilde E, E)$ be such that $g(\cdot, y) \in \mathcal{C}^1_{\omega,\alpha}(E)$, $\partial_y g(\cdot, y) \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^{1\times|\tilde E|})$, for each $y \in \tilde E$, the mappings $y \mapsto g(\cdot, y)$ and $y \mapsto \partial_y g(\cdot, y)$ are continuous under $\|\!|\cdot|\!\|_{\omega,\boldsymbol{\alpha}}$, and $\partial_\omega g \in \mathcal{C}^1_{\beta,loc}(\tilde E, E^{1\times d})$. Moreover, let $\mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha}}(\tilde E, E) \subset \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E)$ be such that

$$\|g\|_{2,\boldsymbol{\omega},\boldsymbol{\alpha}} := \|g\|_2 + \|\partial_\omega g\|_1 + \sup_{y\in\tilde E}[\|g(\cdot, y)\|_{\boldsymbol{\omega},\boldsymbol{\alpha}} + \|\partial_y g(\cdot, y)\|_{\boldsymbol{\omega},\boldsymbol{\alpha}}] < \infty. \tag{3.4}$$

(iv) Let $\mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E) \subset \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E)$ be such that $\partial_\omega g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E^{1\times d})$, $\partial_y g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E^{1\times|\tilde E|})$, $g(\cdot, y) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$ for every $y \in \tilde E$ and there exists $D^\omega_t g \in \mathcal{C}^1_{\beta,loc}(\tilde E, E^{d\times d})$. Moreover, let $\mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(\tilde E, E) \subset \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E)$ be such that

$$\|g\|_{3,\boldsymbol{\omega},\boldsymbol{\alpha}} := \|g\|_{2,\boldsymbol{\omega},\boldsymbol{\alpha}} + \|\partial_\omega g\|_{2,\boldsymbol{\omega},\boldsymbol{\alpha}} + \|\partial_y g\|_{2,\boldsymbol{\omega},\boldsymbol{\alpha}} < \infty. \tag{3.5}$$

(v) Let $\mathcal{C}^{3,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E) \subset \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E)$ be such that $\partial_\omega g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde E, E^{1\times d})$.

(vi) For $\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}} \in \boldsymbol{\Omega_a}$, and $g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha}}(\tilde E, E)$, $\tilde g \in \mathcal{C}^{1,2}_{\tilde\omega,\boldsymbol{\alpha}}(\tilde E, E)$ define

$$\begin{aligned} d^{\omega,\tilde\omega}_{2,\boldsymbol{\alpha}}(g, \tilde g) &:= \|g - \tilde g\|_2 + \|\partial_\omega g - \partial_{\tilde\omega}\tilde g\|_1 \\ &\quad + \sup_{y\in\tilde E}\Big[d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(g(\cdot, y), \tilde g(\cdot, y)) + d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\partial_y g(\cdot, y), \partial_y \tilde g(\cdot, y))\Big]. \end{aligned} \tag{3.6}$$

Remark 3.2. (i) For $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(\tilde E, E)$, by (2.18) we have

$$\begin{aligned} &dg(t, y) = h(t, y) \cdot d\boldsymbol{\omega}_t + f(t, y) : d\langle\boldsymbol{\omega}\rangle_t, \quad \text{where} \\ &h := (\partial_\omega g)^* \in \mathcal{C}^{1,2}_{\boldsymbol{\omega},\boldsymbol{\alpha},loc}(\tilde E, E^d), \qquad f := D^\omega_t g + \frac{1}{2}\partial_\omega h \in \mathcal{C}^1_{\beta,loc}(\tilde E, E^{d\times d}). \end{aligned} \tag{3.7}$$

(ii) In (3.4), we need only $\|\partial_\omega g\|_1$ instead of $\|\partial_\omega g\|_{1,\beta}$, and in (3.5), we do not need $\|D^\omega_t g\|_{1,\beta}$. The latter is particularly convenient because $D^\omega_t g$ may not be unique.

(iii) It is clear that $d^{\omega}_{2,\boldsymbol{\alpha}}(g, \tilde g) := d^{\omega,\omega}_{2,\boldsymbol{\alpha}}(g, \tilde g) = \|g - \tilde g\|_{2,\boldsymbol{\omega},\boldsymbol{\alpha}}$. ■

3.1. Commutativity of ∂_y and path derivatives

Lemma 3.3. (i) *Let $g \in \mathcal{C}^{2,3}_{\boldsymbol{\omega},\boldsymbol{\alpha}}(\tilde E, E)$. Then $\partial_{\omega y} g = [\partial_{y\omega} g]^* \in E^{|\tilde E|\times d}$, namely*

$$\partial_\omega \partial_{y_i} g = \partial_{y_i}\partial_\omega g, \quad i = 1, \ldots, |\tilde E|. \tag{3.8}$$

(ii) *Let $g \in \mathcal{C}^{3,3}_{\boldsymbol{\omega},\boldsymbol{\alpha}}(\tilde E, E)$. Then, for appropriate D^ω_t and for each $i = 1, \ldots, |\tilde E|$,*

$$\partial^2_{\omega\omega}\partial_{y_i} g = \partial_{y_i}\partial^2_{\omega\omega} g \quad \text{and} \quad D^{\boldsymbol{\omega}}_t \partial_{y_i} g = \partial_{y_i} D^{\boldsymbol{\omega}}_t g. \tag{3.9}$$

Proof. Without loss of generality, we assume $|\tilde E| = 1$, namely $\tilde E = \mathbb{R}$. Recall (3.7).

(i) Fix $y \in \mathbb{R}$ and denote, for $0 \neq \Delta y \in \mathbb{R}$,

$$\nabla\varphi_t(y) := \frac{\varphi(t, y + \Delta y) - \varphi(t, y)}{\Delta y}, \quad \varphi = g, h, f.$$

It is straightforward to check that

$$\nabla g_t(y) = \int_0^t \nabla h_s(y) \cdot d\boldsymbol{\omega}_s + \int_0^t \nabla f_s(y) : d\langle\boldsymbol{\omega}\rangle_s$$

$$\nabla h_t(y) = \int_0^1 \partial_y h(t, y + \lambda \Delta y) d\lambda, \qquad \nabla f_t(y) = \int_0^1 \partial_y f(t, y + \lambda \Delta y) d\lambda,$$

and thus, as $|\Delta y| \to 0$,

$$\|\!|\nabla h(y) - \partial_y h(y)\|\!|_{\omega,\boldsymbol{\alpha}} \le \int_0^1 \|\!|\partial_y h(y + \lambda\Delta y) - \partial_y h(y)\|\!|_{\omega,\boldsymbol{\alpha}} d\lambda \to 0,$$

$$\|\nabla f(y) - \partial_y f(y)\|_\beta \le \int_0^1 \|\partial_y f(y + \lambda\Delta y) - \partial_y f(y)\|_\beta d\lambda \to 0.$$

Then it follows from Lemmas 2.6 and 2.7(ii) that

$$\partial_y g(t, y) = \int_0^t \partial_y h(s, y) \cdot d\boldsymbol{\omega}_s + \int_0^t \partial_y f(s, y) : d\langle \boldsymbol{\omega} \rangle_s. \tag{3.10}$$

This implies (3.8) immediately.

(ii) Since $h \in \mathcal{C}^{2,3}_{\boldsymbol{\omega},\boldsymbol{\alpha}}(\tilde{E}, E^{1\times d})$, by (i) we have $\partial_y \partial_\omega h = \partial_\omega \partial_y h$ and thus $\partial_y \partial^2_{\omega\omega} g = \partial^2_{\omega\omega} \partial_y g$. Now applying the convention (2.20) for D^ω_t on (3.10) and by (3.7), we have

$$\begin{aligned} 2D^{\boldsymbol{\omega}}_t(\partial_y g) &= \Big(\partial_y f - \frac{1}{2}\partial_{\omega y} h\Big) + \Big(\partial_y f - \frac{1}{2}\partial_{\omega y} h\Big)^* \\ &= \partial_y\Big[\Big(f - \frac{1}{2}\partial_\omega h\Big) + \Big(f - \frac{1}{2}\partial_\omega h\Big)^*\Big] \\ &= \Big(\partial_y f - \frac{1}{2}\partial_{y\omega} h\Big) + \Big(\partial_y f - \frac{1}{2}\partial_{y\omega} h\Big)^* = 2\partial_y D^{\boldsymbol{\omega}}_t g. \end{aligned}$$

This completes the proof. ■

3.2. Chain rule of path derivatives

Theorem 3.4. (i) *Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{a}}$, $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(\tilde{E})$, $g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde{E}, E)$, and $\eta_t := g(t, \theta_t)$. Then*

$$\eta \in \mathcal{C}^1_{\omega,\alpha}(E) \quad \textit{with } \partial_\omega \eta_t = (\partial_\omega g)(t, \theta_t) + \partial_y g(t, \theta_t)\partial_\omega \theta_t. \tag{3.11}$$

(ii) *Assume further that $\theta \in \mathcal{C}^2_{\omega,\alpha}(\tilde{E})$ and $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(\tilde{E}, E)$. Then, for appropriate $D^{\boldsymbol{\omega}}_t$,*

$$\eta \in \mathcal{C}^2_{\omega,\alpha}(E) \quad \textit{with } D^{\boldsymbol{\omega}}_t \eta_t = (D^{\boldsymbol{\omega}}_t g)(t, \theta_t) + \partial_y g(t, \theta_t) D^{\boldsymbol{\omega}}_t \theta_t. \tag{3.12}$$

Remark 3.5. Similar to [6, Proposition 2.7], the chain rule of pathwise derivatives is equivalent to the Itô–Ventzell formula, which extends the Itô formula in [18, Proposition 5.6]. Indeed, note that $\theta \in \mathcal{C}^2_{\omega,\alpha}(\tilde{E})$ takes the form:

$$d\theta_t = a_t \cdot d\boldsymbol{\omega}_t + b_t : d\langle \boldsymbol{\omega} \rangle_t \quad \text{where } a := (\partial_\omega \theta)^*,\ b := D^{\boldsymbol{\omega}}_t \theta + \frac{1}{2}\partial_\omega a. \tag{3.13}$$

Recall (3.7) again. It follows from Lemma 3.3(i) that $\partial_\omega \partial_y g = (\partial_y h)^*$. Then, noticing that $h \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde{E}, E^d)$, $\partial_y g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha},loc}(\tilde{E}, E^{1\times|\tilde{E}|})$, by applying (3.11) several times and by (3.12), we have

$$\partial_\omega \eta_t = h^*(t, \theta_t) + \partial_y g(t, \theta_t) a^*_t,$$

$$\begin{aligned}\partial^2_{\omega\omega}\eta_t &= \partial_\omega[h(t,\theta_t)+\partial_y g(t,\theta_t)a_t]\\ &= \Big[\partial_\omega h+\partial_y h a^* + (\partial_y h a^*)^* + \partial^2_{yy} g\ [a,a] + \partial_y g \partial_\omega a\Big](t,\theta_t);\\ D^{\boldsymbol{\omega}}_t \eta_t &= \frac{1}{2}\Big[\Big[\Big(f-\frac{1}{2}\partial_\omega h\Big)+\Big(f-\frac{1}{2}\partial_\omega h\Big)^*\Big]\\ &\quad + \partial_y g\Big[\Big(b-\frac{1}{2}\partial_\omega a\Big)+\Big(b-\frac{1}{2}\partial_\omega a\Big)^*\Big]\Big](t,\theta_t).\end{aligned}$$

This, together with (2.18) and the symmetry of $\langle\boldsymbol{\omega}\rangle$, implies:

$$\begin{aligned} d[g(t,\theta_t)] &= \Big[h(t,\theta_t)+\partial_y g(t,\theta_t)a_t\Big]\cdot d\boldsymbol{\omega}_t\\ &\quad + \Big[f+\partial_y g b_t + \frac{1}{2}\partial^2_{yy} g\ [a_t,a_t] + \partial_y h a_t^*\Big](t,\theta_t) : d\langle\boldsymbol{\omega}\rangle_t, \end{aligned} \tag{3.14}$$

which we call the pathwise Itô–Ventzell formula. ■

Proof of Theorem 3.4. (i) For $(s,t)\in\mathbb{T}^2$, we have

$$\begin{aligned}\eta_{s,t} &= g(t,\theta_t)-g(s,\theta_s) = g(t,\theta_t)-g(s,\theta_t)+g(s,\theta_t)-g(s,\theta_s)\\ &= [\partial_\omega g](s,\theta_t)\omega_{s,t} + R^{\omega,g(\cdot,\theta_t)}_{s,t} + \int_0^1 \partial_y g(s,\theta_s+\lambda\theta_{s,t})d\lambda\theta_{s,t}\\ &= \Big[(\partial_\omega g)(s,\theta_s)+\partial_y g(s,\theta_s)\partial_\omega\theta_s\Big]\omega_{s,t} + R^{\omega,\eta}_{s,t},\end{aligned} \tag{3.15}$$

where

$$\begin{aligned} R^{\omega,\eta}_{s,t} &:= \Big[[\partial_\omega g](s,\theta_t)-[\partial_\omega g](s,\theta_s)\Big]\omega_{s,t} + R^{\omega,g(\cdot,\theta_t)}_{s,t}\\ &\quad + \int_0^1 [\partial_y g(s,\theta_s+\lambda\theta_{s,t})-\partial_y g(s,\theta_s)]d\lambda\partial_\omega\theta_s\omega_{s,t}\\ &\quad + \int_0^1 \partial_y g(s,\theta_s+\lambda\theta_{s,t})d\lambda R^{\omega,\theta}_{s,t}.\end{aligned}$$

Then clearly

$$\|R^{\omega,\eta}\|_{\alpha+\beta} \le \|g\|_{2,\omega,\boldsymbol{\alpha}}\Big[\|\theta\|_\beta\|\omega\|_\alpha + 1 + \|\theta\|_\beta\|\partial_\omega\theta\|_\infty\|\omega\|_\alpha + \|\theta\|_{\omega,\boldsymbol{\alpha}}\Big] < \infty. \tag{3.16}$$

Moreover, under our conditions it is clear that $(\partial_\omega g)(t,\theta_t)+\partial_y g(t,\theta_t)\partial_\omega\theta_t$ is Hölder-β-continuous. This proves (3.11).

(ii) Recall (3.7) and (3.13). By reversing the arguments in Remark 3.5, it suffices to prove (3.14). Denote $\delta := t-s$. Recall the first line of (3.15) and note that

$$\begin{aligned}&\theta_{s,t} = a_s\cdot\omega_{s,t} + \partial_\omega a_s : \underline{\omega}_{s,t} + b_s : \langle\boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta});\\ &g(t,y)-g(s,y) = h(s,y)\cdot\omega_{s,t} + \partial_\omega h(s,y) : \underline{\omega}_{s,t} + f(s,y) : \langle\boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta}).\end{aligned}$$

Then, by the standard Taylor expansion and applying Lemma 3.3(i) on g, we have

$$\begin{aligned} g(t,\theta_t)-g(t,\theta_s) &= \partial_y g(t,\theta_s)\theta_{s,t} + \frac{1}{2}\partial^2_{yy} g(t,\theta_s)\ [\theta_{s,t},\theta_{s,t}] + O(\delta^{3\alpha})\\ &= \Big[\partial_y g(s,\theta_s)+\partial_y h(s,\theta_s)\cdot\omega_{s,t}\Big]\theta_{s,t}\\ &\quad + \frac{1}{2}\partial^2_{yy} g(s,\theta_s)\ [\theta_{s,t},\theta_{s,t}] + O(\delta^{2\alpha+\beta});\end{aligned}$$

$$g(t,\theta_s) - g(s,\theta_s) = h(s,\theta_s)\cdot \omega_{s,t} + [\partial_\omega h](s,\theta_s) : \underline{\omega}_{s,t} + f(s,\theta_s) : \langle \boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta}).$$

On the other hand,

$$\begin{aligned}&\int_s^t [h(r,\theta_r) + \partial_y g(r,\theta_r) a_r]\cdot d\boldsymbol{\omega}_r \\ &\quad = [h(s,\theta_s) + \partial_y g(s,\theta_s)a_s]\cdot \omega_{s,t} + \partial_\omega[h(s,\theta_s) + \partial_y g(s,\theta_s)a_s] : \underline{\omega}_{s,t} + O(\delta^{2\alpha+\beta});\\ &\int_s^t [f(r,\theta_r) + \partial_y g(r,\theta_r) b_r] : d\langle\boldsymbol{\omega}\rangle_r = [f(s,\theta_s) + \partial_y g(s,\theta_s) b_s] : \langle\boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta}).\end{aligned}$$

By Lemma 3.3(i) we have $\partial_{\omega y} g = [\partial_{y\omega} g]^* = \partial_y h^*$. Then it follows from (3.11) that

$$\begin{aligned}&\partial_\omega[h(s,\theta_s) + \partial_y g(s,\theta_s)a_s]\\ &\quad = \Big[\partial_\omega h + \partial_y h a_s^* + \partial_y h^* a_s + \partial^2_{yy} g\ [a_s^*, a_s^*] + \partial_y g \partial_\omega a_s\Big](s,\theta_s).\end{aligned} \tag{3.17}$$

Noting that $\omega_{s,t} = O(\delta^\alpha)$, $\underline{\omega}_{s,t} = O(\delta^{2\alpha})$, and $\langle\boldsymbol{\omega}\rangle_{s,t} = O(\delta^{2\alpha})$, then we have

$$\begin{aligned}&\eta_{s,t} - \int_s^t [h(r,\theta_r) + \partial_y g(r,\theta_r)a_r]\cdot d\boldsymbol{\omega}_r - \int_s^t [f(r,\theta_r) + \partial_y g(r,\theta_r) b_r] : d\langle\boldsymbol{\omega}\rangle_r\\ &\quad = \Big[[\partial_y h(s,\theta_s)\cdot\omega_{s,t}][a_s\cdot\omega_{s,t}] + \frac{1}{2}\partial^2_{yy} g(t,\theta_s)\ [(a_s\cdot\omega_{s,t})^*, (a_s\cdot\omega_{s,t})^*]\\ &\qquad - \Big[\partial_y h(s,\theta_s)a_s^* + [\partial_y h(s,\theta_s)a_s^*]^* + \partial^2_{yy} g(s,\theta_s)\ [a_s^*, a_s^*]\Big]\Big] : \underline{\omega}_{s,t} + O(\delta^{2\alpha+\beta})\\ &\quad = \Big[\frac{1}{2}\partial^2_{yy} g(t,\theta_s)\ [\partial_\omega\theta_s, \partial_\omega\theta_s] + \partial_y h(s,\theta_s)\partial_\omega\theta_s\Big] : \langle\boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta}).\end{aligned}$$

This proves (3.14), and hence (3.12). ■

3.3. Some estimates

In this subsection we provide some estimates for $\eta = g(t,\theta_t)$, which will be crucial for studying rough differential equations in next section. These results correspond to [18, Lemma 7.3 and Theorem 7.5], where g does not depend on t.

Lemma 3.6. (i) *Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{a}}$, $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$, $g \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha}}(\tilde{E}, E)$, $\eta_t := g(t,\theta_t)$, and denote*

$$M_1 := \|\boldsymbol{\omega}\|_\alpha + |||\theta|||_{\omega,\boldsymbol{\alpha}}.$$

Then for any $T_0 > 0$ and any $T \le T_0$, there exists a constant $C_{\boldsymbol{\alpha},M_1,T_0}$, depending only on $\boldsymbol{\alpha}$, M_1, T_0, and $|E|$, $|\tilde{E}|$, such that

$$\|\eta\|_{\omega,\boldsymbol{\alpha}} \le C_{\boldsymbol{\alpha},M_1,T_0}\|g\|_{2,\omega,\boldsymbol{\alpha}}. \tag{3.18}$$

(ii) *Assume further that $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(\tilde{E}, E)$, and $(\tilde{\boldsymbol{\omega}}, \tilde{\theta}, \tilde{g}, \tilde{\eta})$ satisfies the same conditions. Denote $\Delta\varphi := \tilde{\varphi} - \varphi$ for appropriate φ, and*

$$M_2 := |||\theta|||_{\omega,\boldsymbol{\alpha}} + |||\tilde{\theta}|||_{\tilde{\omega},\boldsymbol{\alpha}} + \|\boldsymbol{\omega}\|_\alpha + \|\tilde{\boldsymbol{\omega}}\|_\alpha + \|g\|_{3,\omega,\boldsymbol{\alpha}} + \|\tilde{g}\|_{3,\tilde{\omega},\boldsymbol{\alpha}}.$$

Then, for any $T \le T_0$ as in (i)*, there exists a constant $C_{\boldsymbol{\alpha},M_2,T_0}$ such that*

$$d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\eta,\tilde{\eta}) \le C_{\boldsymbol{\alpha},M_2,T_0}\Big[d^{\omega,\tilde{\omega}}_{2,\boldsymbol{\alpha}}(g,\tilde{g}) + \boldsymbol{d}^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\theta,\tilde{\theta}) + |\Delta\theta_0| + \|\Delta\boldsymbol{\omega}\|_\alpha\Big]. \tag{3.19}$$

Proof. (i) First, by (2.2) and (2.12) we have $\|\partial_\omega\theta\|_\infty + \|\theta\|_\beta \le C$. By the first line of (3.15) it is clear that

$$\|\eta\|_\beta \le C\Big[\|g\|_{0,\beta} + \|g\|_1\Big]. \tag{3.20}$$

Next, recall (3.11) and note that

$$\begin{aligned}|\partial_\omega\eta_{s,t}| \le\ & |\partial_\omega g(t,\theta_t) - \partial_\omega g(s,\theta_s)| + |\partial_y g(t,\theta_t) - \partial_y g(s,\theta_s)|\,|\partial_\omega\theta_t| \\ & + |\partial_y g(s,\theta_s)|\,|\partial_\omega\theta_{s,t}|.\end{aligned}$$

Applying (3.20) on $\partial_\omega g$ and $\partial_y g$ we obtain $\|\partial_\omega\eta\|_\beta \le C\|g\|_{2,\omega,\boldsymbol{\alpha}}$. Moreover, by (3.16) we have $\|R^{\omega,\eta}\|_{\alpha+\beta} \le C\|g\|_{2,\omega,\boldsymbol{\alpha}}$. Putting together we prove (3.18).

(ii) First, note that

$$\begin{aligned}\Delta\eta_{s,t} &= \tilde g(t,\tilde\theta_t) - g(t,\theta_t) - \tilde g(s,\tilde\theta_s) + g(s,\theta_s) \\ &= [\Delta g(t,\tilde\theta_t) - \Delta g(s,\tilde\theta_s)] + \int_0^1 \partial_y g(s,\theta_s + \lambda\Delta\theta_s)d\lambda\Delta\theta_{s,t} \\ &\quad + \int_0^1 [\partial_y g(t,\theta_t+\lambda\Delta\theta_t) - \partial_y g(s,\theta_s+\lambda\Delta\theta_s)]d\lambda\Delta\theta_t.\end{aligned}$$

Apply (3.20) on Δg and $\partial_y g$, we obtain

$$\|\Delta\eta\|_\beta \le C\Big[\|\Delta g\|_{0,\beta} + \|\Delta g\|_1 + \|\Delta\theta\|_\beta + |\Delta\theta_0|\Big].$$

Note that $\theta_{s,t} = \partial_\omega\theta_s\omega_{s,t} + R^{\omega,\theta}_{s,t}$, and similarly for $\tilde\theta$. Then, by (2.2),

$$\begin{aligned}\|\Delta\theta\|_\beta &\le \|\partial_{\tilde\omega}\tilde\theta - \partial_\omega\theta\|_\infty\|\tilde\omega\|_\beta + \|\partial_\omega\theta\|_\infty\|\Delta\omega\|_\beta + \|R^{\tilde\omega,\tilde\theta} - R^{\omega,\theta}\|_\beta \\ &\le C\Big[\boldsymbol{d}^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) + \|\Delta\omega\|_\alpha\Big].\end{aligned} \tag{3.21}$$

Thus

$$\|\Delta\eta\|_\beta \le C\Big[\|\Delta g\|_{0,\beta} + \|\Delta g\|_1 + |\Delta\theta_0| + \boldsymbol{d}^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) + \|\Delta\omega\|_\alpha\Big]. \tag{3.22}$$

We shall emphasize that the above C depends on $\|g\|_{2,\omega,\boldsymbol{\alpha}} + \|\tilde g\|_{2,\tilde\omega,\boldsymbol{\alpha}}$, not $\|g\|_{3,\omega,\boldsymbol{\alpha}} + \|\tilde g\|_{3,\tilde\omega,\boldsymbol{\alpha}}$.

Next, note that

$$\begin{aligned}\partial_{\tilde\omega}\tilde\eta_t - \partial_\omega\eta_t &= [\partial_{\tilde\omega}\tilde g(t,\tilde\theta_t) - \partial_\omega g(t,\theta_t)] + [\partial_y\tilde g(t,\tilde\theta_t) - \partial_y g(t,\theta_t)]\partial_{\tilde\omega}\tilde\theta_t \\ &\quad + \partial_y g(t,\theta_t)[\partial_{\tilde\omega}\tilde\theta_t - \partial_\omega\theta_t]. \\ [\partial_{\tilde\omega}\tilde\eta - \partial_\omega\eta]_{s,t} &= [\partial_{\tilde\omega}\tilde g(\cdot,\tilde\theta_\cdot) - \partial_\omega g(\cdot,\theta_\cdot)]_{s,t} + [\partial_y\tilde g(\cdot,\tilde\theta_\cdot) - \partial_y g(\cdot,\theta_\cdot)]_{s,t}\partial_\omega\tilde\theta_t \\ &\quad + [\partial_y\Delta g(s,\tilde\theta_s) + \partial_y g(s,\tilde\theta_s) - \partial_y g(s,\theta_s)]\partial_{\tilde\omega}\tilde\theta_{s,t} \\ &\quad + [\partial_y g(\cdot,\theta_\cdot)]_{s,t}\Delta\partial_\omega\theta_t + \partial_y g(s,\theta_s)\Delta\partial_\omega\theta_{s,t}.\end{aligned}$$

Apply (3.22) on $\partial_\omega g$ and $\partial_y g$, and (3.20) on $\partial_y g$, we obtain from (3.21) that

$$\|\Delta\partial_\omega\eta\|_\alpha \le C\Big[d^{\omega,\tilde\omega}_{2,\boldsymbol{\alpha}}(g,\tilde g) + |\Delta\theta_0| + \boldsymbol{d}^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) + \|\Delta\omega\|_\alpha\Big]. \tag{3.23}$$

Finally, recall (3.16) and note that

$$\begin{aligned}R^{\tilde\omega,\tilde g(\cdot,\tilde y)}_{s,t} - R^{\omega,g(\cdot,y)}_{s,t} &= R^{\tilde\omega,\tilde g(\cdot,\tilde y)}_{s,t} - R^{\omega,g(\cdot,\tilde y)}_{s,t} + \Big[[g(\cdot,\tilde y)]_{s,t} - \partial_\omega g(s,\tilde y)\omega_{s,t}\Big]\\ &\quad -\Big[[g(\cdot,y)]_{s,t} - \partial_\omega g(s,y)\omega_{s,t}\Big]\\ &= R^{\tilde\omega,\tilde g(\cdot,\tilde y)}_{s,t} - R^{\omega,g(\cdot,\tilde y)}_{s,t} + \int_0^1 R^{\omega,\partial_y g(\cdot,y+\lambda\Delta y)}_{s,t} d\lambda \Delta y,\end{aligned}$$

one can obtain the desired estimate for $\|R^{\tilde\omega,\tilde\eta} - R^{\omega,\eta}\|_{\alpha+\beta}$ straightforwardly. This, together with (3.23), completes the proof. ■

Moreover, we have the following simpler results whose proof is omitted.

Lemma 3.7. (i) *Let $\theta \in \Omega_\beta(E)$, $f \in \mathcal{C}^1_\beta(\tilde E, E)$, and $\eta_t := f(t,\theta_t)$. Then $\eta \in \Omega_\beta(E)$ and*

$$\|\eta\|_\beta \le \|f\|_{0,\beta} + \|f\|_1\|\theta\|_\beta \le \|f\|_{1,\beta}[1+\|\theta\|_\beta]. \tag{3.24}$$

(ii) *Let $\theta, \tilde\theta \in \Omega_\beta(E)$, $f, \tilde f \in \mathcal{C}^2_\beta(\tilde E, E)$, and $\eta_t := f(t,\theta_t)$, $\tilde\eta := \tilde f(t,\tilde\theta_t)$. Then*

$$\|\tilde\eta - \eta\|_\beta \le [1+\|\theta\|_\beta + \|\tilde\theta\|_\beta]\Big[\|\tilde f - f\|_{1,\beta} + \|f\|_2[|\tilde\theta_0 - \theta_0| + \|\tilde\theta - \theta\|_\beta]\Big]. \tag{3.25}$$

4. Rough differential equations

In this section we study rough path differential equations with coefficients less regular in the time variable t, motivated from our study of stochastic differential equations with random coefficients in next section. Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{\alpha}}$, $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(E, E^d)$, $f \in \mathcal{C}^2_\beta(E, E^{d\times d})$, and $y_0 \in E$. Consider the following RDE:

$$\theta_t = y_0 + \int_0^t g(s,\theta_s)\cdot d\boldsymbol{\omega}_s + \int_0^t f(s,\theta_s) : d\langle\boldsymbol{\omega}\rangle_s, \quad t \in \mathbb{T}. \tag{4.1}$$

Our goal is to find solution $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$. By Theorem 3.4 and Lemma 3.7, in this case $g(\cdot,\theta) \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$, $f(\cdot,\theta) \in \Omega_\beta(E^{d\times d})$, and thus the right side of (4.1) is well defined.

Remark 4.1. When $\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ is a solution, clearly $\partial_\omega\theta_t = g(t,\theta_t)$, then by Theorem 3.4(i) it is clear that $\theta \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$. So a solution to RDE (4.1) is automatically in $\mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$. We shall use this fact without mentioning it. ■

In standard rough path theory the vector field g of RDE (4.1) is independent of t. In Lejay and Victoir [28], g may depend on t, but is required to be Hölder-$(1-\alpha)$ continuous, which is violated for $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(E, E^d)$ (since $\alpha < \frac{1}{2}$). This relaxation of regularity in t is crucial for studying SDEs and SPDEs with random coefficients, see Remark 5.7. We also refer to Gubinelli, Tindel and Torrecilla [23] for some discussion along this direction.

Theorem 4.2. *Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\boldsymbol{\alpha}}$, $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(E, E^d)$, $f \in \mathcal{C}^2_\beta(E, E^{d\times d})$, and $y_0 \in E$. Then RDE (4.1) has a unique solution $\theta \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$. Moreover, there exists a constant $C_{\boldsymbol{\alpha}}$, depending only on $\boldsymbol{\alpha}$, d, $|E|$, T, $\|f\|_{2,\beta}$, $\|g\|_{3,\omega,\boldsymbol{\alpha}}$, and $\|\boldsymbol{\omega}\|_\alpha$, such that*

$$\|\theta\|_\alpha + \|\theta\|_{\omega,\boldsymbol{\alpha}} \le C_{\boldsymbol{\alpha}}. \tag{4.2}$$

Furthermore, the constant $C_{\boldsymbol{\alpha}}$ is bounded for $d, |E|, T, \|f\|_{2,\beta}, \|g\|_{3,\omega,\boldsymbol{\alpha}}$, and $\|\boldsymbol{\omega}\|_\alpha$ bounded from above and for α and β bounded from below.

Proof. We proceed in three steps.

Step 1. Denote $M := [\|\partial_\omega g\|_0 + \|g\|_1^2]\|\boldsymbol{\omega}\|_\alpha + \|f\|_0\|\boldsymbol{\omega}\|_\alpha[2 + \|\boldsymbol{\omega}\|_\alpha]$ and

$$\mathcal{A}_{\boldsymbol{\alpha}} := \Big\{\theta \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E) : \theta_0 = y_0, \partial_\omega\theta_0 = g^*(0, y_0), \|\theta\|_{\omega,\boldsymbol{\alpha}} \le M + 1\Big\}, \tag{4.3}$$

equipped with the norm $\|\cdot\|_{\omega,\boldsymbol{\alpha}}$. Note that $\mathcal{A}_{\boldsymbol{\alpha}}$ contains $\theta_t := y_0 + g(0, y_0) \cdot \omega_{0,t}$ and thus is not empty. Define a mapping Φ on $\mathcal{A}_{\boldsymbol{\alpha}}$:

$$\Phi(\theta) := \Theta$$
$$\text{where } \Theta_t := y_0 + \Theta^1_t + \Theta^2_t := y_0 + \int_0^t g(s, \theta_s) \cdot d\boldsymbol{\omega}_s + \int_0^t f(s, \theta_s) : d\langle\boldsymbol{\omega}\rangle_s.$$

We show that, there exists $0 < \delta \le 1$, which depends on $\boldsymbol{\alpha}, d, |E|, T, \|f\|_{2,\beta}, \|g\|_{3,\omega,\boldsymbol{\alpha}}$, and $\|\boldsymbol{\omega}\|_\alpha$, but not on y_0, such that whenever $T \le \delta$, Φ is a contraction mapping on $\mathcal{A}_{\boldsymbol{\alpha}}$. One can easily check that $\mathcal{A}_{\boldsymbol{\alpha}}$ is complete under $d^{\omega,\omega}_{\boldsymbol{\alpha}}$, then Φ has a unique fixed point $\theta \in \mathcal{A}_{\boldsymbol{\alpha}}$ which is clearly the unique solution of RDE (4.1).

To prove that Φ is a contraction mapping, let C denote a generic constant which depends only on the above parameters, but not on y_0. We first show that $\Phi(\theta) \in \mathcal{A}_{\boldsymbol{\alpha}}$ for all $\theta \in \mathcal{A}_{\boldsymbol{\alpha}}$. Indeed, clearly $\Theta_0 = y_0$ and $\partial_\omega\theta_0 = g^*(0, y_0)$. For any $\theta \in \mathcal{A}_{\boldsymbol{\alpha}}$, denote $\eta_t := g(t, \theta_t)$. Applying Lemma 3.6 and then Lemma 2.5, we have,

$$\|\eta\|_{\omega,\boldsymbol{\alpha}} \le C, \qquad |\partial_\omega\eta_0| \le \|\partial_\omega g\|_0 + \|\partial_\omega g\|_1^2, \quad \text{and thus}$$
$$\|\Theta^1\|_{\omega,\boldsymbol{\alpha}} \le \|\boldsymbol{\omega}\|_\alpha|\partial_\omega\eta_0| + C\delta^\alpha[1 + \|\boldsymbol{\omega}\|_\alpha]\|\eta\|_{\omega,\boldsymbol{\alpha}} \le [\|\partial_\omega g\|_0 + \|g\|_1^2]\|\boldsymbol{\omega}\|_\alpha + C\delta^\alpha.$$

Similarly, it follows from Lemmas 2.7 and 3.7(i) that

$$\|\Theta^2\|_{\omega,\boldsymbol{\alpha}} = \|\Theta^2\|_{\alpha+\beta} \le \|f\|_0\|\boldsymbol{\omega}\|_\alpha[2 + \|\boldsymbol{\omega}\|_\alpha] + C\delta^\alpha,$$
$$\text{and thus } \|\Theta\|_{\omega,\boldsymbol{\alpha}} \le \|\Theta^1\|_{\omega,\boldsymbol{\alpha}} + \|\Theta^2\|_{\omega,\boldsymbol{\alpha}} \le M + C\delta^\alpha.$$

Set δ small enough we have $\|\Theta\|_{\omega,\boldsymbol{\alpha}} \le M + 1$. That is, $\Theta \in \mathcal{A}_{\boldsymbol{\alpha}}$.

Next, let $\tilde\theta \in \mathcal{A}_{\boldsymbol{\alpha}}$ and denote $\tilde\Theta, \tilde\Theta^1, \tilde\Theta^2, \tilde\eta$ in obvious sense. Let $\Delta\varphi := \tilde\varphi - \varphi$ for appropriate φ. Recall (3.21) we see that

$$\|\Delta\theta\|_\infty \le C\delta^\beta\|\Delta\theta\|_\beta \le C\delta^\beta\|\Delta\theta\|_{\omega,\boldsymbol{\alpha}}. \tag{4.4}$$

Then, applying Lemmas 2.6, 3.6(ii), 2.7(ii), and 3.7(ii), we have

$$\|\Delta\Theta^1\|_{\omega,\boldsymbol{\alpha}} \le C\delta^\alpha\|\Delta\eta\|_{\omega,\boldsymbol{\alpha}} \le C\delta^\alpha\|\Delta\theta\|_{\omega,\boldsymbol{\alpha}}, \qquad \|\Delta\Theta^2\|_{\alpha+\beta} \le C\delta^\alpha\|\Delta\theta\|_\beta,$$
$$\text{and thus } \|\Delta\Theta\|_{\omega,\boldsymbol{\alpha}} \le C\delta^\alpha\|\Delta\theta\|_{\omega,\boldsymbol{\alpha}}.$$

Set δ be small enough such that $C\delta^\alpha \le \frac{1}{2}$, then Φ is a contraction mapping.

Step 2. We now prove the result for general T. Let δ be the constant in Step 1. Let $0 = t_0 < \cdots < t_n = T$ such that $t_{i+1} - t_i \le \delta$, $i = 0, \ldots, n-1$. We may solve the RDE over each interval $[t_i, t_{i+1}]$ with initial condition $(\theta_{t_i}, g(t_i, \theta_{t_i}))$, which is obtained from the previous step by considering the RDE on $[t_{i-1}, t_i]$, and thus we obtain the unique solution over the whole interval $[0, T]$.

Step 3. We now estimate $\|\theta\|_{\omega,\boldsymbol{\alpha}}$. First, when $T \le \delta$ for the constant $\delta = \delta_{\boldsymbol{\alpha}}$ in *Step 1*, we have $\theta \in \mathcal{A}_{\boldsymbol{\alpha}}$ and thus $\|\theta\|_{\omega,\boldsymbol{\alpha}} \le M + 1$. In particular, this implies that

$$|\partial_\omega\theta_{s,t}| \le (M + 1)(t - s)^\beta, \qquad |R^{\omega,\theta}_{s,t}| \le (M + 1)(t - s)^{\alpha+\beta}, \quad \text{whenever } t - s \le \delta.$$

Now for arbitrary s,t, let $k := [\frac{t-s}{\delta}] + 1$ be the smallest integer greater than $\frac{t-s}{\delta}$, and $t_i := s + \frac{i}{k}(t-s),\ i = 0,\dots,k$. Then

$$\begin{aligned}|\partial_\omega \theta_{s,t}| &\le \sum_{i=0}^{k-1} |\partial_\omega \theta_{t_i,t_{i+1}}| \le (M+1)k\Big(\frac{t-s}{k}\Big)^\beta\\ &= (M+1)k^{1-\beta}(t-s)^\beta \le (M+1)(\delta^{-1}T+1)^{1-\beta}(t-s)^\beta.\end{aligned}$$

Thus we have $\|\partial_\omega\theta\|_\beta \le (M+1)(\delta^{-1}T+1)^{1-\beta}$. Similarly we may prove that $\|R^{\omega,\theta}\|_{\alpha+\beta} \le (M+1)(\delta^{-1}T+1)^{1-\alpha-\beta}$.

Finally, note that $\|\partial_\omega\theta\|_\infty \le C$, it is clear that $\|\theta\|_\alpha \le \|\partial_\omega\theta\|_\infty\|\omega\|_\alpha + \|R^{\omega,\theta}\|_\alpha \le C$. ■

We next study the stability of RDEs.

Theorem 4.3. *Let $(y_0, \boldsymbol{\omega}, f, g)$ and $(\tilde y_0, \tilde{\boldsymbol{\omega}}, \tilde f, \tilde g)$ be as in Theorem 4.2, and $\theta, \tilde\theta$ be the corresponding solution of the RDE. Then there exists a constant $C_{\boldsymbol{\alpha}}$, depending only on $\boldsymbol{\alpha}, d, |E|, T, \|f\|_{2,\beta}, \|\tilde f\|_{2,\beta}, \|g\|_{3,\omega,\boldsymbol{\alpha}}, \|\tilde g\|_{3,\tilde\omega,\boldsymbol{\alpha}}$, and $\|\boldsymbol{\omega}\|_\alpha, \|\tilde{\boldsymbol{\omega}}\|_\alpha$, such that, denoting $\Delta\varphi := \varphi - \tilde\varphi$ for appropriate φ,*

$$d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) \le C_{\boldsymbol{\alpha}}[\Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|] \quad \text{where } \Delta I_{\boldsymbol{\alpha}} := d^{\omega,\tilde\omega}_{2,\boldsymbol{\alpha}}(g,\tilde g) + \|\Delta f\|_{1,\beta} + \|\Delta\boldsymbol{\omega}\|_\alpha. \tag{4.5}$$

Proof. First assume $T \le \delta$ for some constant $\delta > 0$ small enough. Use the notations in Step 1 of Theorem 4.2. Applying Lemma 3.6(i) and (4.2) we see that $|\partial_{\tilde\omega}\tilde\eta_0| + \|\tilde\eta\|_{\omega,\beta} \le C$. Then, it follows from Lemmas 2.6 and 3.6(ii) that

$$\begin{aligned}d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\Theta^1,\tilde\Theta^1) &\le C\Big[\delta^\alpha d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\eta,\tilde\eta) + d_\alpha(\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}) + |\eta'_0 - \tilde\eta'_0|\Big]\\ &\le C\Big[\delta^\alpha d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) + \Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|\Big].\end{aligned}$$

Similarly, by Lemmas 2.7 and 3.7, we have

$$\|\Delta\Theta^2\|_{\alpha+\beta} \le C\Big[\delta^\alpha\|\Delta\theta\|_\beta + \Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|\Big].$$

Putting together we get

$$d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) = d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\Theta,\tilde\Theta) \le C\Big[\delta^\alpha d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) + \Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|\Big].$$

Set δ be small enough such that $C\delta^\alpha \le \frac{1}{2}$, we obtain $d^{\omega,\tilde\omega}_{\boldsymbol{\alpha}}(\theta,\tilde\theta) \le C[\Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|]$.

Now for general T, let $k := [\frac{T}{\delta}] + 1$ be the smallest integer greater than $\frac{T}{\delta}$ and $t_i := \frac{i}{k}T,\ i = 0,\dots,k$. Denote

$$\Delta J_i := \sup_{t_i \le s < t \le t_{i+1}} \Big[\frac{|\Delta\partial_\omega\theta_{s,t}|}{(t-s)^\beta} + \frac{|R^{\tilde\omega,\tilde\theta}_{s,t} - R^{\omega,\theta}_{s,t}|}{(t-s)^{\alpha+\beta}}\Big], \quad i = 0,\dots,k-1.$$

By the above arguments we have $\Delta J_i \le C[\Delta I_{\boldsymbol{\alpha}} + |\Delta\theta_{t_i}|]$. Then, applying (3.21) on $[t_i, t_{i+1}]$ and noting that $\partial_\omega\theta_{t_i} = g(t_i,\theta_{t_i})$ and $\partial_\omega\tilde\theta_{t_i} = \tilde g(t_i,\tilde\theta_{t_i})$ are bounded, we have

$$\begin{aligned}|\Delta\theta_{t_{i+1}}| &\le |\Delta\theta_{t_i}| + |\Delta\theta_{t_i,t_{i+1}}| \le |\Delta\theta_{t_i}| + \Delta J_i + C[|\Delta\partial_\omega\theta_{t_i}| + \|\Delta\omega\|_\alpha]\\ &\le C[\Delta I_{\boldsymbol{\alpha}} + |\Delta\theta_{t_i}|].\end{aligned}$$

By induction we get

$$\max_{0\le i\le k} |\Delta\theta_{t_i}| \le C[\Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|], \quad \text{and thus} \quad \max_{0\le i\le k} \Delta J_i \le C[\Delta I_{\boldsymbol{\alpha}} + |\Delta y_0|].$$

Now following the arguments in Theorem 4.2 Step 3 we can prove the desired estimate. ■

Remark 4.4. (i) The uniqueness of RDE solutions does not depend on boundedness of g, $\partial_\omega g$, and f. Indeed, let θ and $\tilde\theta$ be two solutions. Notice that any element of $\mathcal{C}^1_{\omega,\alpha}(E)$ is bounded, and thus we may denote $M_0 := \|\theta\|_\infty + \|\tilde\theta\|_\infty < \infty$. One can see that all the arguments in Theorem 4.2 remain valid if we replace the $\sup_{y\in E}$ in (3.2) with $\sup_{y\in E,|y|\le M_0}$, while the latter is always bounded for g, $\partial_\omega g$, and f.

(ii) If we do not assume boundedness of g, $\partial_\omega g$, and f, in general we can only obtain the local existence, namely the solution exists when T is small. However, if we can construct a solution for large T, as we will see for linear RDEs, then by (ii) above this solution is the unique solution. ■

4.1. Linear RDE

Now consider RDE (4.1) with

$$\begin{gathered} g(t,y) = a_t y + b_t, \qquad f(t,y) = \lambda_t y + l_t, \quad \text{where} \\ y \in E, \quad a \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E^{d\times|E|}), \quad b \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d), \quad \lambda \in \Omega_\beta(E^{d\times d\times|E|}), \quad l \in \Omega_\beta(E^{d\times d}). \end{gathered} \tag{4.6}$$

We remark that the above f and g are not bounded and thus we cannot apply Theorem 4.2 directly. In Friz and Victoir [21], some a priori estimate is provided for linear RDEs and then the global existence follows from the arguments of Theorem 4.2, by replacing the $\sup_{y\in E}$ in (3.2) with the supremum over the a priori bound of the solution, as illustrated in Remark 4.4(ii). Below, we shall construct a solution semi-explicitly. When $|E| = 1$, we have an explicit representation in the spirit of Feyman–Kac formula in stochastic analysis literature, see (4.7). However, the formula fails in the multidimensional case due to the noncommutativity of matrices. Our main idea is to introduce a decoupling strategy, by using the local solution of certain Riccati type of RDEs, so as to reduce the dimension of E. To our best knowledge, such a construction is new even for multidimensional linear SDEs.

Theorem 4.5. *The linear RDE (4.1) with (4.6) has a unique solution.*

Proof. If $b \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E^d)$, under (4.6) it is straightforward to check that $g \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha},loc}(E, E^d)$ and $f \in \mathcal{C}^2_{\beta,loc}(E, E^{d\times d})$, and thus the uniqueness follows from Theorem 4.2 and Remark 4.4(ii). However, in the linear case, by going through the arguments of Theorem 4.2 we can easily see that it is enough to assume the weaker condition $b \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$. We shall construct the solution and thus obtain the existence via induction on $|E|$.

Step 1. We first assume $|E| = 1$, namely $E = \mathbb{R}$. Applying Theorem 3.4 and Remark 3.5 we may verify directly that the following provides a representation of the solution:

$$\theta_t = \Gamma_t^{-1}\Big[\theta_0 + \int_0^t \Gamma_s b_s \cdot d\boldsymbol{\omega}_s + \int_0^t \Gamma_s\big[l_s - a_s b_s^*\big] : d\langle\boldsymbol{\omega}\rangle_s\Big], \tag{4.7}$$

$$\text{where } \Gamma_t := \exp\Big(-\int_0^t a_s \cdot d\boldsymbol{\omega}_s + \int_0^t \Big[\frac{1}{2}a_s a_s^* - \lambda_s\Big] : d\langle\boldsymbol{\omega}\rangle_s\Big).$$

Step 2. In order to show the induction idea clearly, we present the case $|E| = 2$ in detail. With the notations in obvious sense, the linear RDE becomes

$$\begin{aligned} d\theta^1_t &= [a^{11}_t\theta^1_t + a^{12}_t\theta^2_t + b^1_t]\cdot d\boldsymbol{\omega}_t + [\lambda^{11}_t\theta^1_t + \lambda^{12}_t\theta^2_t + l^1_t] : d\langle\boldsymbol{\omega}\rangle_t;\\ d\theta^2_t &= [a^{21}_t\theta^1_t + a^{22}_t\theta^2_t + b^2_t]\cdot d\boldsymbol{\omega}_t + [\lambda^{21}_t\theta^1_t + \lambda^{22}_t\theta^2_t + l^2_t] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.8}$$

Clearly, if the system is decoupled, for example if $a^{12} = 0$ and $\lambda^{12} = 0$, one can easily solve the system by first solving for θ^1 and then solving for θ^2. In the general case, we introduce a decoupling strategy as follows. Consider an auxiliary RDE:

$$d\overline{\Gamma}_t = \overline{a}_t\cdot d\boldsymbol{\omega}_t + \overline{\lambda}_t : d\langle\boldsymbol{\omega}\rangle_t \tag{4.9}$$

where $\overline{a}, \overline{\lambda}$ will be specified later. Denote $\overline{\theta}_t := \theta^2_t + \overline{\Gamma}_t\theta^1_t$. Then, applying the Itô–Ventzell formula (3.14) we have

$$\begin{aligned} d\overline{\theta}_t &= \Big[[a^{22}_t\theta^2_t + a^{21}_t\theta^1_t + b^2_t] + \overline{\Gamma}_t[a^{12}_t\theta^2_t + a^{11}_t\theta^1_t + b^1_t] + \overline{a}_t\theta^1_t\Big]\cdot d\boldsymbol{\omega}_t\\ &\quad + \Big[[\lambda^{22}_t\theta^2_t + \lambda^{21}_t\theta^1_t + l^2_t] + \overline{\Gamma}_t[\lambda^{12}_t\theta^2_t + \lambda^{11}_t\theta^1_t + l^1_t] + \overline{\lambda}_t\theta^1_t\\ &\quad + \overline{a}_t[a^{11}_t\theta^1_t + a^{12}_t\theta^2_t + b^1_t]^*\Big] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.10}$$

We want to choose $\overline{a}, \overline{\lambda}$ so that the right side above involves only $\overline{\theta}$. That is,

$$\begin{aligned} &a^{21} + \overline{\Gamma}_t a^{11} + \overline{a} = \overline{\Gamma}[a^{22} + \overline{\Gamma}a^{12}],\\ &\lambda^{21} + \overline{\Gamma}\lambda^{11} + \overline{\lambda} + \overline{a}(a^{11})^* = \overline{\Gamma}_t[\lambda^{22} + \overline{\Gamma}\lambda^{12} + \overline{a}(a^{12})^*]. \end{aligned}$$

This implies

$$\begin{aligned} &\overline{a} = a^{12}(\overline{\Gamma})^2 + [a^{22} - a^{11}]\overline{\Gamma} - a^{21}; \qquad (4.11)\\ &\overline{\lambda} \;=\; \lambda^{12}(\overline{\Gamma})^2 + [\lambda^{22} - \lambda^{11}]\overline{\Gamma} - \lambda^{21} + \overline{a}[a^{12}\overline{\Gamma} - a^{11}]^*\\ &\quad\;= c^3(\overline{\Gamma})^3 + c^2(\overline{\Gamma})^2 + c^1\overline{\Gamma} + c^0, \quad \text{where}\\ &c^3 := a^{12}(a^{12})^*, \qquad c^2 := \lambda^{12} - a^{12}(a^{11})^* + (a^{22} - a^{11})(a^{12})^*\\ &c^1 := \lambda^{22} - \lambda^{11} - (a^{22} - a^{11})(a^{11})^* - a^{21}(a^{12})^*, \qquad c^0 := a^{21}(a^{11})^* - \lambda^{21}. \end{aligned}$$

Plugging this into (4.9) we obtain the following Riccati type of RDE:

$$\begin{aligned} d\overline{\Gamma}_t &= \Big[a^{12}_t(\overline{\Gamma})^2_t + [a^{22}_t - a^{11}_t]\overline{\Gamma}_t - a^{21}_t\Big]\cdot d\boldsymbol{\omega}_t + \Big[c^3_t(\overline{\Gamma})^3_t + c^2_t(\overline{\Gamma})^2_t\\ &\quad + c^1_t\overline{\Gamma}_t + c^0_t\Big] : d\langle\boldsymbol{\omega}\rangle_t, \end{aligned} \tag{4.12}$$

and the RDE (4.10) becomes:

$$\begin{aligned} d\overline{\theta}_t &= \Big[[a^{22} + \overline{\Gamma}a^{12}]\overline{\theta}_t + [b^2_t + \overline{\Gamma}_t b^1_t]\Big]\cdot d\boldsymbol{\omega}_t\\ &\quad + \Big[[\lambda^{22} + \overline{\Gamma}\lambda^{12} + \overline{a}(a^{12})^*]\overline{\theta}_t + [l^2_t + \overline{\Gamma}_t l^1_t + \overline{a}_t(b^1_t)^*]\Big] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.13}$$

Moreover, plug $\theta^2 = \overline{\theta} - \overline{\Gamma}\theta^1$ into the second equation of (4.8), we have

$$\begin{aligned} d\theta^1_t &= \Big[[a^{11}_t - a^{12}_t\overline{\Gamma}_t]\theta^1_t + [a^{12}_t\overline{\theta}_t + b^1_t]\Big]\cdot d\boldsymbol{\omega}_t\\ &\quad + \Big[[\lambda^{11}_t - \lambda^{12}_t\overline{\Gamma}_t]\theta^1_t + [\lambda^{12}_t\overline{\theta}_t + l^1_t]\Big] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.14}$$

Now the RDEs (4.12), (4.13), and (4.14) are decoupled. We shall emphasize though the Riccati RDE (4.12) typically does not have a global solution on $[0, T]$. However, following the arguments in Theorem 4.2, there exists a constant $\delta > 0$, which depends only on the coefficients a, λ and the rough path $\boldsymbol{\omega}$, such that the Riccati RDE (4.12) with initial value 0 has a solution whenever the time interval is smaller than δ. We now set $0 = t_0 < \cdots < t_n = T$ such that $t_i - t_{i-1} \leq \delta$ for $i = 1, \ldots, n$, and we solve the system (4.8) as follows. First, we solve RDE (4.12) on $[t_0, t_1]$ with initial value $\overline{\Gamma}_{t_0} = 0$. Plug this into (4.13), where $\overline{a}$ is determined by (4.11), we solve (4.13) on $[t_0, t_1]$ with initial value $\overline{\theta}_0 = \theta^2_0$. Plug $\overline{\Gamma}$ and $\overline{\theta}$ into (4.14), we may solve (4.14) on $[t_0, t_1]$ with initial value θ^1_0. Moreover, $\theta^2 := \overline{\theta} - \overline{\Gamma}\theta^1$ satisfies the second equation of (4.8) on $[t_0, t_1]$ with initial value θ^2_0. Next, we solve the Riccati RDE (4.12) on $[t_1, t_2]$, again with initial value $\overline{\Gamma}_{t_1} = 0$. Then we solve (4.13) on $[t_1, t_2]$ with initial value $\overline{\theta}_{t_1} = \theta^2_{t_1}$. Plug $\overline{\Gamma}$ and $\overline{\theta}$ into (4.14), we may solve (4.14) on $[t_1, t_2]$ with initial value $\theta^1_{t_1}$. Moreover, $\theta^2 := \overline{\theta} - \overline{\Gamma}\theta^1$ satisfies the second equation of (4.8) on $[t_1, t_2]$ with initial value $\theta^2_{t_1}$. Repeat the arguments we solve the system (4.8) over the whole interval $[0, T]$.

Step 3. We now assume the result is true for $|E| = n - 1$ and we shall prove the case $|E| = n$. With obvious notations, we consider

$$d\theta^i_t = \Big[\sum_{j=1}^{n} a^{ij}_t \theta^j_t + b^i_t\Big] \cdot d\boldsymbol{\omega}_t + \Big[\sum_{j=1}^{n} \lambda^{ij}_t \theta^j_t + l^i_t\Big] : d\langle\boldsymbol{\omega}\rangle_t, \quad i = 1, \ldots, n. \tag{4.15}$$

Denote $\overline{\theta} := \theta^n + \sum_{i=1}^{n-1} \overline{\Gamma}^i \theta^i$, where, for $i = 1, \ldots, n-1$,

$$\begin{aligned} d\overline{\Gamma}^i_t = &\Big[\sum_{j=1}^{n-1} [a^{jn}\overline{\Gamma}^i_t - a^{ji}_t]\overline{\Gamma}^j_t + [a^{nn}_t \overline{\Gamma}^i_t - a^{ni}_t]\Big] \cdot d\boldsymbol{\omega}_t \\ &+ \Big[[\overline{\Gamma}^i_t \lambda^{nn}_t - \lambda^{ni}_t] + \sum_{j=1}^{n-1} \overline{\Gamma}^j_t [\overline{\Gamma}^i_t \lambda^{jn}_t - \lambda^{ji}_t] \\ &+ \sum_{j=1}^{n-1}\Big[\sum_{k=1}^{n-1}[a^{kn}\overline{\Gamma}^j_t - a^{kj}_t]\overline{\Gamma}^k_t + [a^{nn}_t\overline{\Gamma}^j_t - a^{nj}_t]\Big] \\ &\times [\overline{\Gamma}^i_t (a^{jn}_t)^* - (a^{ji}_t)^*]\Big] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.16}$$

Then

$$\begin{aligned} d\overline{\theta}_t = &\Big[[a^{nn}_t + \sum_{i=1}^{n-1} \overline{\Gamma}^i_t a^{in}_t]\overline{\theta}_t + [b^n_t + \sum_{i=1}^{n-1} \overline{\Gamma}^i_t b^i_t]\Big] \cdot d\boldsymbol{\omega}_t \\ &+ \Big[\Big[\lambda^{nn}_t + \sum_{i=1}^{n-1}[\overline{\Gamma}^i_t \lambda^{in}_t + \overline{a}^i_t (a^{in})^*]\Big]\overline{\theta}_t \\ &+ \Big[l^n + \sum_{i=1}^{n-1}[\overline{\Gamma}^i_t l^i_t + \overline{a}^i_t (b^i_t)^*]\Big]\Big] : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{4.17}$$

where $\overline{a}^i_t := \sum_{j=1}^{n-1} [a^{jn}\overline{\Gamma}^i_t - a^{ji}_t]\overline{\Gamma}^j_t + [a^{nn}_t \overline{\Gamma}^i_t - a^{ni}_t]$.

Plug this into (4.15), we obtain

$$d\theta^i_t = \Big[\sum_{j=1}^{n-1}[a^{ij}_t - a^{in}_t\overline{\Gamma}^j_t]\theta^j_t + [b^i_t + a^{in}_t\overline{\theta}_t]\Big]\cdot d\boldsymbol{\omega}_t$$

$$+\Big[\sum_{j=1}^{n-1}[\lambda^{ij}_t - \lambda^{in}_t\overline{\Gamma}^j_t]\theta^j_t + [l^i_t + \lambda^{in}_t\overline{\theta}_t]\Big] : d\langle\boldsymbol{\omega}\rangle_t, \quad i = 1,\dots,n-1. \tag{4.18}$$

Now similarly, there exists $\delta > 0$, depending only on a, λ, and the rough path $\boldsymbol{\omega}$, such that the system of Riccati type RDE (4.16) with initial condition 0 has a solution whenever the time interval is smaller than δ. Now set $0 = t_0 < \cdots < t_n = T$ such that $t_i - t_{i-1} \le \delta$. As in Step 2, we may first solve (4.16) on $[t_0, t_1]$ with initial condition $\overline{\Gamma}^i_0 = 0$. We then solve (4.17) on $[t_0, t_1]$ with initial condition $\overline{\theta}_0 = \theta^n_0$. Now notice that the linear system (4.18) has only dimension $n-1$, then by induction assumption, we may solve (4.18) on $[t_0, t_1]$ with initial condition θ^i_0, $i = 1,\dots,n-1$, which further provides $\theta^n := \overline{\theta} - \sum_{i=1}^{n-1}\overline{\Gamma}^i\theta^i$. Now repeat the arguments as in Step 2, we obtain the solution over the whole interval $[0, T]$. ■

Remark 4.6. (i) When $E = \mathbb{R}$, the representation formula (4.7) actually holds under weaker conditions: $a, b \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(\mathbb{R}^d)$. Moreover, uniqueness also holds under this weaker condition. Indeed, for any arbitrary solution $\theta \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$ and for the Γ defined in (4.7), by applying the Itô–Ventzell formula (3.14) we see that

$$\Gamma_t\theta_t = \theta_0 + \int_0^t \Gamma_s b_s \cdot d\boldsymbol{\omega}_s + \int_0^t \Gamma_s\big[l_s - a_s b^*_s\big] : d\langle\boldsymbol{\omega}\rangle_s.$$

Then θ has to be the one in (4.7).

(ii) In the multidimensional case, we note that the Riccati RDE (4.12) does not involve b. Then we may also obtain the uniqueness, under our weaker condition $b \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d)$, from the strategy in this proof. ■

Applying Theorem 4.3 and following the arguments in the beginning of the proof for Theorem 4.5 (or Remark 4.6(ii)) concerning the weaker condition on b, the following result is immediate.

Corollary 4.7. *Let $\boldsymbol{\omega}, a, b, \lambda, l, \theta$ be as in Theorem 4.5 and $\tilde{\boldsymbol{\omega}}, \tilde{a}, \tilde{b}, \tilde{\lambda}, \tilde{l}, \tilde{\theta}$. Denote $\Delta\varphi := \varphi - \tilde{\varphi}$ for appropriate φ. Then*

$$d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(\theta,\tilde{\theta}) \le C\Big[d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(a,\tilde{a}) + d^{\omega,\tilde{\omega}}_{\boldsymbol{\alpha}}(b,\tilde{b}) + \|\Delta\lambda\|_\beta + \|\Delta l\|_\beta + \|\Delta\boldsymbol{\omega}\|_\alpha$$

$$+ |\Delta a_0| + |\partial_\omega a_0 - \partial_{\tilde{\omega}}\tilde{a}_0| + |\Delta b_0| + |\partial_\omega b_0 - \partial_{\tilde{\omega}}\tilde{b}_0|\Big].$$

5. Pathwise solutions of stochastic differential equations

5.1. The rough path setting for Brownian motion

Let $\Omega_0 := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space, B the canonical process, $\mathbb{F} = \mathbb{F}^B$ the natural filtration, and $\mathbb{P}_0$ the Wiener measure. Following Föllmer [17] (or see Bichteler [1] and Karandikar [25] for more general results on pathwise stochastic integration), we may construct pathwise Itô integration as follows:

$$\Phi_t(\omega) := \overline{\lim_{n\to\infty}} \sum_{i=0}^{2^n-1} \omega_{t_i^n}(\omega_{t_i^n\wedge t, t_{i+1}^n\wedge t})^* \quad \text{where } t_i^n := \frac{iT}{2^n},\ i = 0, \ldots, 2^n. \tag{5.1}$$

Then Φ is $\mathbb{F}$-adapted and $\Phi_t = \int_0^t B_s d_{Ito} B_s^*, 0 \le t \le T, \mathbb{P}_0$-a.s. Here d_{Ito} stands for Itô integration. Define

$$\begin{gathered} \underline{\Phi}_{s,t}(\omega) := \Phi_t(\omega) - \Phi_s(\omega) - \omega_s\omega_{s,t}^*, \qquad \underline{\Phi}_{s,t}^{Str}(\omega) := \underline{\Phi}_{s,t}(\omega) + \frac{1}{2}(t-s)I_d; \\ \langle\omega\rangle_t := \omega_t\omega_t^* - \Phi_t(\omega) - [\Phi_t(\omega)]^*. \end{gathered} \tag{5.2}$$

It is straightforward to check that

$$\underline{\Phi}_{s,t}(\omega) - \underline{\Phi}_{s,r}(\omega) - \underline{\Phi}_{r,t}(\omega) = \omega_{s,r}\omega_{r,t}^* = \underline{\Phi}_{s,t}^{Str}(\omega) - \underline{\Phi}_{s,r}^{Str}(\omega) - \underline{\Phi}_{r,t}^{Str}(\omega). \tag{5.3}$$

Moreover, we have the following well known result:

Lemma 5.1. *For any $\frac{1}{3} < \alpha < \frac{1}{2}$, we have $\mathbb{P}_0(A_\alpha) = 1$, where*

$$\begin{aligned} A_\alpha := &\left\{\sup_{(s,t)\in\mathbb{T}^2} \frac{|\underline{\Phi}_{s,t}|}{|t-s|^{2\alpha}} < \infty\right\} \cap \left\{\langle\omega\rangle_t = tI_d, 0 \le t \le T\right\} \\ &\cap \left\{\overline{\lim_{t\downarrow s}} \frac{|v\cdot\omega_{s,t}|}{|t-s|^{2\alpha}} = \infty,\ \forall s \in \mathbb{Q}\cap[0,T),\ v \in \mathbb{R}^d\setminus\{0\}\right\}. \end{aligned} \tag{5.4}$$

Now set, for the A_α defined in (5.4),

$$\begin{aligned} \Omega &:= \left\{\omega \in \Omega_0 : (\omega, \underline{\Phi}(\omega)) \in \boldsymbol{\Omega_\alpha} \text{ and } \omega \in A_\alpha,\ \text{for all } \frac{1}{3} < \alpha < \frac{1}{2}\right\}; \\ d_\alpha(\omega,\tilde\omega) &:= d_\alpha\Big((\omega, \underline{\Phi}(\omega)), (\tilde\omega, \underline{\Phi}(\tilde\omega))\Big), \quad \text{for all } \omega,\tilde\omega \in \Omega \text{ and } \frac{1}{3} < \alpha < \frac{1}{2}. \end{aligned} \tag{5.5}$$

By (5.3) and Lemma 5.1, we see that $\mathbb{P}_0(\Omega) = 1$. From now on, we shall always restrict the sample space to Ω, and we still denote by B the canonical process and $\mathbb{F} := \mathbb{F}^B$. Define

$$\begin{aligned} &\mathcal{C}(\Omega, E) := \bigcup\left\{\mathcal{C}_{\boldsymbol\alpha}(\Omega, E) : \boldsymbol\alpha \text{ satisfies (2.3)}\right\}, \quad \text{where} \\ &\mathcal{C}_{\boldsymbol\alpha}(\Omega, E) := \left\{\theta \in \mathbb{L}^0(\mathbb{F}) : \theta(\omega) \in \mathcal{C}^1_{\omega,\boldsymbol\alpha}(E),\ \forall\omega\in\Omega,\ \text{and } \mathbb{E}^{\mathbb{P}_0}\Big[\|\theta(\omega)\|^2_{\omega,\boldsymbol\alpha}\Big] < \infty\right\}. \end{aligned} \tag{5.6}$$

We now define the pathwise stochastic integral by using the rough path integral: for $\theta \in \mathcal{C}(\Omega, E^d)$,

$$\begin{aligned} \Big(\int_0^t \theta_s \cdot dB_s\Big)(\omega) &:= \int_0^t \theta_s(\omega)\cdot d(\omega, \underline{\Phi}(\omega))_s, \quad \forall\omega\in\Omega; \\ \Big(\int_0^t \theta_s \circ dB_s\Big)(\omega) &:= \int_0^t \theta_s(\omega)\cdot d(\omega, \underline{\Phi}^{Str}(\omega))_s, \quad \forall\omega\in\Omega. \end{aligned} \tag{5.7}$$

The following result can be found in [18, Proposition 5.1 and Corollary 5.2].

Theorem 5.2. *For any $\theta \in \mathcal{C}(\Omega, E^d)$, the above pathwise stochastic integrals $\int_0^t \theta_s \cdot dB_s$ and $\int_0^t \theta_s \circ dB_s$ coincide with the Itô integral and the Stratonovich integral, respectively.*

Remark 5.3. Let X be a semi-martingale with $dX_t = \theta_t \cdot dB_t + \lambda_t dt$, where $\theta \in \mathcal{C}(\Omega, E^d)$ and λ is continuous. Then $X \in \mathcal{C}(\Omega, E)$ with $\partial_\omega X_t(\omega) = \theta_t(\omega)$ for each $\omega \in \Omega$. In the spirit of Dupire [13]'s functional Itô calculus, [6] defines the above θ as the path derivative of the process X. So the Gubinelli's derivative $\partial_\omega X(\omega)$ in Definition 2.3 is consistent with the path derivatives introduced in [6]. ■

Remark 5.4. Let $\omega \in \Omega$ and $\theta \in \mathcal{C}^2_{(\omega,\underline{\Phi}(\omega)),\boldsymbol{\alpha}}(E)$ for certain $\boldsymbol{\alpha}$ satisfying (2.3). Define

$$\partial_t^\omega \theta := \mathrm{Trace}(D_t^\omega \theta). \tag{5.8}$$

Then $\partial_t^\omega \theta$ is unique and is consistent with the time derivative in [6]. Moreover, the pathwise Ito formula (2.18) and the pathwise Taylor expansion (2.22), (2.23) become:

$$\begin{aligned} d\theta_t &= \partial_\omega \theta_t d\boldsymbol{\omega}_t + \Big[\partial_t^\omega \theta_t + \frac{1}{2}\mathrm{Trace}(\partial^2_{\omega\omega}\theta_t)\Big]dt; \\ \theta_{s,t} &= \partial_\omega \theta_s \omega_{s,t} + \frac{1}{2}\partial^2_{\omega\omega}\theta_s : [\omega_{s,t}\omega^*_{s,t} + \underline{\omega}_{s,t} - \underline{\omega}^*_{s,t}] + \partial_t^{\boldsymbol{\omega}}\theta_s(t-s) + O((t-s)^{2\alpha+\beta}); \\ \theta_{s,t} &= \partial_\omega \theta_s \omega_{s,t} + \frac{1}{2}\partial^2_{\omega\omega}\theta_s : [\omega_{s,t}\omega^*_{s,t}] + \partial_t^{\boldsymbol{\omega}}\theta_s(t-s) + O((t-s)^{2\alpha+\beta}), \end{aligned} \tag{5.9}$$

respectively. These are also consistent with [6]. ■

5.2. *Stochastic differential equations with regular solutions*

We now consider the following SDE with random coefficients:

$$X_t = x + \int_0^t \sigma(s, X_s, \omega) \cdot dB_s + \int_0^t b(s, X_s, \omega)ds, \quad \omega \in \Omega, \tag{5.10}$$

where b, σ are $\mathbb{F}$-progressively measurable. Clearly, the above SDE can be rewritten as the following RDE:

$$\begin{aligned} X_t(\omega) &= x + \int_0^t \sigma(s, X_s(\omega), \omega) \cdot d(\omega, \underline{\Phi}(\omega))_s \\ &\quad + \int_0^t b(s, X_s(\omega), \omega)\frac{I_d}{d} : d\langle\omega\rangle_s, \quad \omega \in \Omega. \end{aligned} \tag{5.11}$$

The following result is a direct consequence of Theorems 4.2 and 4.3.

Theorem 5.5. (i) *Assume, for each $\omega \in \Omega$, there exists $\boldsymbol{\alpha}(\omega)$ satisfying (2.3) such that $b(\cdot, \omega) \in \mathcal{C}^2_{\beta(\omega)}(E, E)$ and $\sigma(\cdot, \omega) \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}(\omega)}(E, E^d)$. Then the SDE has a unique solution X such that $X(\omega) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}(\omega)}(E)$ for all $\omega \in \Omega$.*

(ii) *Assume further that b and σ are continuous in ω in the following sense:*

$$\lim_{n\to\infty}\Big[\|b(\cdot,\omega^n) - b(\cdot,\omega)\|_{1,\beta(\omega)} + d^{\omega,\omega^n}_{2,\boldsymbol{\alpha}(\omega)}(\sigma(\cdot,\omega^n),\sigma(\cdot,\omega))\Big] = 0, \tag{5.12}$$

for any $\omega, \omega^n \in \Omega$ such that $\lim_{n\to\infty} d_{\alpha(\omega)}(\omega^n, \omega) = 0$.

Then X is also continuous in ω in the sense that:

$$\lim_{n\to\infty} d^{\omega,\omega^n}_{\boldsymbol{\alpha}(\omega)}(X(\omega), X(\omega^n)) = 0, \quad \textit{and consequently,}$$

$$\lim_{n\to\infty} \|X(\omega) - X(\omega^n)\|_\infty = 0. \tag{5.13}$$

Remark 5.6. The construction of pathwise solutions of SDEs via rough path is standard. However, we remark that our canonical sample space Ω is universal, which particularly does not depend on the integrands θ in (5.7) or the vector fields $\sigma(t, \omega, x)$ in (5.10). Consequently, our solution is constructed indeed for every $\omega \in \Omega$, without the exceptional null set. ■

Remark 5.7. (i) Assume σ is Hölder-$\frac{1}{2}$ continuous in t and Lipschitz continuous in ω in the following sense:

$$|\sigma(t, x, \omega) - \sigma(\tilde{t}, x, \tilde{\omega})| \le C\Big[\sqrt{\tilde{t} - t} + \sup_{0\le s\le T} |\omega_{s\wedge t} - \tilde{\omega}_{s\wedge \tilde{t}}|\Big]. \tag{5.14}$$

Then $\sigma(\cdot, x, \omega)$ is Hölder-α continuous in t for all $\alpha < \frac{1}{2}$. We remark that the distance on the right side of (5.14) is used in Zhang and Zhuo [39] and is equivalent to the metric introduced by Dupire [13].

(ii) As mentioned in Introduction, since ω is only Hölder-α continuous for $\alpha < \frac{1}{2}$, it is not reasonable to assume $\sigma(\cdot, x, \omega)$ is Hölder-$(1 - \alpha)$ continuous as required in Lejay and Victoir [28]. ■

Remark 5.8. Under the Stratonovich integration, the quadratic compensator of the Brownian motion sample path defined in (2.6) vanishes: $\langle(\omega, \underline{\Phi}^{str}(\omega))\rangle_t = 0$. If we want to consider SDE in the form:

$$dX_t = \sigma(t, X_t, \omega) \circ dB_t + b(t, \omega, X_t)dt, \tag{5.15}$$

we cannot simply rewrite it into

$$dX_t(\omega) = \sigma(t, \omega, X_t(\omega)) \cdot d(\omega, \underline{\Phi}^{str}(\omega))_t + b(t, \omega, X_t(\omega))\frac{I_d}{d} : d\langle(\omega, \underline{\Phi}^{str}(\omega))\rangle_t.$$

We can obtain pathwise solution of (5.15) in the following two ways:

(i) We may rewrite (5.15) in Itô form:

$$dX_t = \sigma(t, \omega, X_t) \cdot dB_t + \Big[b + \frac{1}{2}\mathrm{Trace}\big(\partial_\omega\sigma + \partial_y\sigma\sigma^*\big)\Big](t, \omega, X_t)dt, \tag{5.16}$$

which corresponds further to the following RDE:

$$\begin{aligned} dX_t(\omega) &= \sigma(t, \omega, X_t(\omega)) \cdot d(\omega, \underline{\Phi}(\omega))_t \\ &\quad + \Big[\frac{bI_d}{d} + \frac{\partial_\omega\sigma + \partial_y\sigma\sigma^*}{2}\Big](t, \omega, X_t(\omega)) : d\langle\omega\rangle_t. \end{aligned} \tag{5.17}$$

(ii) In Section 4, we may easily extend our results to more general RDEs:

$$d\theta_t = g(t, \theta_t) \cdot d\boldsymbol{\omega}_t + f(t, \theta_t) : d\langle\boldsymbol{\omega}\rangle_t + h(t, \theta_t)dt. \tag{5.18}$$

Then we may deal with (5.15) directly. ■

6. Rough PDEs and stochastic PDEs

In this section, we extend the results in previous sections to rough PDEs (1.3) and stochastic PDEs (1.4) with random coefficients. The well-posedness of such RPDEs and SPDEs, especially in the fully nonlinear case, is very challenging and has received very strong attention. We refer to Lions and Souganidis [29–32], Buckdahn and Ma [3,4], Caruana and Friz [7], Caruana, Friz and Oberhauser [8], Friz and Obhauser [19], Diehl and Friz [11], Oberhauser and Riedel [12], and Gubinelli, Tindel and Torrecilla, [23] for well-posedness of some classes of RPDEs/SPDEs, where various notions of solutions are proposed.

While this section is mainly motivated from the study of pathwise viscosity solutions of SPDEs in Buckdahn, Ma and Zhang [5] and Buckdahn, Keller, Ma and Zhang [2], in this section we shall focus on classical solutions only. In particular, we do not intend to establish strong well-posedness for general f, instead we shall investigate diffusion coefficients σ and g and see when the RPDE/SPDE can be transformed to a deterministic PDE. Again, unlike most results in the standard literature of rough PDEs, we allow the coefficients to depend on (t, ω). The results will require quite high regularity of the coefficients, in the sense of our path regularity. In order to simplify the presentation, for some results we shall not specify the precise regularity conditions.

6.1. RDEs with spatial parameters

Let $u_0 : \tilde{E} \to E$, $g : \mathbb{T} \times \tilde{E} \times E \to E^d$, $f : \mathbb{T} \times \tilde{E} \times E \to E^{d\times d}$, and consider the following RDE with parameter $x \in \tilde{E}$:

$$u_t(x) = u_0(x) + \int_0^t g(s, x, u_s(x)) \cdot d\boldsymbol{\omega}_s + \int_0^t f(s, x, u_s(x)) : d\langle \boldsymbol{\omega} \rangle_s. \tag{6.1}$$

Assume u_0, g and f are differentiable in x, and differentiate (6.1) formally in x_i, $i = 1, \ldots, |\tilde{E}|$, we obtain: denoting $v_t^i(x) := \partial_{x_i} u_t(x)$,

$$\begin{aligned} v_t^i(x) &= \partial_{x_i} u_0(x) + \int_0^t [\partial_{x_i} g(s, x, u_s(x)) + \partial_y g(s, x, u_s(x)) v_s^i(x)] \cdot d\boldsymbol{\omega}_s \\ &\quad + \int_0^t [\partial_{x_i} f(s, x, u_s(x)) + \partial_y f(s, x, u_s(x)) v_s^i(x)] : d\langle \boldsymbol{\omega} \rangle_s. \end{aligned} \tag{6.2}$$

Theorem 6.1. *Assume*

(i) *u_0, g, f are continuously differentiable in x;*
(ii) *for each $x \in \tilde{E}$, $i = 1, \ldots, |\tilde{E}|$, $j = 1, \ldots, |E|$,*

$$\begin{gathered} g(x, \cdot) \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(E, E^d), \quad f(x, \cdot) \in \mathcal{C}^2_\beta(E, E^{d\times d}); \\ \partial_{x_i} g(x, \cdot) \in \mathcal{C}^{1,2}_{\omega,\boldsymbol{\alpha}}(E, E^d), \quad \partial_{y_j} g(x, \cdot) \in \mathcal{C}^{2,3}_{\omega,\boldsymbol{\alpha}}(E, E^d), \quad \partial_{x_i} f(x, \cdot) \in \mathcal{C}^0_\beta(E, E^{d\times d}). \end{gathered} \tag{6.3}$$

(iii) *for any $x \in \tilde{E}$, denoting $\Delta\varphi := \varphi(x + \Delta x, \cdot) - \varphi(x, \cdot)$ for appropriate φ,*

$$\begin{gathered} \lim_{|\Delta x|\to 0} \Big[\|\Delta g\|_{2,\omega,\boldsymbol{\alpha}} + \|\Delta f\|_{1,\beta} \Big] = 0; \\ \lim_{|\Delta x|\to 0} \Big[\|\Delta[\partial_x g]\|_{2,\omega,\boldsymbol{\alpha}} + \|\Delta[\partial_y g]\|_{2,\omega,\boldsymbol{\alpha}} + \|\Delta[\partial_x f]\|_{0,\beta} + \|\Delta[\partial_y f]\|_{0,\beta} \Big] = 0. \end{gathered} \tag{6.4}$$

Moreover, $\partial_{\omega x} g$ and $\partial_{\omega y} g$ are continuous.

Then, for each $x \in \tilde{E}$, RDEs (6.1) and (6.2) have unique solution $u(x, \cdot), v^i(x, \cdot) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$, respectively. Moreover, u is differentiable in x with $\partial_{x_i} u = v^i$.

Proof. First, without loss of generality we may assume $|\tilde{E}| = 1$, namely $\tilde{E} = \mathbb{R}$. For each $x \in \tilde{E}$, by the first line of (6.3) and applying Theorem 4.2, we see that RDE (4.1) has a unique solution $u(x) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$. By the second line of (6.3) and applying Theorem 3.4 and Lemma 3.7, we see that, for $j = 1, \ldots, |E|$,

$$\partial_x g(x, u(x)) \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E^d), \quad \partial_{y_j} g(x, u(x)) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E^d), \quad \partial_x f(x, u(x)),$$
$$\partial_{y_j} f(x, u(x)) \in \Omega_\beta(E^{d\times d}).$$

Then by Theorem 4.5 the linear RDE (6.2) has a unique solution $v(x) \in \mathcal{C}^2_{\omega,\boldsymbol{\alpha}}(E)$.

It remains to prove $\partial_x u = v$. Given $x \in \mathbb{R}$, $\Delta x \in \mathbb{R} \setminus \{0\}$ and $\lambda \in [0, 1]$, denote

$$\Delta u_t := u_t(x + \Delta x) - u_t(x), \qquad \nabla u_t := \frac{\Delta u_t}{\Delta x},$$
$$\varphi_t(\lambda) := \varphi(t, x + \lambda\Delta x, u_t(x) + \lambda\Delta u_t(x)), \qquad \Delta\varphi_t(\lambda) := \varphi_t(\lambda) - \varphi_t(0),$$

for appropriate φ.

By the first line of (6.4), it follows from Theorem 4.3 that:

$$\lim_{|\Delta x|\to 0} \|\Delta u\|_{\omega,\boldsymbol{\alpha}} = 0. \tag{6.5}$$

Moreover, one can easily check that,

$$d\nabla u_t = \int_0^1 [\partial_x g_t(\lambda) + \partial_y g_t(\lambda)\nabla u_t]d\lambda \cdot d\boldsymbol{\omega}_t + \int_0^1 [\partial_x f_t(\lambda) + \partial_y f_t(\lambda)\nabla u_t]d\lambda : d\langle\boldsymbol{\omega}\rangle_t;$$
$$dv_t(x) = [\partial_x g_t(0) + \partial_y g_t(0)v_t(x)] \cdot d\boldsymbol{\omega}_t + [\partial_x f_t(0) + \partial_y f_t(0)v_t(x)] : d\langle\boldsymbol{\omega}\rangle_t.$$

By the second line of (6.4) and (6.5), it follows from Lemmas 3.6(ii) and 3.7(ii) that

$$\lim_{|\Delta x|\to 0}\Big[\|\partial_x g_t(\lambda) - \partial_x g(0)\|_{\omega,\boldsymbol{\alpha}} + \|\partial_y g_t(\lambda) - \partial_y g(0)\|_{\omega,\boldsymbol{\alpha}}\Big] = 0,$$
$$\lim_{|\Delta x|\to 0}\Big[\|\partial_x f_t(\lambda) - \partial_x f(0)\|_\beta + \|\partial_y f_t(\lambda) - \partial_y f(0)\|_\beta\Big] = 0,$$

for any $\lambda \in [0, 1]$. Furthermore, by Theorem 3.4(i) we have

$$\partial_\omega[\partial_x g_0(\lambda)] = \partial_{\omega x} g_0(\lambda) + \partial_{yx} g_0(\lambda) g_0(\lambda),$$
$$\partial_\omega[\partial_y g_0(\lambda)] = \partial_{\omega y} g_0(\lambda) + \partial_{yy} g_0(\lambda) g_0(\lambda).$$

Recalling the continuity of $\partial_{\omega x} g$, $\partial_{\omega y} g$ in (iii) we see that, for any $\lambda \in [0, 1]$,

$$\lim_{|\Delta x|\to 0}\Big[|\partial_\omega[\partial_x g_0(\lambda)] - \partial_\omega[\partial_x g_0(0)]| + |\partial_\omega[\partial_y g_0(\lambda)] - \partial_\omega[\partial_y g_0(0)]|\Big] = 0.$$

Now by Corollary 4.7 we have $\lim_{|\Delta x|\to 0} \|\nabla u - v(x)\|_{\omega,\boldsymbol{\alpha}} = 0$. That is, $\partial_x u_t(x) = v_t(x)$. ■

6.2. Pathwise characteristics

As standard in the literature, see e.g. Kunita [26] for Stochastic PDEs and [18, Chapter 12] for rough PDEs, the main tool for dealing with semilinear RPDEs/SPDEs is the characteristics, which we shall introduce below by using RDEs against rough paths and backward rough paths.

Let $\sigma : \mathbb{T} \times \tilde{E} \to \tilde{E}^d$ and $g : \mathbb{T} \times \tilde{E} \times E \to E^{d\times d}$. Fix $t_0 \in \mathbb{T}$ and denote

$$\overset{\leftarrow t_0}{\sigma}(t, y) := \sigma(t_0 - t, y), \qquad \overset{\leftarrow t_0}{g}(t, x, y) := g(t_0 - t, x, y). \tag{6.6}$$

Consider the following characteristic RDEs:

$$\theta^x_t = x - \int_0^t \sigma(s,\theta^x_s)\cdot d\boldsymbol{\omega}_s, \qquad \overleftarrow{\theta}^{t_0,x}_t = x + \int_0^t \overleftarrow{\sigma}^{t_0}(s,\overleftarrow{\theta}^{t_0,x}_s)\cdot d\overleftarrow{\boldsymbol{\omega}}^{t_0}_s; \tag{6.7}$$

$$\begin{aligned} \eta^{x,y}_t &= y + \int_0^t g(s,\theta^x_s,\eta^{x,y}_s)\cdot d\boldsymbol{\omega}_s,\\ \overleftarrow{\eta}^{t_0,x,y}_t &= y - \int_0^t \overleftarrow{g}^{t_0}(s,\overleftarrow{\theta}^{t_0,x}_s,\overleftarrow{\eta}^{t_0,x,y}_s)\cdot d\overleftarrow{\boldsymbol{\omega}}^{t_0}_s. \end{aligned} \tag{6.8}$$

By Lemma 2.11 and Theorem 4.2, the following result is obvious.

Lemma 6.2. (i) *Assume $\sigma \in \mathcal{C}^{2,3}_{\omega,\alpha}(\tilde{E},\tilde{E}^d)$. Then, for each $x \in \tilde{E}$, the RDEs (6.7) have unique solution $\theta^x \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(\tilde{E})$ and $\overleftarrow{\theta}^{t_0,x} \in \mathcal{C}^1_{\overleftarrow{\omega}^{t_0},\boldsymbol{\alpha}}([0,t_0],\tilde{E})$ satisfying $\overleftarrow{\theta}^{t_0,\theta^x_{t_0}}_t = \theta^x_{t_0-t}$, $t \in [0,t_0]$. In particular, the mapping $x \mapsto \theta^x_{t_0}$ is one to one with inverse function $x \mapsto \overleftarrow{\theta}^{t_0,x}_{t_0}$.*

(ii) *Assume further that, for each $x \in \tilde{E}$ and for the above solution θ^x, the mapping $(t,y) \mapsto g(t,\theta^x_t,y)$ is in $\mathcal{C}^{2,3}_{\omega,\alpha}(E,E^{d\times d})$. Then the RDEs (6.8) have unique solution $\eta^{x,y} \in \mathcal{C}^1_{\omega,\boldsymbol{\alpha}}(E)$ and $\overleftarrow{\eta}^{t_0,x,y} \in \mathcal{C}^1_{\overleftarrow{\omega}^{t_0},\boldsymbol{\alpha}}(E)$ satisfying $\overleftarrow{\eta}^{t_0,\theta^x_{t_0},\eta^x_{t_0}}_t = \eta^{x,y}_{t_0-t}$, $t \in [0,t_0]$. In particular, the mapping $(x,y) \mapsto (\theta^x_{t_0},\eta^{x,y}_{t_0})$ is one to one with inverse functions $(x,y) \mapsto (\overleftarrow{\theta}^{t_0,x}_{t_0},\overleftarrow{\eta}^{t_0,x,y}_{t_0})$.*

Now define

$$\begin{aligned} &\varphi(t,x) := \overleftarrow{\theta}^{t,x}_t, \qquad \psi(t,x,y) := \overleftarrow{\eta}^{t,\theta^x_t,y}_t, \qquad \zeta(t,x,y) := \eta^{\varphi(t,x),y}_t,\\ &\widehat{g}(t,x,y) := g(t,\theta^x_t,y). \end{aligned} \tag{6.9}$$

Lemma 6.3. *Assume σ and g are smooth enough in the sense of Theorem 6.1. Then φ,ψ are twice differentiable in (x,y), and for any fixed (x,y), $\varphi(\cdot,x),\psi(\cdot,x,y) \in \mathcal{C}^\omega_\alpha$. Moreover, they satisfy the following RDEs:*

$$\varphi(t,x) = x + \int_0^t \partial_x\varphi\sigma(s,x)\cdot d\boldsymbol{\omega}_s + \int_0^t \Big[\frac{1}{2}\partial^2_{xx}\varphi\ [\sigma,\sigma] + \partial_x\varphi[\partial_x\sigma\sigma^*]\Big](s,x) : d\langle\boldsymbol{\omega}\rangle_s;$$

$$\begin{aligned} \psi(t,x,y) \ &= \ y - \int_0^t [\partial_y\psi\widehat{g}](s,x,y)\cdot d\boldsymbol{\omega}_s\\ &\quad + \int_0^t \Big[\frac{1}{2}\partial^2_{yy}\psi\ [\widehat{g},\widehat{g}] + \partial_y\psi[\partial_y\widehat{g}\widehat{g}^*]\Big](s,x,y) : d\langle\boldsymbol{\omega}\rangle_s. \end{aligned}$$

Proof. By Theorem 6.1, $\theta^x, \overleftarrow{\theta}^{t,x}, \eta^{x,y}, \overleftarrow{\eta}^{t,x,y}$ are sufficiently differentiable in (x,y). This implies the desired differentiability of φ,ψ. We now check the RDEs.

First, given $(s,t) \in \mathbb{T}^2$ and denote $\delta := t-s$. Note that

$$\varphi(t,x) = \overleftarrow{\theta}^{t,x}_t = \overleftarrow{\theta}^{s,\overleftarrow{\theta}^{t,x}_\delta}_s = \varphi(s,\overleftarrow{\theta}^{t,x}_\delta);$$

and that, applying Lemma 2.11,

$$
\begin{aligned}
\overleftarrow{\theta}^{t,x}_{\delta} - x &= \int_0^{\delta} \overleftarrow{\sigma}^{t}(r, \overleftarrow{\theta}^{t,x}_{r}) \cdot d\overleftarrow{\boldsymbol{\omega}}^{t}_{r} \\
&= \overleftarrow{\sigma}^{t}(0,x)\cdot \overleftarrow{\omega}^{t}_{0,\delta} + [\partial_{\overleftarrow{\omega}^{t}} \overleftarrow{\sigma}^{t} + \partial_x \overleftarrow{\sigma}^{t} (\overleftarrow{\sigma}^{t})^*](0,x) : \underline{\overleftarrow{\omega}}^{t}_{0,\delta} + O(\delta^{2\alpha+\beta}) \\
&= \sigma(t,x)\cdot\omega_{s,t} + [-\partial_\omega \sigma + \partial_x \sigma\sigma^*](t,x) : [\omega_{s,t}\omega^*_{s,t} - \underline{\omega}_{s,t}] + O(\delta^{2\alpha+\beta}) \\
&= \sigma(s,x)\cdot\omega_{s,t} + \partial_\omega\sigma(s,x) : \underline{\omega}_{s,t} \\
&\quad + \partial_x\sigma\sigma^*(s,x) : [\omega_{s,t}\omega^*_{s,t} - \underline{\omega}_{s,t}] + O(\delta^{2\alpha+\beta}).
\end{aligned}
$$

Then, applying Taylor expansion,

$$
\begin{aligned}
\varphi(t,x) - \varphi(s,x) &= \varphi(s, \overleftarrow{\theta}^{t,x}_{\delta}) - \varphi(s,x) \\
&= \partial_x\varphi(s,x)[\overleftarrow{\theta}^{t,x}_{\delta} - x] + \frac{1}{2}\partial^2_{xx}\varphi(s,x)\,[\overleftarrow{\theta}^{t,x}_{\delta} - x, \overleftarrow{\theta}^{t,x}_{\delta} - x] + O(\delta^{3\alpha}) \\
&= \partial_x\varphi(s,x)\Big[\sigma(s,x)\cdot\omega_{s,t} + \partial_\omega\sigma(s,x) : \underline{\omega}_{s,t} \\
&\quad + \partial_x\sigma\sigma^*(s,x) : [\omega_{s,t}\omega^*_{s,t} - \underline{\omega}_{s,t}]\Big] \\
&\quad + \frac{1}{2}\partial^2_{xx}\varphi(s,x)\,[\sigma(s,x)\cdot\omega_{s,t}] + O(\delta^{2\alpha+\beta}).
\end{aligned}
$$

In particular, this implies

$$\partial_\omega\varphi = \partial_x\varphi\sigma.$$

On the other hand, by applying Theorem 6.1 on (6.7) and view $(\theta^x, \partial_x\theta^x)$ as the solution to a higher dimensional RDE, one can check similarly that

$$\partial_\omega[\partial_x\varphi] = \partial_x[(\partial_x\varphi\sigma)^*].$$

Denote $\tilde{\varphi}$ as the right side of the RDE for φ. Then, taking values at (s,x),

$$
\begin{aligned}
[\tilde{\varphi}(\cdot,x)]_{s,t} &= \partial_x\varphi\sigma\cdot\omega_{s,t} + \partial_\omega[\partial_x\varphi\sigma] : \underline{\omega}_{s,t} \\
&\quad + \Big[\frac{1}{2}\partial^2_{xx}\varphi\,[\sigma,\sigma] + \partial_x\varphi[\partial_x\sigma\sigma^*]\Big] : \langle\boldsymbol{\omega}\rangle_{s,t} + O(\delta^{2\alpha+\beta}) \\
&= \partial_x\varphi\sigma\cdot\omega_{s,t} + \Big[\partial_x[\partial_x\varphi\sigma]\sigma^* + \partial_x\varphi\partial_\omega\sigma\Big] : \underline{\omega}_{s,t} \\
&\quad + \Big[\frac{1}{2}\partial^2_{xx}\varphi\,[\sigma,\sigma] + \partial_x\varphi[\partial_x\sigma\sigma^*]\Big] : [\omega_{s,t}\omega^*_{s,t} - \underline{\omega}_{s,t} - \underline{\omega}^*_{s,t}] + O(\delta^{2\alpha+\beta}).
\end{aligned}
$$

It is straightforward to check that $[\varphi(\cdot,x)]_{s,t} = [\tilde{\varphi}(\cdot,x)]_{s,t} + O(\delta^{2\alpha+\beta})$, implying $\varphi = \tilde{\varphi}$.

Similarly, notice that

$$\psi(t,x,y) = \overleftarrow{\eta}^{t,\theta^x_t,y}_{t} = \overleftarrow{\eta}^{s,\overleftarrow{\theta}^{t,\theta^x_t}_{\delta},\overleftarrow{\eta}^{t,\theta^x_t,y}_{\delta}}_{s} = \overleftarrow{\eta}^{s,\theta^x_s,\overleftarrow{\eta}^{t,\theta^x_t,y}_{\delta}}_{s} = \psi(s,x,\overleftarrow{\eta}^{t,\theta^x_t,y}_{\delta}).$$

Following similar arguments one can verify the RDE for ψ. ∎

6.3. Rough PDEs

Now consider RPDE:

$$u_t(x) = u_0(x) + \int_0^t [\partial_x u_s(x)\sigma_s(x) + g_s(x, u_s(x))] \cdot d\boldsymbol{\omega}_s + \int_0^t f_s(x, u_s(x), \partial_x u_s(x), \partial^2_{xx} u_s(x)) : d\langle\boldsymbol{\omega}\rangle_s. \tag{6.10}$$

Define

$$v(t,x) := \psi(t, x, u(t, \theta^x_t)) \quad \text{and equivalently } u(t,x) = \zeta(t, x, v(t, \varphi(t,x))).$$

Theorem 6.4. *Assume the coefficients and u are smooth enough. Then u is a solution of RPDE (6.10) if and only if v satisfies:*

$$dv_t(x) = \widehat{f}(t, x, v_t(x), \partial_x v_t(x), \partial^2_{xx} v_t(x)) : d\langle\boldsymbol{\omega}\rangle_t, \tag{6.11}$$

$$\text{or equivalently, } D^\omega_t v_t(x) = \widehat{f}(t, x, v_t(x), \partial_x v_t(x), \partial^2_{xx} v_t(x)),$$

where

$$\begin{aligned} \widehat{f}(t,x,y,z,\gamma) \;&:=\; \partial_y\psi(t,x,\widehat{y})\Big[f(t,\theta^x_t,\widehat{y},\widehat{z},\widehat{\gamma}) - \frac{1}{2}\widehat{\gamma} : [\sigma,\sigma](t,\theta^x_t) \\ &\qquad - \big[\widehat{z}\partial_x\sigma + \partial_x g + \partial_y g\widehat{z}\big]\sigma^*\Big](t,\theta^x_t,\widehat{y}); \end{aligned} \tag{6.12}$$

$$\widehat{y} = \zeta(t, \theta^x_t, y);$$

$$\widehat{z} = \partial_x\zeta(t,\theta^x_t,y) + \partial_y\zeta(t,\theta^x_t,y)z\partial_x\varphi(t,\theta^x_t);$$

$$\begin{aligned} \widehat{\gamma} \;=\; &\partial^2_{xx}\xi(t,\theta^x_t,y) + [\partial_{xy}\zeta(t,\theta^x_t,y) + \partial_{yx}\sigma(t,\theta^x_t)]\,[z, \partial_x\varphi(t,\theta^x_t)] \\ &+ \partial^2_{yy}\zeta(t,\theta^x_t,y)\,[\partial_x\varphi\partial_x\varphi, \partial_x\varphi\partial_x\varphi](t,\theta^x_t) \\ &+ \partial_y\zeta(t,\theta^x_t,y)\Big[\gamma\,[\partial_x\varphi,\partial_x\varphi](t,\theta^x_t) + z\partial^2_{xx}\varphi(t,\theta^x_t)\Big]. \end{aligned}$$

Proof. Applying the Itô–Ventzell formula (3.14) we have

$$\begin{aligned} du(t,\theta^x_t) \;=\; &g(t,\theta^x_t,u(t,\theta^x_t))d\boldsymbol{\omega}_t + \Big[f(\cdot,u,\partial_x u,\partial^2_{xx}u) \\ &- \Big[\frac{1}{2}\partial^2_x u : [\sigma,\sigma] + \partial_x u\partial_x\sigma\sigma^* + \partial_x g(\cdot,u)\sigma^* + \partial_y g\partial_x u\sigma^*\Big]\Big] \\ &\times (t,\theta^x_t) : d\langle\boldsymbol{\omega}\rangle_t; \end{aligned}$$

$$\begin{aligned} dv(t,x) \;=\; &d[\psi(t,x,u(t,\theta^x_t))] = \partial_y\psi(t,x,u(t,\theta^x_t))\Big[f(\cdot,u,\partial_x u,\partial^2_{xx}u) \\ &- \frac{1}{2}\partial^2_x u : [\sigma,\sigma] - \big[\partial_x u\partial_x\sigma + \partial_x g + \partial_y g\partial_x u\big]\sigma^*\Big] \\ &\times (t,\theta^x_t,u(t,\theta^x_t)) : d\langle\boldsymbol{\omega}\rangle_t. \end{aligned} \tag{6.13}$$

Now note that

$$u(t,x) = \zeta(t,x,v(t,\varphi(t,x)));$$

$$\partial_x u = \partial_x\zeta + \partial_y\zeta\partial_x v\partial_x\varphi;$$

$$\begin{aligned}\partial^2_{xx}u &= \partial^2_{xx}\xi + [\partial_{xy}\xi + \partial_{yx}\sigma]\ [\partial_x v, \partial_x\varphi] + \partial^2_{yy}\zeta\ [\partial_x\varphi\partial_x\varphi, \partial_x\varphi\partial_x\varphi]\\ &\quad + \partial_y\zeta\partial^2_{xx}v\ [\partial_x\varphi, \partial_x\varphi] + \partial_y\zeta\partial_x v\partial^2_{xx}\varphi.\end{aligned}$$

Then

$$u(t, \theta^x_t) = \zeta(t, \theta^x_t, v(t, x));$$

$$\partial_x u(t, \theta^x_t) = \partial_x\zeta(t, \theta^x_t, v(t, x)) + \partial_y\zeta(t, \theta^x_t, v(t, x))\partial_x v(t, x)\partial_x\varphi(t, \theta^x_t);$$

$$\begin{aligned}\partial^2_{xx}u(t, \theta^x_t) &= \partial^2_{xx}\xi(t, \theta^x_t, v(t, x)) + [\partial_{xy}\zeta(t, \theta^x_t, v(t, x)) + \partial_{yx}\sigma(t, \theta^x_t)]\\ &\quad \times [\partial_x v(t, x), \partial_x\varphi(t, \theta^x_t)]\\ &\quad + \partial^2_{yy}\zeta(t, \theta^x_t, v(t, x))\ [\partial_x\varphi\partial_x\varphi, \partial_x\varphi\partial_x\varphi](t, \theta^x_t)\\ &\quad + \partial_y\zeta(t, \theta^x_t, v(t, x))\partial^2_{xx}v(t, x)\ [\partial_x\varphi, \partial_x\varphi](t, \theta^x_t)\\ &\quad + \partial_y\zeta(t, \theta^x_t, v(t, x))\partial_x v(t, x)\partial^2_{xx}\varphi(t, \theta^x_t).\end{aligned}$$

Plug this into (6.13), we obtain the result immediately. ■

6.4. Pathwise solution of Stochastic PDEs

We now study Stochastic PDE:

$$\begin{aligned}u_t(\omega, x) &= u_0(x) + \int_0^t [\sigma_s(\omega, x)\partial_x u_s(\omega, x) + g_s(\omega, x, u_s(\omega, x))] \cdot dB_s\\ &\quad + \int_0^t f_s(\omega, x, u_s(\omega, x), \partial_x u_s(\omega, x), \partial^2_{xx}u_s(\omega, x))ds, \quad \mathbb{P}_0\text{-a.s.}\end{aligned} \tag{6.14}$$

Clearly, this corresponds to RPDE:

$$\begin{aligned}u_t(\omega, x) &= u_0(x) + \int_0^t [\sigma_s(\omega, x)\partial_x u_s(\omega, x) + g_s(\omega, x, u_s(\omega, x))] \cdot d(\omega, \underline{F}(\omega))_s\\ &\quad + \int_0^t F_s(\omega, x, u_s(\omega, x), \partial_x u_s(\omega, x), \partial^2_{xx}u_s(\omega, x)) : d\langle\omega\rangle_s,\\ &\forall\omega \in \Omega,\end{aligned} \tag{6.15}$$

$$\text{where } F(t, \omega, x, y, z, \gamma) := f(t, \omega, x, y, z, \gamma)\frac{I_d}{d}. \tag{6.16}$$

Define $\theta^{\omega,x}_t$, $\psi(t, \omega, x, y)$, $\widehat{F}(t, \omega, x, y, z, \gamma)$ in obvious sense and

$$\begin{aligned}&v(t, \omega, x) := \psi(t, \omega, x, u(t, \omega, \theta^{\omega,x}_t)),\\ &\widehat{f}(t, \omega, x, y, z, \gamma) := \mathrm{Trace}[\widehat{F}(t, \omega, x, y, z, \gamma)].\end{aligned} \tag{6.17}$$

Then we have, recalling $\partial^\omega_t v$ defined in Remark 5.4,

$$dv(t, \omega, x) = \partial^\omega_t v(t, \omega, x)dt = \widehat{f}_t(\omega, x, v_t(\omega, x), \partial_x v_t(\omega, x), \partial^2_{xx}v_t(\omega, x))dt.$$

Clearly, this implies that $\partial^\omega_t v_t(x) = \partial_t v(t, \omega, x)$, the standard time derivative for fixed (ω, x). We now conclude the paper with the following result:

Theorem 6.5. *Assume the coefficients and u are smooth enough. Then, for each $\omega \in \Omega$, $u(\omega, \cdot)$ is a solution of (6.15) if and only if $v(\omega, \cdot)$ is a solution of the following PDE:*

$$\partial_t v_t(\omega, x) = \widehat{f}_t(\omega, x, v_t(\omega, x), \partial_x v_t(\omega, x), \partial^2_{xx}v_t(\omega, x)). \tag{6.18}$$

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References

[1] K. Bichteler, Stochastic integration and L^p-theory of semimartingales, Ann. Probab. 9 (1981) 48–89.
[2] R. Buckdahn, C. Keller, J. Ma, J. Zhang, Fully nonlinear SPDEs and RPDEs: Classical and viscosity solutions, Working Paper, 2015.
[3] R. Buckdahn, J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I, Stochastic Process. Appl. 93 (2001) 181–204.
[4] R. Buckdahn, J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II, Stochastic Process. Appl. 93 (2001) 205–228.
[5] R. Buckdahn, J. Ma, J. Zhang, Pathwise viscosity solutions of stochastic PDEs and forward path-dependent PDEs, 2014. Preprint, arXiv:1501.06978.
[6] R. Buckdahn, J. Ma, J. Zhang, Pathwise Taylor Expansions for random fields on multiple dimensional paths, Stochastic Process. Appl. 125 (2015) 2820–2855.
[7] M. Caruana, P. Friz, Partial differential equations driven by rough paths, J. Differential Equations 247 (2009) 140–173.
[8] M. Caruana, P. Friz, H. Oberhauser, A (rough) pathwise approach to a class of non-linear stochastic partial differential equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011) 27–46.
[9] R. Cont, D. Fournie, Functional Itô calculus and stochastic integral representation of martingales, Ann. Probab. 41 (1) (2013) 109–133.
[10] A. Cosso, F. Russo, A regularization approach to functional Itô calculus and strong-viscosity solutions to path-dependent PDEs, 2014. Preprint, arXiv:1401.5034.
[11] J. Diehl, P. Friz, Backward stochastic differential equations with rough drivers, Ann. Probab. 40 (2012) 1715–1758.
[12] J. Diehl, H. Oberhauser, S. Riedel, A Levy-area between Brownian motion and rough paths with applications to robust non-linear filtering and RPDEs, 2013. Preprint, arXiv:1301.3799.
[13] B. Dupire, Functional Itô calculus, 2009. papers.ssrn.com.
[14] I. Ekren, C. Keller, N. Touzi, J. Zhang, On viscosity solutions of path dependent PDEs, Ann. Probab. 42 (2014) 204–236.
[15] I. Ekren, N. Touzi, J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I, Ann. Probab. (2012) in press.
[16] I. Ekren, N. Touzi, J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II, Ann. Probab. (2012) in press.
[17] H. Föllmer, Calcul d'Itô sans probabilités, in: Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980), (in French), in: Lecture Notes in Math., vol. 850, Springer, Berlin, 1981, pp. 143–150.
[18] P. Friz, M. Hairer, A Course on Rough Paths: With an Introduction to Regularity Structures, in: Springer Universitext, 2014.
[19] P. Friz, H. Oberhauser, Rough path stability of (semi-)linear SPDEs, Probab. Theory Related Fields 158 (2014) 401–434.
[20] P. Friz, Atul Shekhar, Doob–Meyer for rough paths, 2012. Preprint arXiv:1205.2505.
[21] P. Friz, N. Victoir, Multidimensional Stochastic Processes as Rough Paths: Theory and Applications, Vol. 120, Cambridge University Press, 2010.
[22] M. Gubinelli, Controlling rough paths, J. Funct. Anal. 216 (2004) 86–140.
[23] M. Gubinelli, S. Tindel, I. Torrecilla, Controlled viscosity solutions of fully nonlinear rough PDEs, 2014. Preprint, arXiv:1403.2832.
[24] M. Hairer, Solving the KPZ equation, Ann. of Math. 178 (2013) 559–664.
[25] R. Karandikar, On pathwise stochastic integration, Stochastic Process. Appl. 57 (1995) 11–18.
[26] H. Kunita, Stochastic Flows and Stochastic Differential Equations, in: Cambridge Studies in Advanced Mathematics, vol. 24, Cambridge University Press, Cambridge, 1990.
[27] D. Leao, A. Ohashi, A. Simas, Weak functional Itô calculus and applications, 2014. Preprint, arXiv:1408.1423.
[28] A. Lejay, N. Victoir, On (p, q)-rough paths, J. Differential Equations 225 (2006) 103–133.
[29] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations, C. R. Acad. Sci., Paris I 326 (1998) 1085–1092.

[30] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations: non-smooth equations and applications, C. R. Acad. Sci., Paris I 327 (1998) 735–741.
[31] P.-L. Lions, P.E. Souganidis, Fully nonlinear stochastic PDE with semilinear stochastic dependence, C. R. Acad. Sci., Paris I 331 (2000) 617–624.
[32] P.-L. Lions, P.E. Souganidis, Viscosity solutions of fully nonlinear stochastic partial differential equations, Surikaisekikenkyusho Kokyuroku 1287 (2002) 58–65. (Japanese) (Kyoto, 2001).
[33] T. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoam. 14 (1998) 215–310.
[34] T. Lyons, Rough paths, Signatures and the modelling of functions on streams, in: Proceedings of the International Congress of Mathematicians, Seoul 2014, Vol. IV, pp. 163–184.
[35] T. Lyons, D. Yang, Integration of time-varying cocyclic one-forms against rough paths, 2014. Preprint, arXiv:1408.2785.
[36] H. Oberhauser, An extension of the functional Ito formula under a family of non-dominated measures, 2012. Preprint, arXiv:1212.1414.
[37] S. Peng, F. Wang, BSDE, path-dependent PDE and nonlinear Feynman–Kac formula, 2011. arXiv:1108.4317.
[38] N. Perkowski, D. Prömel, Pathwise stochastic integrals for model free finance, 2013. Preprint, arXiv:1311.6187.
[39] J. Zhang, J. Zhuo, Monotone schemes for fully nonlinear parabolic path dependent PDEs, J. Financ. Eng. 1 (2014) 1450005. http://dx.doi.org/10.1142/S2345768614500056. (23 pages).