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# Optimal stopping under nonlinear expectation

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**Abstract**

Let $X : [0, T] \times \Omega \longrightarrow \mathbb{R}$ be a bounded càdlàg process with positive jumps defined on the canonical space of continuous paths $\Omega$. We consider the problem of optimal stopping the process $X$ under a nonlinear expectation operator $\mathcal{E}$ defined as the supremum of expectations over a weakly compact but nondominated family of probability measures. We introduce the corresponding nonlinear Snell envelope. Our main objective is to extend the Snell envelope characterization to the present context. Namely, we prove that the nonlinear Snell envelope is an $\mathcal{E}$-supermartingale, and an $\mathcal{E}$-martingale up to its first hitting time of the obstacle $X$. This result is obtained under an additional uniform continuity property of $X$. We also extend the result in the context of a random horizon optimal stopping problem.

This result is crucial for the newly developed theory of viscosity solutions of path-dependent PDEs as introduced in Ekren et al. (2014), in the semilinear case, and extended to the fully nonlinear case in the accompanying papers (Ekren et al. [6,7]).


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*Keywords:* Nonlinear expectation; Optimal stopping; Snell envelope

## 1. Introduction

On the canonical space of continuous paths $\Omega$, we consider a bounded càdlàg process $X : [0, T] \times \Omega \longrightarrow \mathbb{R}$, with positive jumps, and satisfying some uniform continuity condition.

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Let $\mathrm{H}_0$ be the first exit time of the canonical process from some convex domain, and $\mathrm{H} := \mathrm{H}_0 \wedge t_0$ for some $t_0 > 0$. This paper focuses on the problem

$$\sup_{\tau \in \mathcal{T}} \mathcal{E}[X_{\tau \wedge \mathrm{H}}], \quad \text{where } \mathcal{E}[\cdot] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot],$$

$\mathcal{T}$ is the collection of all stopping times, relative to the natural filtration of the canonical process, and $\mathcal{P}$ is a weakly compact non-dominated family of probability measures.

Our main result is the following. Similar to the standard theory of optimal stopping, we introduce the corresponding nonlinear Snell envelope $Y$, and we show that the classical Snell envelope characterization holds true in the present context. More precisely, we prove that the Snell envelope $Y$ is an $\mathcal{E}$-supermartingale, and an $\mathcal{E}$-martingale up to its first hitting time $\tau^*$ of the obstacle. Consequently, $\tau^*$ is an optimal stopping time for our problem of optimal stopping under nonlinear expectation.

This result is proved by adapting the classical arguments available in the context of the standard optimal stopping problem under linear expectation. However, such an extension turns out to be highly technical. The first step is to derive the dynamic programming principle in the present context, implying the $\mathcal{E}$-supermartingale property of the Snell envelope $Y$. To establish the $\mathcal{E}$-martingale property on $[0, \tau^*]$, we need to use some limiting argument for a sequence $Y_{\tau_n}$, where $\tau_n$'s are stopping times increasing to $\tau^*$. However, we face one major difficulty related to the fact that in a nonlinear expectation framework the dominated convergence theorem fails in general. It was observed in Denis, Hu and Peng [3] that the monotone convergence theorem holds in this framework if the decreasing sequence of random variables are quasi-continuous. Therefore, one main contribution of this paper is to construct convenient quasi-continuous approximations of the sequence $Y_{\tau_n}$. This allows us to apply the arguments in [3] on $Y_{\tau_n}$, which is decreasing under expectation (but not pointwise!) due to the supermartingale property. The weak compactness of the class $\mathcal{P}$ is crucial for the limiting arguments.

We note that in an one dimensional Markov model with uniformly non-degenerate diffusion, Krylov [10] studied a similar optimal stopping problem in the language of stochastic control (instead of nonlinear expectation). However, his approach relies heavily on the smoothness of the (deterministic) value function, which we do not have here. Indeed, one of the main technical difficulties in our situation is to obtain the locally uniform regularity of the value process.

Our interest in this problem is motivated from the recent notion of viscosity solutions of path-dependent partial differential equations, as developed in [5] and the accompanying papers [6,7]. Our definition is in the spirit of Crandall, Ishii and Lions [2], see also Fleming and Soner [9], but avoids the difficulties related to the fact that our canonical space fails to be locally compact. The key point is that the pointwise maximality condition, in the standard theory of viscosity solution, is replaced by a problem of optimal stopping under nonlinear expectation.

Our previous paper [5] was restricted to the context of semilinear path-dependent partial differential equations. In this special case, our definition of viscosity solutions can be restricted to the context where $\mathcal{P}$ consists of equivalent measures on the canonical space (and hence $\mathcal{P}$ has dominating measures). Consequently, the Snell envelope characterization of the optimal stopping problem under nonlinear expectation is available in the existing literature on reflected backward stochastic differential equations, see e.g. El Karoui et al. [8], Bayraktar, Karatzas and Yao [1]. However, the extension of our definition to the fully nonlinear case requires to consider a nondominated family of measures.

The paper is organized as follows. Section 2 introduces the probabilistic framework. Section 3 formulates the problem of optimal stopping under nonlinear expectation, and contains the statement of our main results. The proof of the Snell envelope characterization in the deterministic maturity case is reported in Section 4. The more involved case of a random maturity is addressed in Section 5. Finally, in the Appendix we present some additional results.

## 2. Nondominated family of measures on the canonical space

### 2.1. The canonical spaces

Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = \mathbf{0}\}$, the set of continuous paths starting from the origin, $B$ the canonical process, $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ the natural filtration generated by $B$, $\mathbb{P}_0$ the Wiener measure, $\mathcal{T}$ the set of $\mathbb{F}$-stopping times, and $\Lambda := [0,T]\times\Omega$. Moreover, for any sub-$\sigma$-field $\mathcal{G} \subset \mathcal{F}_T$, let $\mathbb{L}^0(\mathcal{G})$ denote the set of $\mathcal{G}$-measurable random variables, and $\mathbb{H}^0(\mathbb{F})$ the set of $\mathbb{F}$-progressively measurable processes. Here and in the sequel, for notational simplicity, we use $\mathbf{0}$ to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. We define a seminorm on $\Omega$ and a pseudometric on $\Lambda$ as follows: for any $(t,\omega),(t',\omega') \in \Lambda$,

$$\|\omega\|_t := \sup_{0\le s\le t}|\omega_s|, \qquad \mathbf{d}_\infty\big((t,\omega),(t',\omega')\big) := |t-t'| + \big\|\omega_{\cdot\wedge t} - \omega'_{\cdot\wedge t'}\big\|_T. \tag{2.1}$$

Then $(\Omega, \|\cdot\|_T)$ is a Banach space and $(\Lambda, \mathbf{d}_\infty)$ is a complete pseudometric space. In fact, the subspace $\{(t,\omega_{\cdot\wedge t}) : (t,\omega)\in\Lambda\}$ is a complete metric space under $\mathbf{d}_\infty$.

We next introduce the shifted spaces. Let $0 \le s \le t \le T$.

- Let $\Omega^t := \{\omega \in C([t,T],\mathbb{R}^d) : \omega_t = \mathbf{0}\}$ be the shifted canonical space; $B^t$ the shifted canonical process on $\Omega^t$; $\mathbb{F}^t$ the shifted filtration generated by $B^t$, $\mathbb{P}_0^t$ the Wiener measure on $\Omega^t$, $\mathcal{T}^t$ the set of $\mathbb{F}^t$-stopping times, and $\Lambda^t := [t,T]\times\Omega^t$. Moreover, for any $\mathcal{G}\subset\mathcal{F}^t_T$, $\mathbb{L}^0(\mathcal{G})$ and $\mathbb{H}^0(\mathbb{F}^t)$ are the corresponding sets of measurable random variables and processes, respectively.
- For $\omega\in\Omega^s$ and $\omega'\in\Omega^t$, define the concatenation path $\omega\otimes_t\omega'\in\Omega^s$ by:

$$(\omega\otimes_t\omega')(r) := \omega_r\mathbf{1}_{[s,t)}(r) + (\omega_t+\omega'_r)\mathbf{1}_{[t,T]}(r), \quad \text{for all } r\in[s,T].$$

- Let $0\le s<t\le T$ and $\omega\in\Omega^s$. For any $\xi\in\mathbb{L}^0(\mathcal{F}^s_T)$ and $X\in\mathbb{H}^0(\mathbb{F}^s)$ on $\Omega^s$, define the shifted $\xi^{t,\omega}\in\mathbb{L}^0(\mathcal{F}^t_T)$ and $X^{t,\omega}\in\mathbb{H}^0(\mathbb{F}^t)$ on $\Omega^t$ by:

$$\xi^{t,\omega}(\omega') := \xi(\omega\otimes_t\omega'), \qquad X^{t,\omega}(\omega') := X(\omega\otimes_t\omega'), \quad \text{for all } \omega'\in\Omega^t.$$

### 2.2. Capacity and nonlinear expectation

A probability measure $\mathbb{P}$ on $\Omega$ is called a semimartingale measure if the canonical process $B$ is a semimartingale under $\mathbb{P}$. For every constant $L>0$, we denote by $\mathcal{P}^L$ the collection of all continuous semimartingale measures $\mathbb{P}$ on $\Omega$ whose drift and diffusion characteristics are bounded by $L$ and $\sqrt{2L}$, respectively. To be precise, let $\tilde\Omega := \Omega^3$ be an enlarged canonical space, $\tilde B := (B,A,M)$ be the canonical processes, and $\tilde\omega = (\omega,a,m)\in\tilde\Omega$ be the paths. For any $\mathbb{P}\in\mathcal{P}^L$, there exists an extension $\mathbb{Q}$ on $\tilde\Omega$ such that:

$$\begin{aligned}
&B = A+M, \quad A \text{ is absolutely continuous}, M \text{ is a martingale},\\
&|\alpha^{\mathbb{P}}|\le L, \qquad \frac{1}{2}\mathrm{tr}\,((\beta^{\mathbb{P}})^2)\le L, \quad \text{where } \alpha^{\mathbb{P}}_t := \frac{dA_t}{dt}, \qquad\qquad \mathbb{Q}\text{-a.s.}\\
&\beta^{\mathbb{P}}_t := \sqrt{\frac{d\langle M\rangle_t}{dt}},
\end{aligned} \tag{2.2}$$

Similarly, for any $t\in[0,T)$, we may define $\mathcal{P}^L_t$ on $\Omega^t$.

**Remark 2.1.** Let $\mathbb{S}^d_+$ denote the set of $d \times d$ nonnegative definite matrices.

(i) In $\mathbb{Q}$-a.s. sense, clearly $\beta^{\mathbb{P}} \in \mathbb{L}^0(\mathbb{F}^B)$ and then $\alpha^{\mathbb{P}} \in \mathbb{L}^0(\mathbb{F}^{B,M})$.

(ii) We may also have the following equivalent characterization of $\mathcal{P}^L$. Consider the canonical space $\Omega' := \Omega^2$ with canonical processes $(B, B')$. For any $\mathbb{P} \in \mathcal{P}^L$, there exist a probability measure $\mathbb{Q}'$ and $\alpha^{\mathbb{P}} \in \mathbb{L}^0(\mathbb{F}^{B,B'}, \mathbb{R}^d)$, $\beta^{\mathbb{P}} \in \mathbb{L}^0(\mathbb{F}^B, \mathbb{S}^d_+)$ such that

$$\begin{aligned} &|\alpha^{\mathbb{P}}| \le L, \qquad \frac{1}{2}\mathrm{tr}\,((\beta^{\mathbb{P}})^2) \le L, \qquad \mathbb{Q}'|_{\mathcal{F}^B_T} = \mathbb{P}, \quad \mathbb{Q}'|_{\mathcal{F}^{B'}_T} = \text{Wiener measure}; \\ &dB_t = \alpha^{\mathbb{P}}_t(B, B')dt + \beta^{\mathbb{P}}_t(B)dB'_t, \quad \mathbb{Q}'\text{-a.s.} \end{aligned} \tag{2.3}$$

(iii) For any deterministic measurable functions $\alpha : [0, T] \to \mathbb{R}^d$ and $\beta : [0, T] \to \mathbb{S}^d_+$ satisfying $|\alpha| \le L$, $\frac{1}{2}\mathrm{tr}\,(\beta^2) \le L$, there exists unique $\mathbb{P} \in \mathcal{P}^L$ such that $\alpha^{\mathbb{P}} = \alpha$, $\beta^{\mathbb{P}} = \beta$, $\mathbb{P}$-a.s., where $\alpha^{\mathbb{P}}$, $\beta^{\mathbb{P}}$ can be understood in the sense of either (2.2) or (2.3). ■

Throughout this paper, we shall consider a family $\{\mathcal{P}_t, t \in [0, T]\}$ of semimartingale measures on $\Omega^t$ satisfying:

(P1) *there exists some $L_0$ such that, for all $t$, $\mathcal{P}_t$ is a weakly compact subset of $\mathcal{P}^{L_0}_t$.*

(P2) *For any $0 \le t \le T$, $\tau \in \mathcal{T}^t$, and $\mathbb{P} \in \mathcal{P}_t$, the regular conditional probability distribution $\mathbb{P}^{\tau,\omega} \in \mathcal{P}_{\tau(\omega)}$ for $\mathbb{P}$-a.e. $\omega \in \Omega^t$.*

(P3) *For any $0 \le s \le t \le T$, $\mathbb{P} \in \mathcal{P}_s$, $\{E_i, i \ge 1\} \subset \mathcal{F}^s_t$ disjoint, and $\mathbb{P}^i \in \mathcal{P}_t$, the following $\hat{\mathbb{P}}$ is also in $\mathcal{P}_s$:*

$$\hat{\mathbb{P}} := \mathbb{P} \otimes_t \left[\sum_{i=1}^{\infty} \mathbb{P}^i \mathbf{1}_{E_i} + \mathbb{P}\mathbf{1}_{\cap_{i=1}^{\infty} E_i^c}\right]. \tag{2.4}$$

Here (2.4) means, for any event $E \in \mathcal{F}^s_T$ and denoting $E^{t,\omega} := \{\omega' \in \Omega^t : \omega \otimes_t \omega' \in E\}$:

$$\hat{\mathbb{P}}[E] := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{\infty} \mathbb{P}^i[E^{t,B}]\mathbf{1}_{E_i}(B)\right] + \mathbb{P}\big[E \cap (\cap_{i=1}^{\infty} E_i^c)\big].$$

We refer to the seminal work of Stroock and Varadhan [16] for the introduction of regular conditional probability distribution (r.c.p.d. for short). See also Appendix A.1, in particular (A.2) below for the precise meaning of $\mathbb{P}^{\tau,\omega}$.

**Remark 2.2.** (i) The weak compactness of (P1) is crucial for the existence of the optimal stopping time. As explained in Introduction, the major technical difficulty we face is the failure of the dominated convergence theorem in our nonlinear expectation framework. To overcome this, we shall use the regularity of the processes and the weak compactness of the classes $\mathcal{P}_t$. See e.g. Step 2 in Section 4.4.

(ii) The regular conditional probability distribution is a convenient tool for proving the dynamic programming principle, see e.g. Soner, Touzi, and Zhang [14]. In particular, (P2) is used to prove one inequality in the dynamic programming principle, see e.g. Step 1 in the proof of Lemma 4.1.

(iii) The concatenation Property (P3) is used to prove the opposite inequality in the dynamic programming principle, see e.g. Step 2 in the proof of Lemma 4.1. We remark that this condition can be weakened by using the more abstract framework in Nutz and van Handel [11]. ■

We first observe that

**Lemma 2.3.** *For all $L > 0$, the family $\{\mathcal{P}_t^L, t \in [0, T]\}$ satisfies conditions* (P1)–(P3).

The proof is quite straightforward, by using the definition of r.c.p.d. We nevertheless provide a proof in the Appendix.

The following are some other typical examples of such a family $\{\mathcal{P}_t, t \in [0, T]\}$. Their Properties (P1)–(P3) can be checked similarly.

**Example 2.4.** Let $L, L_1, L_2 > 0$ be some constants.
*Wiener measure* $\mathcal{P}_t^0 := \{\mathbb{P}_0^t\} = \{\mathbb{P} : \alpha^{\mathbb{P}} = 0, \beta^{\mathbb{P}} = I_d\}$.
*Finite variation* $\mathcal{P}_t^{\mathrm{FV}}(L) := \{\mathbb{P} : |\alpha^{\mathbb{P}}| \le L, \beta^{\mathbb{P}} = \mathbf{0}\}$.
*Drifted Wiener measure* $\mathcal{P}_t^{0,\mathrm{ac}}(L) := \{\mathbb{P} : |\alpha^{\mathbb{P}}| \le L, \beta^{\mathbb{P}} = I_d\}$.
*Relaxed bounds* $\mathcal{P}_t(L_1, L_2) := \{\mathbb{P} : |\alpha^{\mathbb{P}}| \le L_1, \mathbf{0} \le \beta^{\mathbb{P}} \le L_2 I_d\}$.
*Relaxed bounds, Uniformly elliptic* $\mathcal{P}_t^{\mathrm{UE}}(L_1, L_2, L) := \{\mathbb{P} : |\alpha^{\mathbb{P}}| \le L_1, L I_d \le \beta^{\mathbb{P}} \le L_2 I_d\}$.
*Equivalent martingale measures* $\mathcal{P}_t^{\mathrm{e}}(L_1, L_2, L) := \{\mathbb{P} \in \mathcal{P}_t(L_1, L_2) : \exists\, |\gamma^{\mathbb{P}}| \le L, \alpha^{\mathbb{P}} = \beta^{\mathbb{P}}\gamma^{\mathbb{P}}\}$.

We denote by $\mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_t)$ the set of all $\xi \in \mathbb{L}^0(\mathcal{F}_T^t)$ with $\sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}[|\xi|] < \infty$. The set $\mathcal{P}_t$ induces the following capacity and nonlinear expectation:

$$\mathcal{C}_t[A] := \sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{P}[A] \quad \text{for } A \in \mathcal{F}_T^t, \quad \text{and} \quad \mathcal{E}_t[\xi] := \sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}[\xi] \\ \text{for } \xi \in \mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_t). \tag{2.5}$$

When $t = 0$, we shall omit $t$ and abbreviate them as $\mathcal{P}, \mathcal{C}, \mathcal{E}$. Clearly $\mathcal{E}$ is a $G$-expectation, in the sense of Peng [12]. We remark that, when $\xi$ satisfies certain regularity condition, then $\mathcal{E}_t[\xi^{t,\omega}]$ can be viewed as the conditional $G$-expectation of $\xi$, and as a process it is the solution of a Second Order BSDE, as introduced by Soner, Touzi and Zhang [13].

We remark that the last three families of measures in Example 2.4 are non-dominated, which are most interesting to us. In particular, in these cases the dominated convergence theorem fails under the corresponding nonlinear expectation as we see in the following simple example.

**Example 2.5.** Consider the relaxed bounds $\mathcal{P}_t(L_1, L_2)$ in Example 2.4 with $d = 1$. Let

$$\xi_n := \mathbf{1}_{\{0<\langle B\rangle_T \le \frac{1}{n}\}},$$

where $\langle B\rangle$ is the pathwise quadratic variation. Then $\xi_n \downarrow 0$ for all $\omega$ as $n \to \infty$, but $\mathcal{E}_0[\xi_n] = 1$ for all $n \ge \frac{1}{2L_2}$. ■

Given a family of probability measures $\mathcal{P}$ on $\Omega$, abusing the terminology of Denis and Martini [4], we say that a property holds $\mathcal{P}$-q.s. (quasi-surely) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$. Moreover, a random variable $\xi : \Omega \to \mathbb{R}$ is

- $\mathcal{P}$-quasi-continuous if for any $\varepsilon > 0$, there exists a closed set $\Omega_\varepsilon \subset \Omega$ such that $\mathcal{C}(\Omega_\varepsilon^c) < \varepsilon$ and $\xi$ is continuous in $\Omega_\varepsilon$,
- $\mathcal{P}$-uniformly integrable if $\mathcal{E}[|\xi|\mathbf{1}_{\{|\xi|\ge n\}}] \longrightarrow 0$, as $n \to \infty$.

Since $\mathcal{P}$ is weakly compact, by Denis, Hu and Peng [3, Lemma 4 and Theorems 22, 28], we have:

**Proposition 2.6.** (i) *Let $(\Omega_n)_{n\ge1}$ be a sequence of open sets with $\Omega_n \uparrow \Omega$. Then $\mathcal{C}(\Omega_n^c) \downarrow 0$.*
(ii) *Let $(\xi_n)_{n\ge1}$ be a sequence of $\mathcal{P}$-quasi-continuous and $\mathcal{P}$-uniformly integrable maps from $\Omega$ to $\mathbb{R}$. If $\xi_n \downarrow \xi$, $\mathcal{P}$-q.s. then $\mathcal{E}[\xi_n] \downarrow \mathcal{E}[\xi]$.*

We finally recall the notion of martingales under nonlinear expectation.

**Definition 2.7.** Let $X \in \mathbb{H}^0(\mathbb{F})$ such that $X_\tau \in \mathbb{L}^1(\mathcal{F}_\tau, \mathcal{P})$ for all $\tau \in \mathcal{T}$. We say that $X$ is a $\mathcal{E}$-supermartingale (resp. submartingale, martingale) if, for any $(t, \omega) \in \Lambda$ and any $\tau \in \mathcal{T}^t$, $\mathcal{E}_t[X^{t,\omega}_\tau] \le$ (resp. $\ge, =$) $X_t(\omega)$ for $\mathcal{P}$-q.s. $\omega \in \Omega$.

We remark that we require the $\mathcal{E}$-supermartingale property holds for stopping times. Under linear expectation $\mathbb{E}^{\mathbb{P}}$, this is equivalent to the $\mathbb{P}$-supermartingale property for deterministic times, due to the Doob's optional sampling theorem. However, under nonlinear expectation, they are in general not equivalent.

## 3. Optimal stopping under nonlinear expectations

We now fix a process $X \in \mathbb{H}^0(\mathbb{F})$.

**Assumption 3.1.** $X$ is a bounded càdlàg process with positive jumps, and there exists a modulus of continuity function $\rho_0$ such that for any $(t, \omega), (t', \omega') \in \Lambda$:

$$X(t, \omega) - X(t', \omega') \le \rho_0\Big(\mathbf{d}_\infty\big((t, \omega), (t', \omega')\big)\Big) \quad \text{whenever } t \le t'. \tag{3.1}$$

**Remark 3.2.** There is some redundancy in the above assumption. Indeed, it is shown in the Appendix that (3.1) *implies that $X$ has left-limits and $X_{t-} \le X_t$ for all $t \in (0, T]$*. Moreover, the fact that $X$ has only positive jumps is important to ensure that the random times $\tau^*$ in (3.2), $\hat{\tau}^*$ in (3.5), and $\tau_n$ in (4.7) and (5.15) are $\mathbb{F}$-stopping times. ■

We define the nonlinear Snell envelope and the corresponding obstacle first hitting time:

$$Y_t(\omega) := \sup_{\tau \in \mathcal{T}^t} \mathcal{E}_t[X^{t,\omega}_\tau], \quad \text{and} \quad \tau^* := \inf\{t \ge 0 : Y_t = X_t\}. \tag{3.2}$$

Our first result is the following *nonlinear Snell envelope* characterization of the deterministic maturity optimal stopping problem $Y_0$.

**Theorem 3.3** (*Deterministic Maturity*)**.** *Under Assumption 3.1, the process $Y$ is an $\mathcal{E}$-supermartingale on $[0, T]$, $Y_{\tau^*} = X_{\tau^*}$, and $Y_{\cdot\wedge\tau^*}$ is an $\mathcal{E}$-martingale. Consequently, $\tau^*$ is an optimal stopping time for the problem $Y_0$.*

To prove the partial comparison principle for viscosity solutions of path-dependent partial differential equations in our accompanying paper [7], we need to consider optimal stopping problems with random maturity time $\mathrm{H} \in \mathcal{T}$ of the form

$$\mathrm{H} := \inf\{t \ge 0 : B_t \in O^c\} \wedge t_0, \tag{3.3}$$

for some $t_0 \in (0, T]$ and some open convex set $O \subset \mathbb{R}^d$ containing the origin. We shall extend the previous result to the following stopped process:

$$\widehat{X}^{\mathrm{H}}_s := X_s \mathbf{1}_{\{s<\mathrm{H}\}} + X_{\mathrm{H}-}\mathbf{1}_{\{s\ge\mathrm{H}\}} \quad \text{for } s \in [0, T]. \tag{3.4}$$

The corresponding Snell envelope and obstacle first hitting time are denoted:

$$\widehat{Y}^{\mathrm{H}}_t(\omega) := \sup_{\tau \in \mathcal{T}^t} \mathcal{E}_t\Big[\big(\widehat{X}^{\mathrm{H}}\big)^{t,\omega}_\tau\Big], \quad \text{and} \quad \widehat{\tau}^* := \inf\{t \ge 0 : \widehat{Y}^{\mathrm{H}}_t = \widehat{X}^{\mathrm{H}}_t\}. \tag{3.5}$$

Our second main result requires the following additional assumption.

**Assumption 3.4.** (i) For some $L > 0, \mathcal{P}_t^{\mathrm{FV}}(L) \subset \mathcal{P}_t$ for all $t \in [0, T]$, where $\mathcal{P}_t^{\mathrm{FV}}(L)$ is defined in Example 2.4.
(ii) For any $0 \le t < t + \delta \le T, \mathcal{P}_t \subset \mathcal{P}_{t+\delta}$ in the following sense: for any $\mathbb{P} \in \mathcal{P}_t$ we have $\tilde{\mathbb{P}} \in \mathcal{P}_{t+\delta}$, where $\tilde{\mathbb{P}}$ is the probability measure on $\Omega^{t+\delta}$ such that the $\tilde{\mathbb{P}}$-distribution of $B^{t+\delta}$ is equal to the $\mathbb{P}$-distribution of $\{B_s^t, t \le s \le T - \delta\}$.

**Remark 3.5.** The above assumption is a technical condition used to prove the dynamic programming principle in Section 5.1 below.

(i) All sets in Example 2.4 satisfy Assumption 3.4(ii), and the relaxed bounds $\mathcal{P}_t(L_1, L_2)$ satisfies Assumption 3.4(i). We remark that, for the viscosity theory of path-dependent partial differential equations in our accompanying papers [6,7], we shall use $\mathcal{P}_t(L, \sqrt{2L})$ which satisfies both (i) and (ii) of Assumption 3.4.
(ii) By a little more involved arguments, we may prove the results in Section 5.1 by replacing Assumption 3.4 (i) with: for $\mathcal{P}_t^{\mathrm{UE}}$ defined in Example 2.4,

$$\text{for some constants } L, L_1, L_2, \quad \mathcal{P}_t^{\mathrm{UE}}(L_1, L_2, L) \subset \mathcal{P}_t \quad \text{for all } t \in [0, T], \tag{3.6}$$

(iii) If $\mathcal{P}_t$ is uniformly nondegenerate, namely

$$\text{there exists } c > 0 \text{ such that } \beta^{\mathbb{P}} \ge c I_d \quad \text{for all } t \text{ and } \mathbb{P} \in \mathcal{P}_t, \tag{3.7}$$

then we shall use (3.6) instead of Assumption 3.4 (i). In this case, under the additional condition that $X$ is uniformly continuous in $(t, \omega)$, $\hat{Y}^{\mathrm{H}}$ is left continuous at $\mathrm{H}$ and the arguments for our main result Theorem 3.6 below can be simplified significantly, see Lemma A.1 and Remark 5.10 below. ∎

**Theorem 3.6** (*Random Maturity*)**.** *Under Assumptions 3.1 and 3.4, the process $\widehat{Y}^{\mathrm{H}}$ is an $\mathcal{E}$-supermartingale on $[0, \mathrm{H}]$, $\widehat{Y}^{\mathrm{H}}_{\widehat{\tau}^*} = \widehat{X}^{\mathrm{H}}_{\widehat{\tau}^*}$, and $\widehat{Y}^{\mathrm{H}}_{\cdot\wedge\widehat{\tau}^*}$ is an $\mathcal{E}$-martingale. In particular, $\widehat{\tau}^*$ is an optimal stopping time for the problem $\widehat{Y}^{\mathrm{H}}_0$.*

**Remark 3.7.** The main idea for proving Theorem 3.6 is to show that $\mathcal{E}[\widehat{Y}^{\mathrm{H}}_{\tau_n}]$ converges to $\mathcal{E}[\widehat{Y}^{\mathrm{H}}_{\widehat{\tau}^*}]$, where $\tau_n$ is defined by (5.15) below and increases to $\widehat{\tau}^*$. However, we face a major difficulty that the dominated convergence theorem fails in our nonlinear expectation framework. Notice that $Y$ is an $\mathcal{E}$-supermartingale and thus $Y_{\tau_n}$ are decreasing under expectation (but not pointwise!). We shall extend the arguments of [3] for the monotone convergence theorem, Proposition 2.6, to our case. For this purpose, we need to construct certain continuous approximations of the stopping times $\tau_n$, and the requirement that the random maturity $\mathrm{H}$ is of the form (3.3) is crucial. We remark that, in his Markov model, Krylov [10] also considers this type of hitting times. We also remark that, in a special case, Song [15] proved that $\mathrm{H}$ is quasi-continuous. ∎

## 4. Deterministic maturity optimal stopping

We now prove Theorem 3.3. Throughout this section, Assumption 3.1 is always in force, and we consider the nonlinear Snell envelope $Y$ together with the first obstacle hitting time $\tau^*$, as defined in (3.2). Assume $|X| \le C_0$, and without loss of generality that $\rho_0 \le 2C_0$. It is obvious that

$$|Y| \le C_0, \qquad Y \ge X, \quad \text{and} \quad Y_T = X_T. \tag{4.1}$$

Throughout this section, we shall use the following modulus of continuity function:

$$\bar{\rho}_0(\delta) := \rho_0(\delta) \vee \Big[\rho_0(\delta^{\frac{1}{3}}) + \delta^{\frac{1}{3}}\Big], \tag{4.2}$$

and we shall use a generic constant $C$ which depends only on $C_0, T, d$, and the $L_0$ in Property (P1), and it may vary from line to line.

### 4.1. Dynamic programming principle

Similar to the standard Snell envelope characterization under linear expectation, our first step is to establish the dynamic programming principle. We start by the case of deterministic times.

**Lemma 4.1.** *For each $t$, the random variable $Y_t$ is uniformly continuous in $\omega$, with the modulus of continuity function $\rho_0$, and satisfies*

$$Y_{t_1}(\omega) = \sup_{\tau \in \mathcal{T}^{t_1}} \mathcal{E}_{t_1}\Big[X_\tau^{t_1,\omega}\mathbf{1}_{\{\tau<t_2\}} + Y_{t_2}^{t_1,\omega}\mathbf{1}_{\{\tau\geq t_2\}}\Big] \quad \textit{for all } 0 \le t_1 \le t_2 \le T, \omega \in \Omega. \tag{4.3}$$

**Proof.** (i) First, for any $t$, any $\omega, \omega' \in \Omega$, and any $\tau \in \mathcal{T}^t$, by (3.1) we have

$$\begin{aligned} |X_\tau^{t,\omega} - X_\tau^{t,\omega'}| &= \big|X(\tau(B^t), \omega \otimes_t B^t) - X(\tau(B^t), \omega' \otimes_t B^t)\big| \\ &\le \rho_0\Big(\mathbf{d}_\infty\big((\tau(B^t), \omega \otimes_t B^t), (\tau(B^t), \omega' \otimes_t B^t)\big)\Big) = \rho_0\big(\|\omega - \omega'\|_t\big). \end{aligned}$$

Since $\tau$ is arbitrary, this proves uniform continuity of $Y_t$ in $\omega$.

(ii) When $t_2 = T$, since $Y_T = X_T$ (4.3) coincides with the definition of $Y$. Without loss of generality we assume $(t_1, \omega) = (0, \mathbf{0})$ and $t := t_2 < T$. Recall that we omit the subscript $_0$.

*Step* 1. We first prove "$\le$". For any $\tau \in \mathcal{T}$ and $\mathbb{P} \in \mathcal{P}$:

$$\mathbb{E}^{\mathbb{P}}[X_\tau] = \mathbb{E}^{\mathbb{P}}\Big[X_\tau \mathbf{1}_{\{\tau<t\}} + \mathbb{E}_t^{\mathbb{P}}[X_\tau]\mathbf{1}_{\{\tau\ge t\}}\Big].$$

By the definition of the regular conditional probability distribution, we have $\mathbb{E}_t^{\mathbb{P}}[X_\tau](\omega) = \mathbb{E}^{\mathbb{P}^{t,\omega}}[X_{\tau^{t,\omega}}^{t,\omega}] \le Y_t(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\tau \ge t\}$, where the inequality follows from Property (P2) of the family $\{\mathcal{P}_t\}$ that $\mathbb{P}^{t,\omega} \in \mathcal{P}_t$. Then:

$$\mathbb{E}^{\mathbb{P}}[X_\tau] \le \mathbb{E}^{\mathbb{P}}\big[X_\tau \mathbf{1}_{\{\tau<t\}} + Y_t \mathbf{1}_{\{\tau\ge t\}}\big].$$

By taking the sup over $\tau$ and $\mathbb{P}$, it follows that:

$$Y_0 = \sup_{\tau\in\mathcal{T}} \mathcal{E}[X_\tau] \le \sup_{\tau\in\mathcal{T}} \mathcal{E}\big[X_\tau \mathbf{1}_{\{\tau<t\}} + Y_t \mathbf{1}_{\{\tau\ge t\}}\big].$$

*Step* 2. We next prove "$\ge$". Fix arbitrary $\tau \in \mathcal{T}$ and $\mathbb{P} \in \mathcal{P}$, we shall prove

$$\mathbb{E}^{\mathbb{P}}\big[X_\tau \mathbf{1}_{\{\tau<t\}} + Y_t \mathbf{1}_{\{\tau\ge t\}}\big] \le Y_0. \tag{4.4}$$

Let $\varepsilon > 0$, and $\{E_i\}_{i\ge1}$ be an $\mathcal{F}_t$-measurable partition of the event $\{\tau \ge t\} \in \mathcal{F}_t$ such that $\|\omega - \tilde{\omega}\|_t \le \varepsilon$ for all $\omega, \tilde{\omega} \in E_i$. For each $i$, fix an $\omega^i \in E_i$, and by the definition of $Y$ we have

$$Y_t(\omega^i) \le \mathbb{E}^{\mathbb{P}^i}\big[X_{\tau^i}^{t,\omega^i}\big] + \varepsilon \quad \text{for some } (\tau^i, \mathbb{P}^i) \in \mathcal{T}^t \times \mathcal{P}_t.$$

By (3.1) and the uniform continuity of $Y$, proved in (i), we have

$$|Y_t(\omega) - Y_t(\omega^i)| \le \rho_0(\varepsilon), \qquad |X_{\tau^i}^{t,\omega} - X_{\tau^i}^{t,\omega^i}| \le \rho_0(\varepsilon), \quad \text{for all } \omega \in E_i.$$

Thus, for $\omega \in E_i$,

$$Y_t(\omega) \le Y_t(\omega^i) + \rho_0(\varepsilon) \le \mathbb{E}^{\mathbb{P}^i}\big[X^{t,\omega^i}_{\tau^i}\big] + \varepsilon + \rho_0(\varepsilon) \le \mathbb{E}^{\mathbb{P}^i}\big[X^{t,\omega}_{\tau^i}\big] + \varepsilon + 2\rho_0(\varepsilon). \tag{4.5}$$

Thanks to Property (P3) of the family $\{\mathcal{P}_t\}$, we may define the following pair $(\tilde\tau, \tilde{\mathbb{P}}) \in \mathcal{T} \times \mathcal{P}$:

$$\tilde\tau := \mathbf{1}_{\{\tau<t\}}\tau + \mathbf{1}_{\{\tau\ge t\}}\sum_{i\ge1}\mathbf{1}_{E_i}\tau^i(B^t); \qquad \tilde{\mathbb{P}} := \mathbb{P}\otimes_t \left[\sum_{i\ge1}\mathbf{1}_{E_i}\mathbb{P}^i + \mathbf{1}_{\{\tau<t\}}\mathbb{P}\right].$$

It is obvious that $\{\tau < t\} = \{\tilde\tau < t\}$. Then, by (4.5),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[X_\tau\mathbf{1}_{\{\tau<t\}} + Y_t\mathbf{1}_{\{\tau\ge t\}}\big] &= \mathbb{E}^{\mathbb{P}}\left[X_\tau\mathbf{1}_{\{\tau<t\}} + \sum_{i\ge1}Y_t\mathbf{1}_{E_i}\right]\\
&\le \mathbb{E}^{\mathbb{P}}\left[X_\tau\mathbf{1}_{\{\tau<t\}} + \sum_{i\ge1}\mathbb{E}^{\mathbb{P}^i}[X^{t,\cdot}_{\tau^i}]\mathbf{1}_{E_i}\right] + \varepsilon + 2\rho_0(\varepsilon)\\
&= \mathbb{E}^{\tilde{\mathbb{P}}}\left[X_{\tilde\tau}\mathbf{1}_{\{\tilde\tau<t\}} + \sum_{i\ge1}X_{\tilde\tau}\mathbf{1}_{E_i}\right] + \varepsilon + 2\rho_0(\varepsilon)\\
&= \mathbb{E}^{\tilde{\mathbb{P}}}\big[X_{\tilde\tau}\big] + \varepsilon + 2\rho_0(\varepsilon) \le Y_0 + \varepsilon + 2\rho_0(\varepsilon),
\end{aligned}$$

which provides (4.4) by sending $\varepsilon \to 0$. ■

We now derive the regularity of $Y$ in $t$.

**Lemma 4.2.** *For each $\omega \in \Omega$ and $0 \le t_1 < t_2 \le T$,*

$$|Y_{t_1}(\omega) - Y_{t_2}(\omega)| \le C\bar\rho_0\Big(\mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big)\Big).$$

**Proof.** Denote $\delta := \mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big)$. If $\delta \ge \frac18$, then clearly $|Y_{t_1}(\omega) - Y_{t_2}(\omega)| \le 2C_0 \le C\bar\rho_0(\delta)$. So we continue the proof assuming $\delta \le \frac18$. First, by setting $\tau = t_2$ in Lemma 4.1,

$$\begin{aligned}
\delta Y &:= Y_{t_2}(\omega) - Y_{t_1}(\omega) \le Y_{t_2}(\omega) - \mathcal{E}_{t_1}\big[Y^{t_1,\omega}_{t_2}\big]\\
&\le \mathcal{E}_{t_1}\big[Y_{t_2}(\omega) - Y_{t_2}(\omega\otimes_{t_1}B^{t_1})\big]\\
&\le \mathcal{E}_{t_1}\big[\rho_0\big(\mathbf{d}_\infty\big((t_2,\omega),(t_2,\omega\otimes_{t_1}B^{t_1})\big)\big)\big]\\
&\le \mathcal{E}_{t_1}\big[\rho_0\big(\delta + \|B^{t_1}\|_{t_1+\delta}\big)\big].
\end{aligned}$$

On the other hand, by the inequality $X \le Y$, Lemma 4.1, and (3.1), we have

$$\begin{aligned}
-\delta Y \le{}& \sup_{\tau\in\mathcal{T}^{t_1}}\mathcal{E}_{t_1}\Big[\big[X^{t_1,\omega}_{t_2} + \rho_0\big(\mathbf{d}_\infty((\tau,\omega\otimes_{t_1}B^{t_1}),(t_2,\omega\otimes_{t_1}B^{t_1}))\big)\big]\mathbf{1}_{\{\tau<t_2\}}\\
&+ Y^{t_1,\omega}_{t_2}\mathbf{1}_{\{\tau\ge t_2\}}\Big] - Y_{t_2}(\omega)\\
\le{}& \mathcal{E}_{t_1}\Big[Y^{t_1,\omega}_{t_2} - Y_{t_2}(\omega) + \rho_0\big(\mathbf{d}_\infty((t_1,\omega),(t_2,\omega\otimes_{t_1}B^{t_1}))\big)\Big]
\end{aligned}$$

$$\le \mathcal{E}_{t_1}\Big[\rho_0\big(\mathbf{d}_\infty((t_2,\omega),(t_2,\omega\otimes_{t_1} B^{t_1}))\big)+\rho_0\big(\mathbf{d}_\infty((t_1,\omega),(t_2,\omega\otimes_{t_1} B^{t_1}))\big)\Big]$$

$$\le 2\mathcal{E}_{t_1}\big[\rho_0\big(\delta+\|B^{t_1}\|_{t_1+\delta}\big)\big].$$

Hence

$$|\delta Y| \le 2\mathcal{E}_{t_1}\big[\rho_0\big(\delta+\|B^{t_1}\|_{t_1+\delta}\big)\big] \le \mathcal{E}_{t_1}\left[\rho_0\left(\delta+\frac{3}{4}\delta^{\frac{1}{3}}\right)+2C_0\mathbf{1}_{\{\|B^{t_1}\|_{t_1+\delta}\ge\frac{3}{4}\delta^{\frac{1}{3}}\}}\right].$$

Since $\delta+\frac{3}{4}\delta^{\frac{1}{3}} \le \delta^{\frac{1}{3}}$ for $\delta \le \frac{1}{8}$, this provides:

$$|\delta Y| \le \rho_0(\delta^{\frac{1}{3}})+C\delta^{-\frac{2}{3}}\mathcal{E}_{t_1}\Big[\|B^{t_1}\|^2_{t_1+\delta}\Big] \le \rho_0(\delta^{\frac{1}{3}})+C\delta^{-\frac{2}{3}}\delta \le C\bar{\rho}_0(\delta). \quad \blacksquare \tag{4.6}$$

We are now ready to prove the dynamic programming principle for stopping times.

**Theorem 4.3.** *For any* $(t,\omega)\in\Lambda$ *and* $\tau\in\mathcal{T}^t$, *we have*

$$Y_t(\omega)=\sup_{\tilde\tau\in\mathcal{T}^t}\mathcal{E}_t\Big[X^{t,\omega}_{\tilde\tau}\mathbf{1}_{\{\tilde\tau<\tau\}}+Y^{t,\omega}_{\tau}\mathbf{1}_{\{\tilde\tau\ge\tau\}}\Big].$$

*Consequently,* $Y$ *is an* $\mathcal{E}$*-supermartingale on* $[0,T]$.

**Proof.** First, follow the arguments in Lemma 4.1(ii) Step 1 and note that Property (P2) of the family $\{\mathcal{P}_t\}$ holds for stopping times, one can prove straightforwardly that

$$Y_t(\omega)\le\sup_{\tilde\tau\in\mathcal{T}^t}\mathcal{E}_t\Big[X^{t,\omega}_{\tilde\tau}\mathbf{1}_{\{\tilde\tau<\tau\}}+Y^{t,\omega}_{\tau}\mathbf{1}_{\{\tilde\tau\ge\tau\}}\Big].$$

On the other hand, let $\tau_k\downarrow\tau$ such that $\tau_k$ takes only finitely many values. By Lemma 4.1 one can easily show that Theorem 4.3 holds for $\tau_k$. Then for any $\mathbb{P}\in\mathcal{P}_t$ and $\tilde\tau\in\mathcal{T}^t$, by denoting $\tilde\tau_m := [\tilde\tau+\frac{1}{m}]\wedge T$ we have

$$\mathbb{E}^{\mathbb{P}}\Big[X^{t,\omega}_{\tilde\tau_m}\mathbf{1}_{\{\tilde\tau_m<\tau_k\}}+Y^{t,\omega}_{\tau_k}\mathbf{1}_{\{\tilde\tau_m\ge\tau_k\}}\Big]\le Y_t(\omega).$$

Sending $k\to\infty$, by Lemma 4.2 and the dominated convergence theorem (under $\mathbb{P}$):

$$\mathbb{E}^{\mathbb{P}}\Big[X^{t,\omega}_{\tilde\tau_m}\mathbf{1}_{\{\tilde\tau_m\le\tau\}}+Y^{t,\omega}_{\tau}\mathbf{1}_{\{\tilde\tau_m>\tau\}}\Big]\le Y_t(\omega).$$

Since the process $X$ is right continuous in $t$, we obtain by sending $m\to\infty$:

$$Y_t(\omega)\ge\mathbb{E}^{\mathbb{P}}\Big[X^{t,\omega}_{\tilde\tau}\mathbf{1}_{\{\tilde\tau<\tau\}}+Y^{t,\omega}_{\tau}\mathbf{1}_{\{\tilde\tau\ge\tau\}}\Big],$$

which provides the required result by the arbitrariness of $\mathbb{P}$ and $\tilde\tau$. $\blacksquare$

### *4.2. Preparation for the $\mathcal{E}$-martingale property*

If $Y_0=X_0$, then $\tau^*=0$ and obviously all the statements of Theorem 3.3 hold true. Therefore, we focus on the non-trivial case $Y_0>X_0$.

We continue following the proof of the Snell envelope characterization in the standard linear expectation context. Let

$$\tau_n := \inf\left\{t\ge 0: Y_t-X_t\le\frac{1}{n}\right\}\wedge T, \quad \text{for } n>(Y_0-X_0)^{-1}. \tag{4.7}$$

**Lemma 4.4.** *The process $Y$ is an $\mathcal{E}$-martingale on* $[0, \tau_n]$.

**Proof.** By the dynamic programming principle of Theorem 4.3,

$$Y_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}\Big[X_\tau \mathbf{1}_{\{\tau<\tau_n\}} + Y_{\tau_n}\mathbf{1}_{\{\tau\geq\tau_n\}}\Big].$$

For any $\varepsilon > 0$, there exist $\tau_\varepsilon \in \mathcal{T}$ and $\mathbb{P}_\varepsilon \in \mathcal{P}$ such that

$$Y_0 \leq \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[X_{\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\tau_n\}} + Y_{\tau_n}\mathbf{1}_{\{\tau_\varepsilon\geq\tau_n\}}\Big] + \varepsilon \leq \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[Y_{\tau_\varepsilon\wedge\tau_n} - \frac{1}{n}\mathbf{1}_{\{\tau_\varepsilon<\tau_n\}}\Big] + \varepsilon, \tag{4.8}$$

where we used the fact that $Y_t - X_t > \frac{1}{n}$ for $t < \tau_n$, by the definition of $\tau_n$. On the other hand, it follows from the $\mathcal{E}$-supermartingale property of $Y$ in Theorem 4.3 that $\mathbb{E}^{\mathbb{P}_\varepsilon}\Big[Y_{\tau_\varepsilon\wedge\tau_n}\Big] \leq \mathcal{E}[Y_{\tau_\varepsilon\wedge\tau_n}] \leq Y_0$, which implies by (4.8) that $\mathbb{P}_\varepsilon[\tau_\varepsilon < \tau_n] \leq n\varepsilon$. We then get from (4.8) that:

$$\begin{aligned} Y_0 &\leq \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[(X_{\tau_\varepsilon} - Y_{\tau_n})\mathbf{1}_{\{\tau_\varepsilon<\tau_n\}} + Y_{\tau_n}\Big] + \varepsilon \leq C\mathbb{P}_\varepsilon[\tau_\varepsilon < \tau_n] + \mathbb{E}^{\mathbb{P}_\varepsilon}[Y_{\tau_n}] + \varepsilon \\ &\leq \mathcal{E}[Y_{\tau_n}] + (Cn+1)\varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, we obtain $Y_0 \leq \mathcal{E}[Y_{\tau_n}]$. Similarly one can prove $Y$ is an $\mathcal{E}$-submartingale on $[0, \tau_n]$. By the $\mathcal{E}$-supermartingale property of $Y$ established in Theorem 4.3, this implies that $Y$ is an $\mathcal{E}$-martingale on $[0, \tau_n]$. ■

By Lemma 4.2 we have

$$Y_0 - \mathcal{E}[Y_{\tau^*}] = \mathcal{E}[Y_{\tau_n}] - \mathcal{E}[Y_{\tau^*}] \leq C\mathcal{E}\Big[\bar\rho_0\Big(\mathbf{d}_\infty\big((\tau_n, \omega), (\tau^*, \omega)\big)\Big)\Big]. \tag{4.9}$$

Clearly, $\tau_n \nearrow \tau^*$, and $\bar\rho_0\Big(\mathbf{d}_\infty\big((\tau_n, \omega), (\tau^*, \omega)\big)\Big) \searrow 0$. However, in general the stopping times $\tau_n, \tau^*$ are not $\mathcal{P}$-quasi-continuous, so we cannot apply Proposition 2.6(ii) to conclude $Y_0 \leq \mathcal{E}[Y_{\tau^*}]$. To overcome this difficulty, we need to approximate $\tau_n$ by continuous random variables.

### *4.3. Continuous approximation*

The following lemma is crucial for us.

**Lemma 4.5.** *Let $\underline{\theta} \leq \theta \leq \overline{\theta}$ be random variables on $\Omega$, with values in a compact interval $I \subset \mathbb{R}$, such that for some $\Omega_0 \subset \Omega$ and $\delta > 0$:*

$$\underline{\theta}(\omega) \leq \theta(\omega') \leq \overline{\theta}(\omega) \quad \text{for all } \omega \in \Omega_0 \quad \text{and} \quad \|\omega - \omega'\| \leq \delta.$$

*Then for any $\varepsilon > 0$, there exists a uniformly continuous function $\hat\theta : \Omega \to I$ and an open subset $\Omega_\varepsilon \subset \Omega$ such that*

$$\mathcal{C}\Big[\Omega_\varepsilon^c\Big] \leq \varepsilon \quad \text{and} \quad \underline{\theta} - \varepsilon \leq \hat\theta \leq \overline{\theta} + \varepsilon \text{ in } \Omega_\varepsilon \cap \Omega_0.$$

**Proof.** If $I$ is a single point set, then $\theta$ is a constant and the result is obviously true. Thus at below we assume the length $|I| > 0$. Let $\{\omega_j\}_{j\geq1}$ be a dense sequence in $\Omega$. Denote $O_j := \{\omega \in \Omega : \|\omega - \omega_j\| < \frac{\delta}{2}\}$ and $\Omega_n := \cup_{j=1}^n O_j$. It is clear that $\Omega_n$ is open and $\Omega_n \uparrow \Omega$ as $n \to \infty$. Let $f_n : [0, \infty) \to [0, 1]$ be defined as follows: $f_n(x) = 1$for $x \in [0, \frac{\delta}{2}]$, $f_n(x) = \frac{1}{n^2|I|}$

for $x \geq \delta$, and $f_n$ is linear in $[\frac{\delta}{2}, \delta]$. Define

$$\theta_n(\omega) := \phi_n(\omega) \sum_{j=1}^{n} \theta(\omega_j)\varphi_{n,j}(\omega) \quad \text{where } \varphi_{n,j}(\omega) := f_n(\|\omega - \omega_j\|)$$
$$\text{and} \quad \phi_n := \left(\sum_{j=1}^{n} \varphi_{n,j}\right)^{-1}.$$

Then clearly $\theta_n$ is uniformly continuous and takes values in $I$. For each $\omega \in \Omega_n \cap \Omega_0$, the set $J_n(\omega) := \{1 \leq j \leq n : \|\omega - \omega_j\| \leq \delta\} \neq \emptyset$ and $\phi_n(\omega) \leq 1$. Then, by our assumption,

$$\begin{aligned}\theta_n(\omega) - \overline{\theta}(\omega) &= \phi_n(\omega)\left(\sum_{j \in J_n(\omega)} [\theta(\omega_j) - \overline{\theta}(\omega)]\varphi_{n,j}(\omega) + \sum_{j \notin J_n(\omega)} [\theta(\omega_j) - \overline{\theta}(\omega)]\varphi_{n,j}(\omega)\right) \\ &\leq \phi_n(\omega) \sum_{j \notin J_n(\omega)} |I|\varphi_{n,j}(\omega) \leq \phi_n(\omega) \sum_{j \notin J_n(\omega)} \frac{1}{n^2} \leq \frac{1}{n}.\end{aligned}$$

Similarly one can show that $\underline{\theta} - \frac{1}{n} \leq \theta_n$ in $\Omega_n \cap \Omega_0$. Finally, since $\Omega_n \uparrow \Omega$ as $n \to \infty$, it follows from Proposition 2.6(i) that $\lim_{n\to\infty} \mathcal{C}[\Omega_n^c] = 0$. ■

### 4.4. *Proof of Theorem 3.3*

We proceed in two steps.

*Step* 1. For each $n$, let $\delta_n > 0$ be such that $3C\bar{\rho}_0(\delta_n) \leq \frac{1}{n(n+1)}$ for the constant $C$ in Lemma 4.2. Now for any $\omega$ and $\omega'$ such that $\|\omega - \omega'\|_T \leq \delta_n$, by (3.1), the uniform continuity of $Y$ in Lemma 4.1, and the fact that $\rho_0 \leq \bar{\rho}_0$, we have

$$(Y - X)_{\tau_{n+1}(\omega)}(\omega') \leq (Y - X)_{\tau_{n+1}(\omega)}(\omega) + 3C\bar{\rho}_0(\delta_n) \leq \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}.$$

Then $\tau_n(\omega') \leq \tau_{n+1}(\omega)$. Since $3C\bar{\rho}_0(\delta_n) \leq \frac{1}{n(n+1)} \leq \frac{1}{n(n-1)}$, similarly we have $\tau_{n-1}(\omega) \leq \tau_n(\omega')$. We may then apply Lemma 4.5 with $\underline{\theta} = \tau_{n-1}, \theta = \tau_n, \overline{\theta} = \tau_{n+1}$, and $\Omega_0 = \Omega$. Thus, there exist an open set $\Omega_n \subset \Omega$ and a continuous random variable $\tilde{\tau}_n$ valued in $[0, T]$ such that

$$\mathcal{C}\big[\Omega_n^c\big] \leq 2^{-n} \quad \text{and} \quad \tau_{n-1} - 2^{-n} \leq \tilde{\tau}_n \leq \tau_{n+1} + 2^{-n} \quad \text{in } \Omega_n.$$

*Step* 2. By Lemma 4.4, for each $n$ large, there exists $\mathbb{P}_n \in \mathcal{P}$ such that

$$Y_0 = \mathcal{E}[Y_{\tau_n}] \leq \mathbb{E}^{\mathbb{P}_n}[Y_{\tau_n}] + 2^{-n}.$$

By Property (P1), $\mathcal{P}$ is weakly compact. Then, there exists a subsequence $\{n_j\}$ and $\mathbb{P}^* \in \mathcal{P}$ such that $\mathbb{P}_{n_j}$ converges weakly to $\mathbb{P}^*$. Now for any $n$ large and any $n_j \geq n$, note that $\tau_{n_j} \geq \tau_n$. Since $Y$ is an $\mathcal{E}$-supermartingale and thus a $\mathbb{P}_{n_j}$-supermartingale, we have

$$Y_0 - 2^{-n_j} \leq \mathbb{E}^{\mathbb{P}_{n_j}}\big[Y_{\tau_{n_j}}\big] \leq \mathbb{E}^{\mathbb{P}_{n_j}}\big[Y_{\tau_n}\big] \leq \mathbb{E}^{\mathbb{P}_{n_j}}\big[Y_{\tilde{\tau}_n}\big] + \mathbb{E}^{\mathbb{P}_{n_j}}\big[|Y_{\tilde{\tau}_n} - Y_{\tau_n}|\big]. \tag{4.10}$$

By the boundedness of $Y$ in (4.1) and the uniform continuity of $Y$ in Lemma 4.2, we have

$$\begin{aligned}|Y_{\tilde{\tau}_n} - Y_{\tau_n}| &\leq C\bar{\rho}_0\Big(\mathbf{d}_\infty\big((\tilde{\tau}_n, \omega), (\tau_n, \omega)\big)\Big) \\ &\leq C\bar{\rho}_0\Big(\mathbf{d}_\infty\big((\tilde{\tau}_n, \omega), (\tau_n, \omega)\big)\Big)\mathbf{1}_{\Omega_{n-1}\cap\Omega_{n+1}} + C\mathbf{1}_{\Omega_{n-1}^c\cup\Omega_{n+1}^c}.\end{aligned}$$

Notice that $\tilde\tau_{n-1} - 2^{1-n} \le \tau_n \le \tilde\tau_{n+1} + 2^{-1-n}$ on $\Omega_{n-1} \cap \Omega_{n+1}$. Then

$$\begin{aligned}|Y_{\tilde\tau_n} - Y_{\tau_n}| &\le C\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n-1}-2^{1-n},\omega)\big)\Big)\mathbf{1}_{\Omega_{n-1}\cap\Omega_{n+1}}\\ &\quad + C\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n+1}+2^{-1-n},\omega)\big)\Big)\mathbf{1}_{\Omega_{n-1}\cap\Omega_{n+1}} + C\mathbf{1}_{\Omega^c_{n-1}\cup\Omega^c_{n+1}}\\ &\le C\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n-1}-2^{1-n},\omega)\big)\Big)\\ &\quad + C\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n+1}+2^{-1-n},\omega)\big)\Big) + C\mathbf{1}_{\Omega^c_{n-1}\cup\Omega^c_{n+1}}.\end{aligned}$$

Then (4.10) together with the estimate $\mathcal{C}[\Omega_n^c] \le 2^{-n}$ lead to

$$\begin{aligned}Y_0 - 2^{-n_j} &\le \mathbb{E}^{\mathbb{P}_{n_j}}\big[Y_{\tilde\tau_n}\big] + C\mathbb{E}^{\mathbb{P}_{n_j}}\Big[\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n-1}-2^{1-n},\omega)\big)\Big)\Big]\\ &\quad + C\mathbb{E}^{\mathbb{P}_{n_j}}\Big[\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n+1}+2^{-1-n},\omega)\big)\Big)\Big] + C2^{-n}.\end{aligned}$$

Notice that $Y$ and $\tilde\tau_{n-1}, \tilde\tau_n, \tilde\tau_{n+1}$ are continuous. Send $j \to \infty$, we obtain

$$\begin{aligned}Y_0 &\le \mathbb{E}^{\mathbb{P}^*}\big[Y_{\tilde\tau_n}\big] + C\mathbb{E}^{\mathbb{P}^*}\Big[\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n-1}-2^{1-n},\omega)\big)\Big)\Big]\\ &\quad + C\mathbb{E}^{\mathbb{P}^*}\Big[\bar\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau_n,\omega),(\tilde\tau_{n+1}-2^{-1-n},\omega)\big)\Big)\Big] + C2^{-n}.\end{aligned} \tag{4.11}$$

Since $\sum_n \mathbb{P}^*\big[|\tilde\tau_n - \tau_n| \ge 2^{-n}\big] \le \sum_n \mathcal{C}\big[|\tilde\tau_n - \tau_n| \ge 2^{-n}\big] \le \sum_n 2^{-n} < \infty$ and $\tau_n \uparrow \tau^*$, by the Borel–Cantelli lemma under $\mathbb{P}^*$ we see that $\tilde\tau_n \to \tau^*$, $\mathbb{P}^*$-a.s. Send $n \to \infty$ in (4.11) and apply the dominated convergence theorem under $\mathbb{P}^*$, we obtain

$$Y_0 \le \mathbb{E}^{\mathbb{P}^*}\big[Y_{\tau^*}\big] \le \mathcal{E}[Y_{\tau^*}].$$

Similarly $Y_t(\omega) \le \mathcal{E}_t[Y^{t,\omega}_{\tau^*}]$ for $t < \tau^*(\omega)$. By the $\mathcal{E}$-supermartingale property of $Y$ established in Theorem 4.3, this implies that $Y$ is an $\mathcal{E}$-martingale on $[0, \tau^*]$. ■

## 5. Random maturity optimal stopping

In this section, we prove Theorem 3.6. The main idea follows that of Theorem 3.3. However, since $\widehat X_{\mathrm{H}}$ is not continuous in $\omega$, the estimates become much more involved.

Throughout this section, let $X, \mathrm{H}, O, t_0, \widehat X := \widehat X^{\mathrm{H}}, \widehat Y := \widehat Y^{\mathrm{H}}$, and $\widehat\tau^*$ be as in Theorem 3.6. Assumptions 3.1 and 3.4 will always be in force. We shall emphasize when the additional Assumption 3.4 is needed, and we fix the constant $L$ as in Assumption 3.4 (i). Assume $|X| \le C_0$, and without loss of generality that $\rho_0 \le 2C_0$ and $L \le 1$. It is clear that

$$|\widehat Y| \le C_0, \widehat X \le \widehat Y, \quad \text{and} \quad \widehat Y_{\mathrm{H}} = \widehat X_{\mathrm{H}} = X_{\mathrm{H}-}. \tag{5.1}$$

By (3.1) and the fact that $X$ has positive jumps, one can check straightforwardly that,

$$\widehat X(t,\omega) - \widehat X(t',\omega') \le \rho_0\big(\mathbf{d}_\infty((t,\omega),(t',\omega'))\big) \quad \text{for } t \le t',\ t \le \mathrm{H}(\omega),\ t' \le \mathrm{H}(\omega') \tag{5.2}$$

except the case $t = t' = \mathrm{H}(\omega') < \mathrm{H}(\omega) \le t_0$.

In particular,

$$\widehat X(t,\omega) - \widehat X(t',\omega) \le \rho_0\big(\mathbf{d}_\infty((t,\omega),(t',\omega))\big) \quad \text{whenever } t \le t' \le \mathrm{H}(\omega). \tag{5.3}$$

Moreover, we define

$$\rho_1(\delta) := \rho_0(\delta) \vee \Big[\rho_0\big((L^{-1}\delta)^{\frac{1}{3}}\big) + \delta^{\frac{1}{3}}\Big], \qquad \rho_2(\delta) := [\rho_1(\delta) + \delta] \vee [\rho_1(\delta^{\frac{1}{3}}) + \delta^{\frac{1}{3}}], \quad (5.4)$$

and in this section, the generic constant $C$ may depend on $L$ as well.

*5.1. Dynamic programming principle*

We start with the regularity in $\omega$.

**Lemma 5.1.** *For any $t < \mathrm{H}(\omega) \wedge \mathrm{H}(\omega')$ we have:*

$$|\widehat{Y}_t(\omega) - \widehat{Y}_t(\omega')| \le C\rho_1\big(\|\omega - \omega'\|_t\big).$$

To motivate our proof, we first follow the arguments in Lemma 4.1(i) and see why it does not work here. Indeed, note that

$$\widehat{Y}_t(\omega) - \widehat{Y}_t(\omega') \le \sup_{\tau\in\mathcal{T}^t} \sup_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\Big[\widehat{X}^{t,\omega}_{\tau\wedge \mathrm{H}^{t,\omega}} - \widehat{X}^{t,\omega'}_{\tau\wedge \mathrm{H}^{t,\omega'}}\Big].$$

Since we do not have $\mathrm{H}^{t,\omega} \le \mathrm{H}^{t,\omega'}$, we cannot apply (5.2) to obtain the required estimate.

**Proof.** Let $\tau \in \mathcal{T}^t$ and $\mathbb{P} \in \mathcal{P}_t$. Denote $\delta := \frac{1}{L}\|\omega-\omega'\|_t$, $t_\delta := [t+\delta]\wedge t_0$ and $\tilde{B}^{t_\delta}_s := B^t_{s+\delta} - B^t_{t_\delta}$ for $s \ge t$. Set $\tau'(B^t) := [\tau(\tilde{B}^{t_\delta}) + \delta] \wedge t_0$, then $\tau' \in \mathcal{T}^t$. Moreover, by Assumption 3.4 and Property (P3), we may choose $\mathbb{P}' \in \mathcal{P}_t$ defined as follows: $\alpha^{\mathbb{P}'} := \frac{1}{\delta}(\omega_t - \omega'_t)$, $\beta^{\mathbb{P}'} := \mathbf{0}$ on $[t, t_\delta]$, and the $\mathbb{P}'$-distribution of $\tilde{B}^{t_\delta}$ is equal to the $\mathbb{P}$-distribution of $B^t$. We claim that

$$I := \mathbb{E}^{\mathbb{P}}[\widehat{X}^{t,\omega}_{\tau\wedge \mathrm{H}^{t,\omega}}] - \mathbb{E}^{\mathbb{P}'}[\widehat{X}^{t,\omega'}_{\tau'\wedge \mathrm{H}^{t,\omega'}}] \le C\rho_1(L\delta). \tag{5.5}$$

Then $\mathbb{E}^{\mathbb{P}}[\widehat{X}^{t,\omega}_{\tau\wedge \mathrm{H}^{t,\omega}}] - \widehat{Y}_t(\omega') \le \mathbb{E}^{\mathbb{P}}[\widehat{X}^{t,\omega}_{\tau\wedge \mathrm{H}^{t,\omega}}] - \mathbb{E}^{\mathbb{P}'}[\widehat{X}^{t,\omega'}_{\tau'\wedge \mathrm{H}^{t,\omega'}}] \le C\rho_1(L\delta)$, and it follows from the arbitrariness of $\mathbb{P} \in \mathcal{P}_t$ and $\tau \in \mathcal{T}^t$ that $\widehat{Y}_t(\omega) - \widehat{Y}_t(\omega') \le C\rho_1(L\delta)$. By exchanging the roles of $\omega$ and $\omega'$, we obtain the required estimate.

It remains to prove (5.5). Denote

$$\tilde{\omega}'_s := \omega'_s\mathbf{1}_{[0,t)}(s) + [\omega'_t + \alpha^{\mathbb{P}'}(s-t)]\mathbf{1}_{[t,T]}(s).$$

Since $t < \mathrm{H}(\omega) \wedge \mathrm{H}(\omega')$, we have $\omega_t, \omega'_t \in O$. By the convexity of $O$, this implies that $\tilde{\omega}'_s \in O$ for $s \in [t, t_\delta]$, and thus $\mathrm{H}^{t,\omega'}(B^t) = (\mathrm{H}^{t,\omega}(\tilde{B}^{t_\delta}) + \delta) \wedge t_0$, $\mathbb{P}'$-a.s. Therefore,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}'}[\widehat{X}^{t,\omega'}_{\tau'\wedge \mathrm{H}^{t,\omega'}}] &= \mathbb{E}^{\mathbb{P}'}\Big[\widehat{X}\Big(\tau'(B^t) \wedge \mathrm{H}^{t,\omega'}(B^t), \omega' \otimes_t B^t\Big)\Big] \\
&= \mathbb{E}^{\mathbb{P}'}\Big[\widehat{X}\Big([\tau(\tilde{B}^{t_\delta}) + \delta] \wedge [\mathrm{H}^{t,\omega}(\tilde{B}^{t_\delta}) + \delta] \wedge t_0, \tilde{\omega}' \otimes_{t_\delta} \tilde{B}^{t_\delta}_{\cdot-\delta}\Big)\Big] \\
&= \mathbb{E}^{\mathbb{P}}\Big[\widehat{X}\Big([\tau(B^t) + \delta] \wedge [\mathrm{H}^{t,\omega}(B^t) + \delta] \wedge t_0, \tilde{\omega}' \otimes_{t_\delta} B^t_{\cdot-\delta}\Big)\Big],
\end{aligned} \tag{5.6}$$

while

$$\mathbb{E}^{\mathbb{P}}[\widehat{X}^{t,\omega}_{\tau\wedge \mathrm{H}^{t,\omega}}] = \mathbb{E}^{\mathbb{P}}\Big[\widehat{X}\Big(\tau(B^t) \wedge \mathrm{H}^{t,\omega}(B^t), \omega \otimes_t B^t\Big)\Big].$$

Notice that, whenever $\tau(B^t) \wedge \mathrm{H}^{t,\omega}(B^t) = [\tau(B^t) + \delta] \wedge [\mathrm{H}^{t,\omega}(B^t) + \delta] \wedge t_0$, we have $\tau(B^t) \wedge \mathrm{H}^{t,\omega}(B^t) = t_0$. This excludes the exceptional case in (5.2). Then it follows from (5.6) and (5.2) that

$$I \le \mathbb{E}^{\mathbb{P}}\Big[\rho_0\Big(\delta + \|(\omega\otimes_t B^t)_{\cdot\wedge\tau(B^t)\wedge \mathrm{H}^{t,\omega}(B^t)} - (\tilde\omega'\otimes_{t_\delta} B^t_{\cdot-\delta})_{\cdot\wedge[\tau(B^t)+\delta]\wedge[\mathrm{H}^{t,\omega}(B^t)+\delta]\wedge t_0}\|_{t_0}\Big)\Big].$$

Note that, denoting $\theta := \tau(B^t)\wedge \mathrm{H}^{t,\omega}(B^t)$,

$$\begin{aligned}
&\|(\omega\otimes_t B^t)_{\cdot\wedge\tau(B^t)\wedge \mathrm{H}^{t,\omega}(B^t)} - (\tilde\omega'\otimes_{t_\delta} B^t_{\cdot-\delta})_{\cdot\wedge[\tau(B^t)+\delta]\wedge[\mathrm{H}^{t,\omega}(B^t)+\delta]\wedge t_0}\|_{t_0}\\
&\quad\le \|\omega\otimes_t B^t - \tilde\omega'\otimes_{t_\delta} B^t_{\cdot-\delta}\|_{t_0} + \sup_{0\le r\le\delta}|(\omega\otimes_t B^t)_{\theta+r} - (\omega\otimes_t B^t)_\theta|\\
&\quad\le \Big[\|\omega-\omega'\|_t\Big]\vee\Big[\sup_{t\le s\le t_\delta}|\omega_t + B^t_s - \tilde\omega'_s|\Big]\vee\Big[\sup_{t_\delta\le s\le t_0}|\omega_t + B^t_s - \tilde\omega'_{t_\delta} - B^t_{s-\delta}|\Big]\\
&\qquad+\sup_{0\le r\le\delta}|(\omega\otimes_t B^t)_{\theta+r} - (\omega\otimes_t B^t)_\theta|\\
&\quad\le 2L\delta + \|B^t\|_{t_\delta} + \sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}| + \sup_{0\le r\le\delta}|B^t_{\theta+r} - B^t_\theta|.
\end{aligned}$$

Since $L\le 1$, we have

$$I \le \mathbb{E}^{\mathbb{P}}\Big[\rho_0\Big(3\delta + \|B^t\|_{t_\delta} + \sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}| + \sup_{0\le r\le\delta}|B^t_{\theta+r} - B^t_\theta|\Big)\Big].$$

If $\delta\ge\frac18$, then $I\le 2C_0\le C\rho_1(L\delta)$. We then continue assuming $\delta\le\frac18$, and thus $3\delta+\frac14\delta^{\frac13}\le\delta^{\frac13}$. Therefore,

$$\begin{aligned}
I &\le \rho_0(\delta^{\frac13}) + C\mathbb{P}\Big(\|B^t\|_{t_\delta} + \sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}| + \sup_{0\le r\le\delta}|B^t_{\theta+r} - B^t_\theta| \ge \frac14\delta^{\frac13}\Big)\\
&\le \rho_0(\delta^{\frac13}) + C\delta^{-\frac83}\mathbb{E}^{\mathbb{P}}\Big[\|B^t\|^8_{t_\delta} + \sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}|^8 + \sup_{0\le r\le\delta}|B^t_{\theta+r} - B^t_\theta|^8\Big]\\
&\le \rho_0(\delta^{\frac13}) + C\delta^{\frac43} + C\delta^{-\frac83}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}|^8\Big].
\end{aligned}$$

Set $t_\delta = s_0 < \cdots < s_n = t_0$ such that $\delta\le s_{i+1}-s_i\le 2\delta$, $i=0,\ldots,n-1$. Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\sup_{t_\delta\le s\le t_0}|B^t_s - B^t_{s-\delta}|^8\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\max_{0\le i\le n-1}\sup_{s_i\le s\le s_{i+1}}|B^t_s - B^t_{s-\delta}|^8\Big]\\
&\le \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}}\Big[\sup_{s_i\le s\le s_{i+1}}[|B^t_s - B^t_{s_i-\delta}| + |B^t_{s-\delta} - B^t_{s_i-\delta}|]^8\Big]\\
&\le C\sum_{i=0}^{n-1}(s_{i+1}-s_i+\delta)^4 \le C\delta^{-1}\delta^4 = C\delta^3.
\end{aligned}$$

Thus $I\le\rho_0(\delta^{\frac13}) + C\delta^{\frac43} + C\delta^{-\frac83}\delta^3 \le \rho_0(\delta^{\frac13}) + C\delta^{\frac13}\le C\rho_1(L\delta)$, proving (5.5) and hence the lemma. ■

We next show that the dynamic programming principle holds along deterministic times.

**Lemma 5.2.** *Let $t_1 < \mathrm{H}(\omega)$ and $t_2\in[t_1,t_0]$. We have:*

$$\widehat Y_{t_1}(\omega) = \sup_{\tau\in\mathcal{T}^{t_1}}\mathcal{E}_{t_1}\Big[\widehat X^{t_1,\omega}_{\tau\wedge\mathrm{H}^{t_1,\omega}}\mathbf{1}_{\{\tau\wedge\mathrm{H}^{t_1,\omega}<t_2\}} + \widehat Y^{t_1,\omega}_{t_2}\mathbf{1}_{\{\tau\wedge\mathrm{H}^{t_1,\omega}\ge t_2\}}\Big].$$

**Proof.** When $t_2 = t_0$, the lemma coincides with the definition of $\widehat{Y}$. Without loss of generality we assume $(t_1, \omega) = (0, \mathbf{0})$ and $t := t_2 < t_0$. First, follow the arguments in Lemma 4.1(ii) Step 1, one can easily prove

$$\widehat{Y}_0 \le \sup_{\tau \in \mathcal{T}} \mathcal{E}\Big[\widehat{X}_{\tau\wedge \mathrm{H}} \mathbf{1}_{\{\tau\wedge \mathrm{H} < t\}} + \widehat{Y}_t \mathbf{1}_{\{\tau \wedge \mathrm{H} \ge t\}}\Big]. \tag{5.7}$$

To show that equality holds in the above inequality, fix arbitrary $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$ satisfying $\tau \le \mathrm{H}$ (otherwise reset $\tau$ as $\tau \wedge \mathrm{H}$), we shall prove

$$\mathbb{E}^{\mathbb{P}}\Big[\widehat{X}_\tau \mathbf{1}_{\{\tau < t\}} + \widehat{Y}_t \mathbf{1}_{\{\tau \ge t\}}\Big] \le \widehat{Y}_0.$$

Since $\widehat{Y}_{\mathrm{H}} = \widehat{X}_{\mathrm{H}}$, this amounts to show that:

$$\mathbb{E}^{\mathbb{P}}\Big[\widehat{X}_\tau \mathbf{1}_{\{\tau < t\}\cup\{\mathrm{H} \le t\}} + \widehat{Y}_t \mathbf{1}_{\{\tau \ge t, \mathrm{H} > t\}}\Big] \le \widehat{Y}_0. \tag{5.8}$$

We adapt the arguments in Lemma 4.1(ii) Step 2 to the present situation. Fix $0 < \delta \le t_0 - t$. Let $\{E_i\}_{i \ge 1}$ be an $\mathcal{F}_t$ measurable partition of the event $\{\tau \ge t, \mathrm{H} > t\} \in \mathcal{F}_t$ such that $\|\omega - \tilde\omega\| \le L\delta$ for all $\omega, \tilde\omega \in E_i$. Fix an $\omega^i \in E_i$ for each $i$. By the definition of $\widehat{Y}$ we have

$$\widehat{Y}_t(\omega^i) \le \mathbb{E}^{\mathbb{P}^i}\Big[\widehat{X}^{t,\omega^i}_{\tau^i \wedge \mathrm{H}^{t,\omega^i}}\Big] + \delta \quad \text{for some } (\tau^i, \mathbb{P}^i) \in \mathcal{T}^t \times \mathcal{P}_t. \tag{5.9}$$

As in Lemma 5.1, we set $t_\delta := t + \delta < t_0$, $\tilde B^{t_\delta}_s := B^t_{s+\delta} - B^t_{t_\delta}$ for $s \ge t$, and $\tilde\tau^i(B^t) := [\tau^i(\tilde B^{t_\delta}) + \delta] \wedge t_0$. Then $\tilde\tau^i \in \mathcal{T}^t$. Moreover by Assumption 3.4 and Property (P3), for each $\omega \in E_i$, we may define $\mathbb{P}^{i,\omega} \in \mathcal{P}_t$ as follows: $\alpha^{\mathbb{P}^{i,\omega}} := \frac{1}{\delta}(\omega^i_t - \omega_t)$, $\beta^{\mathbb{P}^{i,\omega}} := \mathbf{0}$ on $[t, t_\delta]$, and the $\mathbb{P}^{i,\omega}$-distribution of $\tilde B^{t_\delta}$ is equal to the $\mathbb{P}^i$-distribution of $B^t$. By (5.5), we have

$$\mathbb{E}^{\mathbb{P}^i}[\widehat{X}^{t,\omega^i}_{\tau^i\wedge \mathrm{H}^{t,\omega^i}}] - \mathbb{E}^{\mathbb{P}^{i,\omega}}[\widehat{X}^{t,\omega}_{\tilde\tau^i \wedge \mathrm{H}^{t,\omega}}] \le C\rho_1(L\delta). \tag{5.10}$$

Then by Lemma 5.1 and (5.9), (5.10) we have

$$\widehat{Y}_t(\omega) \le \widehat{Y}_t(\omega^i) + C\rho_1(L\delta) \le \mathbb{E}^{\mathbb{P}^{i,\omega}}[\widehat{X}^{t,\omega}_{\tilde\tau^i\wedge \mathrm{H}^{t,\omega}}] + \delta + C\rho_1(L\delta), \quad \text{for all } \omega \in E_i. \tag{5.11}$$

We next define:

$$\tilde\tau := \mathbf{1}_{\{\tau < t\}\cup\{\mathrm{H}\le t\}}\tau + \sum_{i\ge 1} \mathbf{1}_{E_i} \tilde\tau^i(B^t), \quad \text{and then } \{\tau < t\}\cup\{\mathrm{H} \le t\} = \{\tilde\tau < t\} \cup \{\mathrm{H} \le t\}.$$

Since $\tau \le \mathrm{H}$, we see that $\{\tau < t\} \cup \{\mathrm{H} \le t\} = \{\tau < t\} \cup \{\tau = \mathrm{H} = t\}$, and thus it is clear that $\tilde\tau \in \mathcal{T}$. Moreover, we claim that there exists $\tilde{\mathbb{P}} \in \mathcal{P}$ such that

$$\begin{aligned} &\tilde{\mathbb{P}} = \mathbb{P} \text{ on } \mathcal{F}_t \text{ and the regular conditional probability distribution}\\ &(\tilde{\mathbb{P}})^{t,\omega} = \mathbb{P}^{i,\omega} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in E_i, i \ge 1,\\ &(\tilde{\mathbb{P}})^{t,\omega} = \mathbb{P}^{t,\omega} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \{\tau < t\} \cup \{\mathrm{H} \le t\}. \end{aligned} \tag{5.12}$$

Then, by (5.11) we have

$$\widehat{Y}_t(\omega) \le \mathbb{E}^{(\tilde{\mathbb{P}})^{t,\omega}}\big[\widehat{X}^{t,\omega}_{(\tilde\tau\wedge \mathrm{H})^{t,\omega}}\big] + \delta + C\rho_1(L\delta), \quad \mathbb{P}\text{-a.e. } \omega \in \{\tau \ge t, \mathrm{H} > t\}, \tag{5.13}$$

and therefore:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\widehat{X}_\tau \mathbf{1}_{\{\tau<t\}\cup\{\mathrm{H}\le t\}} + \widehat{Y}_t \mathbf{1}_{\{\tau\ge t,\mathrm{H}>t\}}\Big] &\le \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\widehat{X}_{\tilde\tau\wedge\mathrm{H}} \mathbf{1}_{\{\tau<t\}\cup\{\mathrm{H}\le t\}} + \widehat{X}_{\tilde\tau\wedge\mathrm{H}} \mathbf{1}_{\{\tau\ge t,\mathrm{H}>t\}}\Big] \\ &\quad + \delta + C\rho_1(L\delta) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\widehat{X}_{\tilde\tau\wedge\mathrm{H}}\Big] + \delta + C\rho_1(L\delta) \le \widehat{Y}_0 + \delta + C\rho_1(L\delta),\end{aligned}$$

which implies (5.8) by sending $\delta \to 0$. Then the reverse inequality of (5.7) follows from the arbitrariness of $\mathbb{P}$ and $\tau$.

It remains to prove (5.12). For any $\varepsilon > 0$ and each $i \ge 1$, there exists a partition $\{E^i_j, j \ge 1\}$ of $E_i$ such that $\|\omega - \omega'\|_t \le \varepsilon$ for any $\omega, \omega' \in E^i_j$. Fix an $\omega^{ij} \in E^i_j$ for each $(i, j)$. By Property (P3) we may define $\tilde{\mathbb{P}}^\varepsilon \in \mathcal{P}$ by:

$$\tilde{\mathbb{P}}^\varepsilon := \mathbb{P} \otimes_t \left[\sum_{i\ge1}\sum_{j\ge1} \mathbb{P}^{i,\omega^{ij}} \mathbf{1}_{E^i_j} + \mathbb{P}\mathbf{1}_{\{\tau<t\}\cup\{\mathrm{H}\le t\}}\right].$$

By Property (P1), $\mathcal{P}$ is weakly compact. Then $\tilde{\mathbb{P}}^\varepsilon$ has a weak limit $\tilde{\mathbb{P}} \in \mathcal{P}$ as $\varepsilon \to 0$. We now show that $\tilde{\mathbb{P}}$ satisfies all the requirements in (5.12). Indeed, for any partition $0 = s_0 < \cdots < s_m = t < s_{m+1} < \cdots < s_M = t_\delta < s_{M+1} < \cdots < s_N = T$ and any bounded and uniformly continuous function $\varphi : \mathbb{R}^{N\times d} \to \mathbb{R}$, let $\xi := \varphi\big(B_{s_1} - B_{s_0}, \ldots, B_{s_N} - B_{s_{N-1}}\big)$. Then, denoting $\Delta s_k := s_{k+1} - s_k$, $\Delta\omega_k := \omega_{s_k} - \omega_{s_{k-1}}$, we see that

$$\mathbb{E}^{\mathbb{P}^{i,\omega}}[\xi^{t,\omega}] = \eta^i_t(\omega), \qquad \mathbb{E}^{\mathbb{P}^{i,\omega^{ij}}}[\xi^{t,\omega}] = \eta^{i,j}_t(\omega),$$

where:

$$\begin{aligned}&\eta^i_t(\omega) := \mathbb{E}^{\mathbb{P}^i}\Big[\varphi\Big((\Delta\omega_k)_{1\le k\le m}, \frac{\omega^i_t - \omega_t}{\delta}(\Delta s_k)_{m+1\le k\le M}, (B_{s_k-\delta} - B_{s_{k-1}-\delta})_{M+1\le k\le N}\Big)\Big];\\ &\eta^{i,j}_t(\omega)\\ &\quad := \mathbb{E}^{\mathbb{P}^i}\Big[\varphi\Big((\Delta\omega_k)_{1\le k\le m}, \frac{\omega^i_t - \omega^{ij}_t}{\delta}(\Delta s_k)_{m+1\le k\le M}, (B_{s_k-\delta} - B_{s_{k-1}-\delta})_{M+1\le k\le N}\Big)\Big].\end{aligned}$$

Let $\rho$ denote the modulus of continuity function of $\varphi$. Then

$$\Big|\mathbb{E}^{\mathbb{P}^{i,\omega^{ij}}}[\xi^{t,\omega}] - \mathbb{E}^{\mathbb{P}^{i,\omega}}[\xi^{t,\omega}]\Big| \le \rho(\varepsilon) \quad \text{for all } \omega \in E^i_j,$$

and thus

$$\begin{aligned}&\left|\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}[\xi] - \mathbb{E}^{\mathbb{P}}\left[\xi\mathbf{1}_{\{\tau<t\}\cup\{\mathrm{H}\le t\}} + \sum_{i\ge1}\eta^i_t\mathbf{1}_{E_i}\right]\right|\\ &\quad = \left|\mathbb{E}^{\mathbb{P}}\left[\sum_{i,j\ge1}\mathbb{E}^{\mathbb{P}^{i,\omega^{ij}}}[\xi^{t,\cdot}]\mathbf{1}_{E^i_j}\right] - \mathbb{E}^{\mathbb{P}}\left[\sum_{i,j\ge1}\eta^i_t\mathbf{1}_{E^i_j}\right]\right|\\ &\quad \le \mathbb{E}^{\mathbb{P}}\left[\sum_{i,j\ge1}\Big|\mathbb{E}^{\mathbb{P}^{i,\omega^{ij}}}[\xi^{t,\cdot}] - \mathbb{E}^{\mathbb{P}^{i,\cdot}}[\xi^{t,\cdot}]\Big|\mathbf{1}_{E^i_j}\right]\\ &\quad \le \mathbb{E}^{\mathbb{P}}\left[\sum_{i,j\ge1}\rho(\varepsilon)\mathbf{1}_{E^i_j}\right] \le \rho(\varepsilon).\end{aligned}$$

By sending $\varepsilon \to 0$, we obtain $\mathbb{E}^{\tilde{\mathbb{P}}}[\xi] = \mathbb{E}^{\mathbb{P}}\big[\xi \mathbf{1}_{\{\tau<t\}\cup\{\mathrm{H}\le t\}} + \sum_{i\ge 1}\eta_t^i \mathbf{1}_{E_i}\big]$, which proves (5.12) by the arbitrariness of $\xi$. ■

We now prove the regularity in the $t$-variable. Recall the $\rho_2$ defined in (5.4).

**Lemma 5.3.** *Let* $0 \le t_1 < \mathrm{H}(\omega^1)$, $0 \le t_2 < \mathrm{H}(\omega^2)$, *and* $t_1 \le t_2$. *Then we have:*

$$|\widehat{Y}_{t_1}(\omega^1) - \widehat{Y}_{t_2}(\omega^2)| \le C\Big[1 + \frac{1}{d(\omega^1_{t_1}, O^c)}\Big]\rho_2\Big(\mathbf{d}_\infty\big((t_1,\omega^1),(t_2,\omega^2)\big)\Big).$$

**Proof.** Without loss of generality we assume $t_1 < t_2$. Also, in view of the uniform continuity in $\omega$ of Lemma 5.1, it suffices to prove the lemma in the case $\omega^1 = \omega^2 = \omega$.

Denote $\delta := \mathbf{d}_\infty\big((t_1,\omega),(t_2,\omega)\big)$ and $\varepsilon := d(\omega_{t_1}, O^c)$. For $\delta \ge \frac{1}{8}$, we have $|\widehat{Y}_{t_1}(\omega) - \widehat{Y}_{t_2}(\omega)| \le 2C_0 \le C\varepsilon^{-1}\rho_2(\delta)$. So we assume in the rest of this proof that $\delta < \frac{1}{8}$.

First, by Assumption 3.4, we may consider the measure $\mathbb{P} \in \mathcal{P}_{t_1}$ such that $\alpha_t^{\mathbb{P}} := 0$, $\beta_t^{\mathbb{P}} := 0$, $t \in [t_1, t_2]$. Then, by setting $\tau := t_0$ in Lemma 5.2, we see that $\widehat{Y}_{t_1}(\omega) \ge \mathcal{E}_{t_1}[\widehat{Y}^{t_1,\omega}_{t_2}] \ge \mathbb{E}^{\mathbb{P}}[\widehat{Y}^{t_1,\omega}_{t_2}] = \widehat{Y}_{t_2}(\omega_{\cdot\wedge t_1})$. Note that $\mathrm{H}(\omega_{\cdot\wedge t_1}) = t_0 > t_2$. Thus, by Lemma 5.1,

$$\widehat{Y}_{t_2}(\omega) - \widehat{Y}_{t_1}(\omega) \le C\rho_1\Big(\mathbf{d}_\infty\big((t_2,\omega_{\cdot\wedge t_1}),(t_2,\omega)\big)\Big) \le C\rho_1(\delta) \le C\rho_2(\delta). \tag{5.14}$$

Next, for arbitrary $\tau \in \mathcal{T}^{t_1}$, noting that $\widehat{X} \le \widehat{Y}$ we have

$$\begin{aligned}
I(\tau) &:= \mathcal{E}_{t_1}\Big[\widehat{X}^{t_1,\omega}_{\tau\wedge \mathrm{H}^{t_1,\omega}}\mathbf{1}_{\{\tau\wedge\mathrm{H}^{t_1,\omega}<t_2\}} + \widehat{Y}^{t_1,\omega}_{t_2}\mathbf{1}_{\{\tau\wedge\mathrm{H}^{t_1,\omega}\ge t_2\}}\Big] - \widehat{Y}_{t_2}(\omega)\\
&= \mathcal{E}_{t_1}\Big[\widehat{X}^{t_1,\omega}_{\tau}\mathbf{1}_{\{\tau<\mathrm{H}^{t_1,\omega}\wedge t_2\}} + \widehat{X}^{t_1,\omega}_{\mathrm{H}^{t_1,\omega}}\mathbf{1}_{\{\mathrm{H}^{t_1,\omega}<t_2,\mathrm{H}^{t_1,\omega}\le\tau\}} + \widehat{Y}^{t_1,\omega}_{t_2}\mathbf{1}_{\{\tau\wedge\mathrm{H}^{t_1,\omega}\ge t_2\}}\Big] - \widehat{Y}_{t_2}(\omega)\\
&\le \mathcal{E}_{t_1}\Big[\big(\widehat{X}^{t_1,\omega}_{\tau} - \widehat{X}^{t_1,\omega}_{\mathrm{H}^{t_1,\omega}\wedge t_2}\big)\mathbf{1}_{\{\tau<\mathrm{H}^{t_1,\omega}\wedge t_2\}} + \widehat{Y}^{t_1,\omega}_{\mathrm{H}^{t_1,\omega}\wedge t_2}\Big] - \widehat{Y}_{t_2}(\omega)\\
&\le \mathcal{E}_{t_1}\Big[\big(\widehat{X}^{t_1,\omega}_{\tau} - \widehat{X}^{t_1,\omega}_{\mathrm{H}^{t_1,\omega}\wedge t_2}\big)\mathbf{1}_{\{\tau<\mathrm{H}^{t_1,\omega}\wedge t_2\}}\Big] + \mathcal{E}_{t_1}\Big[|\widehat{Y}^{t_1,\omega}_{t_2} - \widehat{Y}_{t_2}(\omega)|\mathbf{1}_{\{\mathrm{H}^{t_1,\omega}>t_2\}}\Big]\\
&\quad + C\,\mathcal{C}_{t_1}\big[\mathrm{H}^{t_1,\omega} \le t_2\big].
\end{aligned}$$

By (5.3) and Lemma 5.1 we have

$$\begin{aligned}
I(\tau) &\le \mathcal{E}_{t_1}\Big[\rho_0\Big(\mathbf{d}_\infty((t_1,\omega),(t_2,\omega\otimes_{t_1}B^{t_1}))\Big)\Big] + C\mathcal{E}_{t_1}\Big[\rho_1\big(\|\omega - \omega\otimes_{t_1}B^{t_1}\|_{t_2}\big)\Big]\\
&\quad + C\mathcal{C}_{t_1}\big[\|B^{t_1}\|_{t_2} \ge \varepsilon\big]\\
&\le \mathcal{E}_{t_1}\Big[\rho_0\big(\delta + \|B^{t_1}\|_{t_2}\big)\Big] + C\mathcal{E}_{t_1}\Big[\rho_1\big(\delta + \|B^{t_1}\|_{t_2}\big)\Big] + C\varepsilon^{-1}\mathcal{E}_{t_1}\Big[\|B^{t_1}\|_{t_2}\Big]\\
&\le C[1+\varepsilon^{-1}]\mathcal{E}_{t_1}\Big[\rho_1\big(\delta + \|B^{t_1}\|_{t_2}\big)\Big].
\end{aligned}$$

Since $\delta \le \frac{1}{8}$, following the proof of (4.6) we have

$$I(\tau) \le C[1+\varepsilon^{-1}]\Big[\rho_1(\delta^{\frac{1}{3}}) + \delta^{\frac{1}{3}}\Big] \le C[1+\varepsilon^{-1}]\rho_2(\delta).$$

By the arbitrariness of $\tau$ and the dynamic programming principle of Theorem 5.4, we obtain $\widehat{Y}_{t_1}(\omega) - \widehat{Y}_{t_2}(\omega) \le C\varepsilon^{-1}\rho_2(\delta)$, and the proof is complete by (5.14). ■

Applying Lemmas 5.1–5.3, and following the same arguments as those of Theorem 4.3, we establish the dynamic programming principle in the present context.

**Theorem 5.4.** *Let $t < \mathrm{H}(\omega)$ and $\tau \in \mathcal{T}^t$. Then*

$$\widehat{Y}_t(\omega) = \sup_{\tilde\tau \in \mathcal{T}^t} \mathcal{E}_t\Big[\widehat{X}^{t,\omega}_{\tilde\tau\wedge \mathrm{H}^{t,\omega}} \mathbf{1}_{\{\tilde\tau\wedge\mathrm{H}^{t,\omega}<\tau\}} + \widehat{Y}^{t,\omega}_\tau \mathbf{1}_{\{\tilde\tau\wedge\mathrm{H}^{t,\omega}\geq\tau\}}\Big].$$

*Consequently, $\widehat{Y}$ is a $\mathcal{E}$-supermartingale on* $[0, \mathrm{H}]$.

By Lemma 5.3, $\widehat{Y}$ is continuous for $t \in [0, \mathrm{H})$. Moreover, since $\hat{Y}$ is an $\mathcal{E}$-supermartingale, we see that $\hat{Y}_{\mathrm{H}-}$ exists. However, Example A.2 below shows that in general $\widehat{Y}$ may be discontinuous at $\mathrm{H}$. This issue is crucial for our purpose, and we will discuss more in Section 5.4 below.

### *5.2. Continuous approximation of the hitting times*

Similar to the proof of Theorem 3.3, we need to apply some limiting arguments. We therefore assume without loss of generality that $\widehat{Y}_0 > \widehat{X}_0$ and introduce the stopping times: for any $m \geq 1$ and $n > (\widehat{Y}_0 - \widehat{X}_0)^{-1}$,

$$\begin{aligned} \mathrm{H}_m &:= \inf\Big\{t \geq 0 : \mathbf{d}(\omega_t, O^c) \leq \frac{1}{m}\Big\} \wedge \Big(t_0 - \frac{1}{m}\Big), \\ \tau_n &:= \inf\Big\{t \geq 0 : \widehat{Y}_t - \widehat{X}_t \leq \frac{1}{n}\Big\}. \end{aligned} \tag{5.15}$$

Here we abuse the notation slightly by using the same notation $\tau_n$ as in (4.7). Our main task in this subsection is to build an approximation of $\mathrm{H}_m$ and $\tau_n$ by continuous random variables. This will be obtained by a repeated use of Lemma 4.5.

We start by a continuous approximation of the sequence $(\mathrm{H}_m)_{m\geq 1}$ defined in (5.15).

**Lemma 5.5.** *For all $m \geq 2$:*

(i) $\mathrm{H}_{m-1}(\omega) \leq \mathrm{H}_m(\omega') \leq \mathrm{H}_{m+1}(\omega)$, *whenever* $\|\omega - \omega'\|_{t_0} \leq \frac{1}{m(m+1)}$,

(ii) *there exists an open subset $\Omega_0^m \subset \Omega$, and a uniformly continuous $\hat{\mathrm{H}}_m$ such that*

$$\mathcal{C}\big[(\Omega_0^m)^c\big] < 2^{-m} \quad \textit{and} \quad \mathrm{H}_{m-1} - 2^{-m} \leq \hat{\mathrm{H}}_m \leq \mathrm{H}_{m+1} + 2^{-m} \quad \textit{on } \Omega_0^m,$$

(iii) *there exist $\delta_m > 0$ such that $|\hat{\mathrm{H}}_m(\omega) - \hat{\mathrm{H}}_m(\omega')| \leq 2^{-m}$ whenever $\|\omega - \omega'\|_{t_0} \leq \delta_m$, and:*

$$\mathcal{C}\big[(\hat\Omega_0^m)^c\big] \leq 2^{-m} \quad \textit{where } \hat\Omega_0^m := \{\omega \in \Omega_0^m : \mathbf{d}(\omega, [\Omega_0^m]^c) > \delta_m\}.$$

**Proof.** Notice that (ii) is a direct consequence of (i) obtained by applying Lemma 4.5 with $\varepsilon = 2^{-m}$. To prove (i), we observe that for $\|\omega - \omega'\|_{t_0} \leq \frac{1}{m(m+1)}$ and $t < \mathrm{H}_m(\omega')$, we have

$$\mathbf{d}(\omega_t, O^c) \geq \mathbf{d}(\omega'_t, O^c) - \frac{1}{m(m+1)} > \frac{1}{m} - \frac{1}{m(m+1)} = \frac{1}{m+1}.$$

This shows that $\mathrm{H}_m(\omega') \leq \mathrm{H}_{m+1}(\omega)$ whenever $\|\omega - \omega'\|_{t_0} \leq \frac{1}{m(m+1)}$. Similarly, $\mathrm{H}_{m-1}(\omega) \leq \mathrm{H}_m(\omega')$ whenever $\|\omega - \omega'\|_{t_0} \leq \frac{1}{m(m-1)}$, and the inequality (i) follows.

It remains to prove (iii). The first claim follows from the uniform continuity of $\hat{\mathrm{H}}_m$. For each $\delta > 0$, define $h_\delta : [0, \infty) \to [0, 1]$ as follows:

$$h_\delta(x) := 1 \quad \text{for } x \leq \delta, \qquad h_\delta(x) = 0 \quad \text{for } x \geq 2\delta, \text{ and } h_\delta \text{ is linear on } [\delta, 2\delta]. \tag{5.16}$$

Then the map $\omega \longmapsto \psi_\delta(\omega) := h_\delta(d(\omega, [\Omega_0^m]^c))$ is continuous, and $\psi_\delta \downarrow \mathbf{1}_{[\Omega_0^m]^c}$ as $\delta \downarrow 0$. Applying Proposition 2.6(ii) we have

$$\lim_{\delta\to 0} \mathcal{E}[\psi_\delta] = \mathcal{E}\big[\mathbf{1}_{(\Omega_0^m)^c}\big] = \mathcal{C}\big[(\Omega_0^m)^c\big] < 2^{-m}.$$

By definition of $\hat{\Omega}_0^m$, notice that $\mathbf{1}_{(\hat{\Omega}_0^m)^c} \le \psi_{\delta_m}$. Then $\mathcal{C}\big[(\hat{\Omega}_0^m)^c\big] \le \mathcal{E}[\psi_{\delta_m}]$, and (iii) holds true for sufficiently small $\delta_m$. ■

We next derive a continuous approximation of the sequences

$$\tau_n^m := \tau_n \wedge \hat{\mathrm{H}}_m, \tag{5.17}$$

where $\tau_n$ and $\hat{\mathrm{H}}_m$ are defined in (5.15) and Lemma 5.5(ii), respectively.

**Lemma 5.6.** *For all $m \ge 2, n > (\widehat{Y}_0 - \widehat{X}_0)^{-1}$, there exists an open subset $\Omega_n^m \subset \Omega$ and a uniformly continuous map $\hat{\tau}_n^m$ such that*

$$\tau_{n-1}^m - 2^{1-m} - 2^{-n} \le \hat{\tau}_n^m \le \tau_{n+1}^m + 2^{1-m} + 2^{-n}$$
$$\text{on } \hat{\Omega}_0^m \cap \Omega_n^m, \quad \text{and} \quad \mathcal{C}\big[(\Omega_n^m)^c\big] \le 2^{-n}.$$

**Proof.** Fix $m$, and recall the modulus of continuity $\rho_1$ introduced in (5.4). For each $n$, let $0 < \delta_n^m < \delta^m$ such that $(\rho_0 + C\rho_1)(\delta_n^m) \le \frac{1}{n(n+1)}$, where $C$ is the constant in Lemma 5.1. We shall prove

$$(\tau_{n-1} \wedge \hat{\mathrm{H}}_m)(\omega) - 2^{1-m} \le (\tau_n \wedge \hat{\mathrm{H}}_m)(\omega') \le (\tau_{n+1} \wedge \hat{\mathrm{H}}_m)(\omega) + 2^{1-m}$$
$$\text{whenever } \omega \in \hat{\Omega}_0^m,\ \|\omega - \omega'\|_{t_0} \le \delta_n^m. \tag{5.18}$$

Then the required statement follows from Lemma 4.5 with $\varepsilon = 2^{-n}$.

We shall prove only the right inequality of (5.18). The left one can be proved similarly. Let $\omega, \omega'$ be as in (5.18). First, by Lemma 5.5(iii) we have

$$\omega' \in \Omega_0^m \quad \text{and} \quad \hat{\mathrm{H}}_m(\omega') \le \hat{\mathrm{H}}_m(\omega) + 2^{-m}. \tag{5.19}$$

We now prove the right inequality of (5.18) in three cases.

*Case* 1. if $\tau_{n+1}(\omega) \ge \hat{\mathrm{H}}_m(\omega') - 2^{-m}$, then $\hat{\mathrm{H}}_m(\omega') \le (\tau_{n+1} \wedge \hat{\mathrm{H}}_m)(\omega) + 2^{-m}$ and thus the result is true.

*Case* 2. If $\tau_{n+1}(\omega) = \mathrm{H}(\omega)$, then by Lemma 5.5(ii) we have $\hat{\mathrm{H}}_m(\omega) \le \mathrm{H}_{m+1}(\omega) + 2^{-m} \le \tau_{n+1}(\omega) + 2^{-m}$, and thus $\hat{\mathrm{H}}_m(\omega') \le \hat{\mathrm{H}}_m(\omega) + 2^{-m} \le \tau_{n+1}(\omega) + 2^{1-m}$. This, together with (5.19), proves the desired inequality.

*Case* 3. We now assume $\tau_{n+1}(\omega) < \hat{\mathrm{H}}_m(\omega') - 2^{-m}$ and $\tau_{n+1}(\omega) < \mathrm{H}(\omega)$. By Lemma 5.5(ii) we have $\tau_{n+1}(\omega) < \mathrm{H}_{m+1}(\omega')$, and thus $\tau_{n+1}(\omega) < \mathrm{H}(\omega')$. Then it follows from Lemma 5.1 that

$$(Y - X)_{\tau_{n+1}(\omega)}(\omega') \le (Y - X)_{\tau_{n+1}(\omega)}(\omega) + (\rho_0 + C\rho_1)(\delta_n^m) \le \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}.$$

That is, $\tau_n(\omega') \le \tau_{n+1}(\omega)$. This, together with (5.19), proves the desired inequality. ■

For our final approximation result, we introduce the notations:

$$\bar{\tau}_n := \tau_n \wedge \mathrm{H}_n, \qquad \underline{\theta}_n^* := \hat{\tau}_{n-1}^{n-1} - 2^{3-n}, \qquad \overline{\theta}_n^* := \hat{\tau}_{n+1}^{n+1} + 2^{1-n}, \tag{5.20}$$

and

$$\Omega_n^* := \hat{\Omega}_0^{n-1} \cap \Omega_{n-1}^{n-1} \cap \hat{\Omega}_0^{n+1} \cap \Omega_{n+1}^{n+1}. \tag{5.21}$$

**Lemma 5.7.** *For all $n \geq (\widehat{Y}_0 - \widehat{X}_0)^{-1} \vee 2$, $\underline{\theta}_n^*, \overline{\theta}_n^*$ are uniformly continuous, and $\underline{\theta}_n^* \leq \bar{\tau}_n \leq \overline{\theta}_n^*$ on $\Omega_n^*$.*

**Proof.** This is a direct combination of Lemmas 5.5 and 5.6.

*5.3. Proof of Theorem 3.6*

We first prove the $\mathcal{E}$-martingale property under an additional condition.

**Lemma 5.8.** *Let $\tau \in \mathcal{T}$ such that $\tau \leq \tau^*$ and $\mathcal{E}[Y_{\tau-}] = \mathcal{E}[Y_\tau]$ (in particular if $\tau < \mathrm{H}$). Then $\widehat{Y}$ is an $\mathcal{E}$-martingale on $[0, \tau]$.*

**Proof.** If $\widehat{Y}_0 = \widehat{X}_0$, then $\widehat{\tau}^* = 0$ and obviously the statement is true. We then assume $\widehat{Y}_0 > \widehat{X}_0$, and prove the lemma in several steps.

*Step* 1. Let $n$ be sufficiently large so that $\frac{1}{n} < \widehat{Y}_0 - \widehat{X}_0$. Follow the same arguments as that of Lemma 4.4, one can easily prove:

$$\widehat{Y} \text{ is an } \mathcal{E}\text{-martingale on } [0, \tau_n]. \tag{5.22}$$

*Step* 2. Recall the sequence of stopping times $(\bar{\tau}_n)_{n\geq 1}$ introduced in (5.20). By Step 1 we have $\widehat{Y}_0 = \mathcal{E}[\widehat{Y}_{\bar{\tau}_n}]$. Then for any $\varepsilon > 0$, there exists $\mathbb{P}_n \in \mathcal{P}$ such that $\widehat{Y}_0 - \varepsilon < \mathbb{E}^{\mathbb{P}_n}[\widehat{Y}_{\bar{\tau}_n}]$. Since $\mathcal{P}$ is weakly compact, there exists subsequence $\{n_j\}$ and $\mathbb{P}^* \in \mathcal{P}$ such that $\mathbb{P}_{n_j}$ converges weakly to $\mathbb{P}^*$. Now for any $n$ and $n_j \geq n$, since $Y$ is a supermartingale under each $\mathbb{P}_{n_j}$ and $(\bar{\tau}_n)_{n\geq 1}$ is increasing, we have

$$\widehat{Y}_0 - \varepsilon < \mathbb{E}^{\mathbb{P}_{n_j}}\big[\widehat{Y}_{\bar{\tau}_{n_j}}\big] \leq \mathbb{E}^{\mathbb{P}_{n_j}}\big[\widehat{Y}_{\bar{\tau}_n}\big]. \tag{5.23}$$

Our next objective is to send $j \nearrow \infty$, for fixed $n$, and use the weak convergence of $\mathbb{P}^{n_j}$ towards $\mathbb{P}^*$. To do this, we need to approximate $\widehat{Y}_{\bar{\tau}_n}$ with continuous random variables. Denote

$$\psi_n(\omega) := h_n\Big(\inf_{0\leq t\leq \overline{\theta}_n^*(\omega)} \mathbf{d}(\omega_t, O^c)\Big)$$
$$\text{with } h_n(x) := 1 \wedge [(n+3)(n+4)x - (n+3)]^+. \tag{5.24}$$

Then $\psi_n$ is continuous in $\omega$, and

$$\{\psi_n > 0\} \subset \Big\{\inf_{0\leq t\leq \overline{\theta}_n^*(\omega)} \mathbf{d}(\omega_t, O^c) > \frac{1}{n+4}\Big\} \subset \{\overline{\theta}_n^* < \mathrm{H}_{n+4}\}. \tag{5.25}$$

In particular, this implies that $\widehat{Y}_{\underline{\theta}_n^*}\psi_n$ and $\widehat{Y}_{\overline{\theta}_n^*}\psi_n$ are continuous in $\omega$. We now decompose the right hand-side term of (5.23) into:

$$\widehat{Y}_0 - \varepsilon \leq \mathbb{E}^{\mathbb{P}_{n_j}}\Big[\big[\widehat{Y}_{\underline{\theta}_n^*} + (\widehat{Y}_{\bar{\tau}_n} - \widehat{Y}_{\underline{\theta}_n^*})\mathbf{1}_{\Omega_n^*}\big]\big(\psi_n + (1-\psi_n)\big) + (\widehat{Y}_{\bar{\tau}_n} - \widehat{Y}_{\underline{\theta}_n^*})\mathbf{1}_{(\Omega_n^*)^c}\Big].$$

Note that $\underline{\theta}_n^* \leq \bar{\tau}_n \leq \overline{\theta}_n^*$ on $\Omega_n^*$. Then

$$\widehat{Y}_0 - \varepsilon \leq \mathbb{E}^{\mathbb{P}_{n_j}}\Big[\Big(\widehat{Y}_{\underline{\theta}_n^*} + \sup_{\underline{\theta}_n^*\leq t\leq \overline{\theta}_n^*}(\widehat{Y}_t - \widehat{Y}_{\underline{\theta}_n^*})\Big)\psi_n\Big] + C\mathcal{C}[\psi_n < 1] + C\mathcal{C}\big[(\Omega_n^*)^c\big].$$

Send $j \to \infty$, we obtain

$$\widehat{Y}_0 - \varepsilon \leq \mathbb{E}^{\mathbb{P}^*}\Big[\psi_n \widehat{Y}_{\underline{\theta}_n^*}\Big] + \mathbb{E}^{\mathbb{P}^*}\Big[\psi_n \sup_{\underline{\theta}_n^* \leq t \leq \overline{\theta}_n^*} (\widehat{Y}_t - \widehat{Y}_{\underline{\theta}_n^*})\Big] + C\mathcal{C}[\psi_n < 1] + C\mathcal{C}[(\Omega_n^*)^c]. \quad (5.26)$$

*Step* 3. In this step we show that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^*}\Big[\psi_n \sup_{\underline{\theta}_n^* \leq t \leq \overline{\theta}_n^*} (\widehat{Y}_t - \widehat{Y}_{\underline{\theta}_n^*})\Big] = \lim_{n\to\infty} \mathcal{C}[\psi_n < 1] = \lim_{n\to\infty} \mathcal{C}[(\Omega_n^*)^c] = 0. \quad (5.27)$$

(i) First, by the definition of $\Omega_n^*$ in (5.21) together with Lemmas 5.5(iii) and 5.6, it follows that $\mathcal{C}\big[(\Omega_n^*)^c\big] \leq C2^{-n} \longrightarrow 0$ as $n \to \infty$.

(ii) Next, notice that

$$\{\psi_n < 1\} = \Big\{\inf_{0 \leq t \leq \overline{\theta}_n^*(\omega)} \mathbf{d}(\omega_t, O^c) < \frac{1}{n+3}\Big\} \subset \{\overline{\theta}_n^* > \mathrm{H}_{n+3}\}.$$

Moreover, by (5.20) and Lemma 5.7,

$$\overline{\theta}_n^* = \hat{\tau}_{n+1}^{n+1} + 2^{1-n} = \underline{\theta}_{n+2}^* + 2^{2-n} \leq \bar{\tau}_{n+2} + 2^{2-n} \leq \mathrm{H}_{n+2} + 2^{2-n}, \quad \text{on } \Omega_{n+2}^*.$$

Then

$$\begin{aligned}\{\psi_n < 1\} &\subset (\Omega_{n+2}^*)^c \cup \{\mathrm{H}_{n+3} < \mathrm{H}_{n+2} + 2^{2-n}\} \\ &\subset (\Omega_{n+2}^*)^c \cup \Big\{\sup_{\mathrm{H}_{n+2} \leq t \leq \mathrm{H}_{n+2} + 2^{2-n}} |B_t - B_{\mathrm{H}_{n+2}}| \geq \frac{1}{(n+2)(n+3)}\Big\}.\end{aligned}$$

Then one can easily see that $\mathcal{C}[\psi_n < 1] \to 0$, as $n \to \infty$.

(iii) Finally, it is clear that $\underline{\theta}_n^* \to \widehat{\tau}^*, \overline{\theta}_n^* \to \widehat{\tau}^*$. Recall that $\widehat{Y}_{\widehat{\tau}^*-}$ exists. By (5.25), we see that $\psi_n \sup_{\underline{\theta}_n^* \leq t \leq \overline{\theta}_n^*} (\widehat{Y}_t - \widehat{Y}_{\underline{\theta}_n^*}) \to 0$, $\mathbb{P}^*$-a.s. as $n \to \infty$. Then by applying the dominated convergence theorem under $\mathbb{P}^*$ we obtain the first convergence in (5.27).

*Step* 4. By the dominated convergence theorem under $\mathbb{P}^*$ we obtain $\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^*}[\psi_n \widehat{Y}_{\underline{\theta}_n^*}] = \mathbb{E}^{\mathbb{P}^*}[\widehat{Y}_{\widehat{\tau}^*-}]$. This, together with (5.26) and (5.27), implies that

$$\widehat{Y}_0 \leq \mathbb{E}^{\mathbb{P}^*}[\widehat{Y}_{\widehat{\tau}^*-}] + \varepsilon.$$

Note that $\widehat{Y}$ is an $\mathbb{P}^*$-supermartingale and $\tau \leq \widehat{\tau}^*$, then

$$\widehat{Y}_0 \leq \mathbb{E}^{\mathbb{P}^*}[\widehat{Y}_{\tau-}] + \varepsilon.$$

Since $\varepsilon$ is arbitrary, we obtain $\widehat{Y}_0 \leq \mathcal{E}[\widehat{Y}_{\tau-}]$, and thus by the assumption $\mathcal{E}[\widehat{Y}_{\tau-}] = \mathcal{E}[\widehat{Y}_\tau]$ we have $\widehat{Y}_0 \leq \mathcal{E}[\widehat{Y}_\tau]$. This, together with the fact that $\widehat{Y}$ is a $\mathcal{E}$-supermartingale, implies that

$$\widehat{Y}_0 = \mathcal{E}[\widehat{Y}_\tau]. \quad (5.28)$$

Similarly, one can prove $\widehat{Y}_t(\omega) = \mathcal{E}_t[\widehat{Y}^{t,\omega}_{\tau^{t,\omega}}]$ for $t < \tau(\omega)$, and thus $\widehat{Y}_{\cdot\wedge\tau}$ is a $\mathcal{E}$-martingale. ■

In light of Lemma 5.8, the following result is obviously important for us.

**Proposition 5.9.** *It holds that* $\mathcal{E}[\widehat{Y}_{\widehat{\tau}^*-}] = \mathcal{E}[\widehat{Y}_{\widehat{\tau}^*}]$.

We recall again that $\widehat{Y}_{\widehat{\tau}^*-} = \widehat{Y}_{\widehat{\tau}^*}$ whenever $\widehat{\tau}^* < \mathrm{H}$. So the only possible discontinuity is at $\mathrm{H}$. The proof of Proposition 5.9 is reported in Section 5.4 below. Let us first show how it allows to complete the

**Proof of Theorem 3.6.** By Lemma 5.8 and Proposition 5.9, $\widehat{Y}$ is an $\mathcal{E}$-martingale on $[0, \widehat{\tau}^*]$. Moreover, since $\widehat{X}_{\widehat{\tau}^*} = \widehat{Y}_{\widehat{\tau}^*}$, then $\widehat{Y}_0 = \mathcal{E}[\widehat{X}_{\widehat{\tau}^*}]$ and thus $\widehat{\tau}^*$ is an optimal stopping time. ■

**Remark 5.10.** Assume Assumption 3.4(ii) and the conditions of Lemma A.1 below hold, by Remark 3.5(iii) and Lemma A.1 we see that Proposition 5.9 and hence Theorem 3.6 hold. That is, in this case the Section 5.4 below is not needed. ■

### 5.4. $\mathcal{E}$-continuity of $\widehat{Y}$ at the random maturity

This subsection is dedicated to the proof of Proposition 5.9. We first reformulate some pathwise properties established in previous subsections. For that purpose, we introduce the following additional notation: for any $\mathbb{P} \in \mathcal{P}$, $\tau \in \mathcal{T}$, and $E \in \mathcal{F}_\tau$

$$\mathcal{P}(\mathbb{P}, \tau, E) := \Big\{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \otimes_\tau \big[\mathbb{P}'\mathbf{1}_E + \mathbb{P}\mathbf{1}_{E^c}\big]\Big\}, \quad \mathcal{P}(\mathbb{P}, \tau) := \mathcal{P}(\mathbb{P}, \tau, \Omega). \tag{5.29}$$

That is, $\mathbb{P}' \in \mathcal{P}(\mathbb{P}, \tau, E)$ means $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_\tau$ and $(\mathbb{P}')^{\tau,\omega} = \mathbb{P}^{\tau,\omega}$ for $\mathbb{P}$-a.e. $\omega \in E^c$.

The first result corresponds to Theorem 5.4.

**Lemma 5.11.** *Let $\mathbb{P} \in \mathcal{P}$, $\tau_1, \tau_2 \in \mathcal{T}$, and $E \in \mathcal{F}_{\tau_1}$. Assume $\tau_1 \le \tau_2 \le \mathrm{H}$, and $\tau_1 < \mathrm{H}$ on $E$. Then for any $\varepsilon > 0$, there exist $\mathbb{P}_\varepsilon \in \mathcal{P}(\mathbb{P}, \tau_1, E)$ and $\tau_\varepsilon \in \mathcal{T}$ with values in $[\tau_1, \tau_2]$, s.t.*

$$\mathbb{E}^{\mathbb{P}}\Big[\widehat{Y}_{\tau_1}\mathbf{1}_E\Big] \le \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\big[\widehat{X}_{\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\tau_2\}} + \widehat{Y}_{\tau_2}\mathbf{1}_{\{\tau_\varepsilon=\tau_2\}}\big]\mathbf{1}_E\Big] + \varepsilon.$$

**Proof.** Let $\tau_1^n$ be a sequence of stopping times such that $\tau_1^n \downarrow \tau$ and each $\tau_1^n$ takes only finitely many values. Applying Lemma 5.3 together with the dominated convergence theorem under $\mathbb{P}$, we see that $\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\Big[|\widehat{Y}_{\tau_1^n\wedge\tau_2} - \widehat{Y}_{\tau_1}|\Big] = 0$. Fix $n$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[|\widehat{Y}_{\tau_1^n\wedge\tau_2} - \widehat{Y}_{\tau_1}|\Big] \le \frac{\varepsilon}{2}. \tag{5.30}$$

Assume $\tau_1^n$ takes values $\{t_i, i = 1, \dots, m\}$, and for each $i$, denote $E_i := E \cap \{\tau_1^n = t_i < \tau_2\} \in \mathcal{F}_{t_i}$. By (5.13), there exists $\tilde{\tau}_i \in \mathcal{T}$ and $\tilde{\mathbb{P}}_i \in \mathcal{P}(\mathbb{P}, t_i)$ such that $\tilde{\tau}_i \ge t_i$ on $E_i$ and

$$\widehat{Y}_{t_i} \le \mathbb{E}^{\tilde{\mathbb{P}}_i}_{t_i}\Big[\widehat{X}_{\tilde{\tau}_i\wedge\mathrm{H}}\Big] + \frac{\varepsilon}{2}, \quad \mathbb{P}\text{-a.s. on } E_i. \tag{5.31}$$

Here $\mathbb{E}^{\tilde{\mathbb{P}}_i}_{t_i}[\cdot] := \mathbb{E}^{\tilde{\mathbb{P}}_i}[\cdot|\mathcal{F}_{t_i}]$ denotes the conditional expectation. Define

$$\tilde{\tau} := \tau_2\mathbf{1}_{E^c\cup\{\tau_2\le\tau_1^n\}} + \sum_{i=1}^m \tilde{\tau}_i\mathbf{1}_{E_i}, \qquad \tilde{\mathbb{P}} := \mathbb{P}\mathbf{1}_{E^c\cup\{\tau_2\le\tau_1^n\}} + \sum_{i=1}^m \tilde{\mathbb{P}}_i\mathbf{1}_{E_i}. \tag{5.32}$$

Then one can check straightforwardly that

$$\tilde{\tau} \in \mathcal{T} \quad \text{and} \quad \tilde{\tau} \ge \tau_2 \wedge \tau_1^n; \tag{5.33}$$

and $\tilde{\mathbb{P}} \in \mathcal{P}(\mathbb{P}, \tau_2 \wedge \tau_1^n, E) \subset \mathcal{P}(\mathbb{P}, \tau_1, E)$. Moreover, by (5.31) and (5.32),

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\widehat{Y}_{\tau_2\wedge\tau_1^n}\mathbf{1}_E\big] &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left[\widehat{Y}_{\tau_2}\mathbf{1}_{\{\tau_2\le\tau_1^n\}} + \sum_{i=1}^m \widehat{Y}_{t_i}\mathbf{1}_{E_i}\right]\mathbf{1}_E\right]\\ &\le \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\Big[\widehat{Y}_{\tau_2}\mathbf{1}_{\{\tau_2\le\tau_1^n\}} + \Big(\widehat{X}_{\tilde{\tau}\wedge\mathrm{H}} + \frac{\varepsilon}{2}\Big)\mathbf{1}_{\{\tau_1^n<\tau_2\}}\Big]\mathbf{1}_E\Big].\end{aligned}$$

This, together with (5.30) and (5.33), leads to

$$\begin{aligned}
&\mathbb{E}^{\tilde{\mathbb{P}}}\Big[\big(\widehat{Y}_{\tau_1} - \widehat{X}_{\tilde{\tau}}\mathbf{1}_{\{\tilde{\tau}<\tau_2\}} - \widehat{Y}_{\tau_2}\mathbf{1}_{\{\tilde{\tau}\geq\tau_2\}}\big)\mathbf{1}_E\Big]\\
&\quad\leq \varepsilon + \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\big(\widehat{Y}_{\tau_2}\mathbf{1}_{\{\tau_2\leq\tau_1^n\}} + \widehat{X}_{\tilde{\tau}\wedge \mathrm{H}}\mathbf{1}_{\{\tau_1^n<\tau_2\}} - \widehat{X}_{\tilde{\tau}}\mathbf{1}_{\{\tilde{\tau}<\tau_2\}} - \widehat{Y}_{\tau_2}\mathbf{1}_{\{\tilde{\tau}\geq\tau_2\}}\big)\mathbf{1}_E\Big]\\
&\quad= \varepsilon + \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\big(\widehat{X}_{\tilde{\tau}\wedge \mathrm{H}} - \widehat{Y}_{\tau_2}\big)\mathbf{1}_{\{\tau_1^n<\tau_2\leq\tilde{\tau}\}}\mathbf{1}_E\Big]\\
&\quad= \varepsilon + \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\big(\mathbb{E}^{\tilde{\mathbb{P}}}_{\tau_2}[\widehat{X}_{\tilde{\tau}\wedge \mathrm{H}}] - \widehat{Y}_{\tau_2}\big)\mathbf{1}_{\{\tau_1^n<\tau_2\leq\tilde{\tau}\}}\mathbf{1}_E\Big] \leq \varepsilon,
\end{aligned}$$

where the last inequality follows from the definition of $\widehat{Y}$. Then, by setting $\tau_\varepsilon := \tilde{\tau}\wedge\tau_2$ we prove the result. ■

Next result corresponds to Lemma 5.8.

**Lemma 5.12.** *Let $\mathbb{P}\in\mathcal{P}$, $\tau\in\mathcal{T}$, and $E\in\mathcal{F}_\tau$ such that $\tau\leq\widehat{\tau}^*$ on $E$. Then for all $\varepsilon>0$:*

$$\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_E\widehat{Y}_\tau\big] \leq \mathbb{E}^{\mathbb{P}_\varepsilon}\big[\mathbf{1}_E\widehat{Y}_{\widehat{\tau}^*-}\big] + \varepsilon \quad \textit{for some } \mathbb{P}_\varepsilon\in\mathcal{P}(\mathbb{P},\tau,E).$$

**Proof.** We proceed in three steps.

*Step* 1. We first assume $\tau = t < \widehat{\tau}^*$ on $E$. We shall prove the result following the arguments in Lemma 5.8. Recall the notations in Section 5.2 and the $\psi_n$ defined in (5.24), and let $\rho_n$ denote the modulus of continuity functions of $\underline{\theta}_n^*$, $\overline{\theta}_n^*$, and $\psi_n$.

Denote $\bar{\tau}_n := 0$ for $n \leq (\widehat{Y}_0 - \widehat{X}_0)^{-1}$. For any $n$ and $\delta>0$, let $\{E_i^{n,\delta}, i\geq 1\}\subset\mathcal{F}_t$ be a partition of $E\cap\{\bar{\tau}_{n-1}\leq t<\bar{\tau}_n\}$ such that $\|\omega-\omega'\|_t\leq\delta$ for any $\omega,\omega'\in E_i^{n,\delta}$. For each $(n,i)$, fix $\omega^{n,i} := \omega^{n,\delta,i}\in E_i^{n,\delta}$. By Lemma 5.8, $\widehat{Y}\mathbf{1}_{E_i^{n,\delta}}$ is an $\mathcal{E}$-martingale on $[t,\bar{\tau}_n]$. Then $\widehat{Y}_t(\omega^{n,i}) = \mathcal{E}_t[\widehat{Y}^{t,\omega^{n,i}}_{\bar{\tau}_n^{t,\omega^{n,i}}}]$, and thus there exists $\mathbb{P}_i^{n,\delta}\in\mathcal{P}_t$ such that

$$\widehat{Y}_t(\omega^{n,i}) \leq \mathbb{E}^{\mathbb{P}_i^{n,\delta}}\Big[\widehat{Y}^{t,\omega^{n,i}}_{\bar{\tau}_n^{t,\omega^{n,i}}}\Big] + \varepsilon. \tag{5.34}$$

Note that $\cup_{m=1}^n\cup_{i\geq 1}E_i^{m,\delta} = E\cap\{t<\bar{\tau}_n\}$. Set

$$\mathbb{P}^{n,\delta} := \mathbb{P}\otimes_t\left[\sum_{m=1}^n\sum_{i\geq 1}\mathbb{P}_i^{m,\delta}\mathbf{1}_{E_i^{m,\delta}} + \mathbb{P}\mathbf{1}_{E^c\cup\{t\geq\bar{\tau}_n\}}\right]\in\mathcal{P}(\mathbb{P},t,E). \tag{5.35}$$

Recall the $h_\delta$ defined by (5.16). We claim that, for any $N\geq n$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t\mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^{N,\delta}}[\widehat{Y}_{t\vee\underline{\theta}_n^*}\psi_n\mathbf{1}_E]\\
&\quad\leq Cn\mathcal{E}\Big[\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)\Big] + C\rho_n(\delta) + \varepsilon + C2^{-n} + C\mathcal{C}(\psi_n<1)\\
&\qquad + 2\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\sup_{\underline{\theta}_n^*\leq s\leq\overline{\theta}_n^*}|\widehat{Y}_s - \widehat{Y}_{\underline{\theta}_n^*}|\psi_n\mathbf{1}_E\Big] + C\mathcal{E}\Big[h_\delta\big(d\big(\omega,(\Omega_n^*)^c\big)\big)\Big],
\end{aligned}\tag{5.36}$$

where $\eta_n(\delta) := \sup_{t\leq s_1<s_2\leq t_0, s_2-s_1\leq\rho_n(\delta)}|B_{s_1}^t - B_{s_2}^t|$.

Moreover, one can easily find $\mathcal{F}_t$-measurable continuous random variables $\varphi_k$ such that $|\varphi_k| \le 1$ and $\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}}[|\mathbf{1}_E - \varphi_k|] = 0$. Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t \mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^{N,\delta}}[\widehat{Y}_{t\vee \underline{\theta}^*_n} \psi_n \varphi_k] \\
&\quad \le Cn\mathcal{E}\Big[\rho_2\Big(\delta + \rho_n(\delta) + 2\eta_n(\delta)\Big)\Big] + C\rho_n(\delta) + \varepsilon + C2^{-n} + C\mathcal{C}(\psi_n < 1) \\
&\qquad + C\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\sup_{\underline{\theta}^*_n \le s \le \overline{\theta}^*_n} |\widehat{Y}_s - \widehat{Y}_{\underline{\theta}^*_n}|\psi_n\varphi_k\Big] + C\mathcal{E}\Big[h_\delta\Big(d\Big(\omega, (\Omega^*_n)^c\Big)\Big)\Big] + C\mathbb{E}^{\mathbb{P}}[|\mathbf{1}_E - \varphi_k|].
\end{aligned}$$

Send $\delta \to 0$. First note that $[\delta + \rho_n(\delta) + 2\eta_n(\delta)] \downarrow 0$ and $h_\delta \downarrow \mathbf{1}_{\{0\}}$, then by Proposition 2.6(ii) we have

$$\begin{aligned}
&\lim_{\delta\to 0} \mathcal{E}\Big[\rho_2\Big(\delta + \rho_n(\delta) + 2\eta_n(\delta)\Big)\Big] = 0; \\
&\lim_{\delta\to 0} \mathcal{E}\Big[h_\delta\Big(d\Big(\omega, (\Omega^*_n)^c\Big)\Big)\Big] = \mathcal{C}\Big[d\Big(\omega, (\Omega^*_n)^c\Big) = 0\Big] = \mathcal{C}[(\Omega^*_n)^c] \le C2^{-n}.
\end{aligned}$$

Moreover, for each $N$, by the weak compactness assumption (P1) we see that $\mathbb{P}^{N,\delta}$ has a weak limit $\mathbb{P}^N \in \mathcal{P}$. It is straightforward to check that $\mathbb{P}^N \in \mathcal{P}(\mathbb{P}, t, E)$. Note that the random variables $\widehat{Y}_{t\vee\underline{\theta}^*_n}\psi_n\varphi_k$ and $\sup_{\underline{\theta}^*_n \le s \le \overline{\theta}^*_n} |\widehat{Y}_s - \widehat{Y}_{\underline{\theta}^*_n}|\psi_n\varphi_k$ are continuous. Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t \mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^{N}}[\widehat{Y}_{t\vee \underline{\theta}^*_n} \psi_n \varphi_k] \\
&\quad \le \varepsilon + C2^{-n} + C\mathcal{C}(\psi_n < 1) + C\mathbb{E}^{\mathbb{P}^{N}}\Big[\sup_{\underline{\theta}^*_n \le s \le \overline{\theta}^*_n} |\widehat{Y}_s - \widehat{Y}_{\underline{\theta}^*_n}|\psi_n\varphi_k\Big] + C\mathbb{E}^{\mathbb{P}}[|\mathbf{1}_E - \varphi_k|].
\end{aligned}$$

Again by the weak compactness assumption (P1), $\mathbb{P}^N$ has a weak limit $\mathbb{P}^* \in \mathcal{P}(\mathbb{P}, t, E)$ as $N \to \infty$. Now send $N \to \infty$, by the continuity of the random variables we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t \mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^{*}}[\widehat{Y}_{t\vee \underline{\theta}^*_n} \psi_n \varphi_k] \\
&\quad \le \varepsilon + C2^{-n} + C\mathcal{C}(\psi_n < 1) + C\mathbb{E}^{\mathbb{P}^{*}}\Big[\sup_{\underline{\theta}^*_n \le s \le \overline{\theta}^*_n} |\widehat{Y}_s - \widehat{Y}_{\underline{\theta}^*_n}|\psi_n\varphi_k\Big] + C\mathbb{E}^{\mathbb{P}}[|\mathbf{1}_E - \varphi_k|].
\end{aligned}$$

Send $k \to \infty$ and recall that $\mathbb{P}^* = \mathbb{P}$ on $\mathcal{F}_t$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t \mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^{*}}[\widehat{Y}_{t\vee \underline{\theta}^*_n} \psi_n 1_E] \\
&\quad \le \varepsilon + C2^{-n} + C\mathcal{C}(\psi_n < 1) + 2\mathbb{E}^{\mathbb{P}^{*}}\Big[\sup_{\underline{\theta}^*_n \le s \le \overline{\theta}^*_n} |\widehat{Y}_s - \widehat{Y}_{\underline{\theta}^*_n}|\psi_n\mathbf{1}_E\Big].
\end{aligned}$$

Finally send $n \to \infty$, by (5.27) and applying the dominated convergence theorem under $\mathbb{P}$ and $\mathbb{P}^*$ we have

$$\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t \mathbf{1}_E] - \mathbb{E}^{\mathbb{P}^*}[\widehat{Y}_{\widehat{\tau}^*-} 1_E] \le \varepsilon.$$

That is, $\mathbb{P}_\varepsilon := \mathbb{P}^*$ satisfies the requirement in the case $\tau = t < \widehat{\tau}^*$ on $E$.

*Step* 2. We now prove Claim (5.36). Indeed, for any $m \le n$ and any $\omega \in E^{m,\delta}_i$, by Lemma 5.1 we have

$$\begin{aligned}
&\widehat{Y}_t(\omega) - \mathbb{E}^{\mathbb{P}^{m,\delta}_i}\Big[\widehat{Y}^{t,\omega}_{\bar{\tau}^{t,\omega}_n}\Big] \\
&\quad = \widehat{Y}_t(\omega) - \widehat{Y}_t(\omega^{m,i}) + \widehat{Y}_t(\omega^{m,i}) - \mathbb{E}^{\mathbb{P}^{m,\delta}_i}\Big[\widehat{Y}^{t,\omega^{m,i}}_{\bar{\tau}^{t,\omega^{m,i}}_n}\Big] + \mathbb{E}^{\mathbb{P}^{m,\delta}_i}\Big[\widehat{Y}^{t,\omega^{m,i}}_{\bar{\tau}^{t,\omega^{m,i}}_n} - \widehat{Y}^{t,\omega}_{\bar{\tau}^{t,\omega}_n}\Big]
\end{aligned}$$

$$\leq C\rho_1(\delta) + \varepsilon + \mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[\big|\widehat{Y}^{t,\omega^{m,i}}_{\bar{\tau}_n^{t,\omega^{m,i}}} - \widehat{Y}^{t,\omega}_{\bar{\tau}_n^{t,\omega}}\big|\mathbf{1}_{(\Omega_n^*)^{t,\omega^{m,i}}\cap(\Omega_n^*)^{t,\omega}}\psi_n^{t,\omega^{m,i}}\psi_n^{t,\omega}\Big]$$
$$+ C\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega^{m,i}}]^c \cup [(\Omega_n^*)^{t,\omega}]^c\Big] + C\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[1 - \psi_n^{t,\omega^{m,i}} + 1 - \psi_n^{t,\omega}\Big]. \tag{5.37}$$

Note that

$$\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[1 - \psi_n^{t,\omega^{m,i}} + 1 - \psi_n^{t,\omega}\Big] \leq 2\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[1 - \psi_n^{t,\omega}\Big] + \rho_n(\delta);$$

$$\begin{aligned}
\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega^{m,i}}]^c \cup [(\Omega_n^*)^{t,\omega}]^c\Big] &\leq 2\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega}]^c\Big] \\
&\quad + \mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega^{m,i}}]^c \cap (\Omega_n^*)^{t,\omega}\Big] \\
&\leq 2\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega}]^c\Big] \\
&\quad + \mathbb{P}_i^{m,\delta}\Big[0 < d\big(\omega\otimes_t B^t, (\Omega_n^*)^c\big) < \delta\Big] \\
&\leq 2\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega}]^c\Big] \\
&\quad + \mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[h_\delta\big(d\big(\omega\otimes_t B^t, (\Omega_n^*)^c\big)\big)\Big].
\end{aligned} \tag{5.38}$$

Moreover, on $(\Omega_n^*)^{t,\omega^{m,i}} \cap (\Omega_n^*)^{t,\omega} \cap \{\psi_n^{t,\omega^{m,i}} > 0\} \cap \{\psi_n^{t,\omega} > 0\}$, by Lemma 5.7 and (5.25) we have

$$(\underline{\theta}_n^*)^{t,\omega^{m,i}} \leq \bar{\tau}_n^{t,\omega^{m,i}} \leq (\overline{\theta}_n^*)^{t,\omega^{m,i}} < \mathrm{H}_{n+4}^{t,\omega^{m,i}}; \qquad (\underline{\theta}_n^*)^{t,\omega} \leq \bar{\tau}_n^{t,\omega} \leq (\overline{\theta}_n^*)^{t,\omega} < \mathrm{H}_{n+4}^{t,\omega}.$$

Then

$$\begin{aligned}
\big|\widehat{Y}^{t,\omega^{m,i}}_{\bar{\tau}_n^{t,\omega^{m,i}}} - \widehat{Y}^{t,\omega}_{\bar{\tau}_n^{t,\omega}}\big| &\leq \big|\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}\big| \\
&\quad + \sup_{(\underline{\theta}_n^*)^{t,\omega^{m,i}} \leq s \leq (\overline{\theta}_n^*)^{t,\omega^{m,i}}} |\widehat{Y}_s^{t,\omega^{m,i}} - \widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}}| + \sup_{(\underline{\theta}_n^*)^{t,\omega} \leq s \leq (\overline{\theta}_n^*)^{t,\omega}} |\widehat{Y}_s^{t,\omega} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}| \\
&= \big|\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}\big| + 2\sup_{(\underline{\theta}_n^*)^{t,\omega} \leq s \leq (\overline{\theta}_n^*)^{t,\omega}} |\widehat{Y}_s^{t,\omega} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}| \\
&\quad + \sup_{(\underline{\theta}_n^*)^{t,\omega^{m,i}} \leq s \leq (\overline{\theta}_n^*)^{t,\omega^{m,i}}} |\widehat{Y}_s^{t,\omega^{m,i}} - \widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}}| - \sup_{(\underline{\theta}_n^*)^{t,\omega} \leq s \leq (\overline{\theta}_n^*)^{t,\omega}} |\widehat{Y}_s^{t,\omega} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}|.
\end{aligned}$$

Applying Lemma 5.3 we get

$$\begin{aligned}
\big|\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}} - \widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}\big| &\leq Cn\rho_2\Big(\mathbf{d}_\infty\big(((\underline{\theta}_n^*)^{t,\omega^{m,i}}, \omega^{m,i}\otimes_t B^t), ((\underline{\theta}_n^*)^{t,\omega}, \omega\otimes_t B^t)\big)\Big) \\
&\leq Cn\rho_2\Big(\delta + \rho_n(\delta) + 2\sup_{(\underline{\theta}_n^*)^{t,\omega} - \rho_n(\delta) \leq s \leq (\underline{\theta}_n^*)^{t,\omega} + \rho_n(\delta)} |B_s^t - B^t_{(\underline{\theta}_n^*)^{t,\omega}}|\Big) \\
&\leq Cn\rho_2\Big(\delta + \rho_n(\delta) + 2\eta_n(\delta)\Big),
\end{aligned}$$

and, similarly,

$$
\begin{aligned}
&\sup_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}\le s\le(\overline{\theta}_n^*)^{t,\omega^{m,i}}}|\widehat{Y}_s^{t,\omega^{m,i}}-\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}}|-\sup_{(\underline{\theta}_n^*)^{t,\omega}\le s\le(\overline{\theta}_n^*)^{t,\omega}}|\widehat{Y}_s^{t,\omega}-\widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}|\\
&\le \sup_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}\le s\le(\underline{\theta}_n^*)^{t,\omega^{m,i}}\vee(\underline{\theta}_n^*)^{t,\omega}}|\widehat{Y}_s^{t,\omega^{m,i}}-\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}}|\\
&\quad+\sup_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}\vee(\underline{\theta}_n^*)^{t,\omega}\le s\le(\overline{\theta}_n^*)^{t,\omega^{m,i}}\wedge(\overline{\theta}_n^*)^{t,\omega}}|\widehat{Y}_s^{t,\omega^{m,i}}-\widehat{Y}_s^{t,\omega}|+|\widehat{Y}^{t,\omega^{m,i}}_{(\underline{\theta}_n^*)^{t,\omega^{m,i}}}-\widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}|\\
&\quad+\sup_{(\overline{\theta}_n^*)^{t,\omega^{m,i}}\wedge(\overline{\theta}_n^*)^{t,\omega}\le s\le(\overline{\theta}_n^*)^{t,\omega^{m,i}}}|\widehat{Y}_s^{t,\omega^{m,i}}-\widehat{Y}^{t,\omega^{m,i}}_{(\overline{\theta}_n^*)^{t,\omega^{m,i}}}|+|\widehat{Y}^{t,\omega^{m,i}}_{(\overline{\theta}_n^*)^{t,\omega^{m,i}}}-\widehat{Y}^{t,\omega}_{(\overline{\theta}_n^*)^{t,\omega}}|\\
&\le Cn\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)+C\rho_1(\delta)\le Cn\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big).
\end{aligned}
$$

Then

$$
\big|\widehat{Y}^{t,\omega^{m,i}}_{\bar{\tau}_n^{t,\omega^{m,i}}}-\widehat{Y}^{t,\omega}_{\bar{\tau}_n^{t,\omega}}\big|\le Cn\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)+2\sup_{(\underline{\theta}_n^*)^{t,\omega}\le s\le(\overline{\theta}_n^*)^{t,\omega}}|\widehat{Y}_s^{t,\omega}-\widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}|.
$$

Plug this and (5.38) into (5.37), for $\omega\in E_i^{m,\delta}$ we obtain

$$
\begin{aligned}
\widehat{Y}_t(\omega)-\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[\widehat{Y}^{t,\omega}_{\bar{\tau}_n^{t,\omega}}\Big]\le{}& Cn\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)\Big]+C\rho_n(\delta)+\varepsilon\\
&+2\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[\sup_{(\underline{\theta}_n^*)^{t,\omega}\le s\le(\overline{\theta}_n^*)^{t,\omega}}|\widehat{Y}_s^{t,\omega}-\widehat{Y}^{t,\omega}_{(\underline{\theta}_n^*)^{t,\omega}}|\psi_n^{t,\omega}\Big]\\
&+C\mathbb{P}_i^{m,\delta}\Big[[(\Omega_n^*)^{t,\omega}]^c\Big]\\
&+C\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[1-\psi_n^{t,\omega}\Big]+C\mathbb{E}^{\mathbb{P}_i^{m,\delta}}\Big[h_\delta\big(d\big(\omega\otimes_t B^t,(\Omega_n^*)^c\big)\big)\Big].
\end{aligned}
$$

Then by (5.35) we have, for any $N\ge n$,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\widehat{Y}_t\mathbf{1}_E]-\mathbb{E}^{\mathbb{P}^{N,\delta}}[\widehat{Y}_{t\vee\bar{\tau}_n}\mathbf{1}_E]=\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[[\widehat{Y}_t-\widehat{Y}_{\bar{\tau}_n}]\mathbf{1}_{E\cap\{t<\bar{\tau}_n\}}\Big]\\
&\le Cn\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)\Big]+C\rho_n(\delta)+\varepsilon+C\mathbb{P}^{N,\delta}\Big[[\Omega_n^*]^c\Big]\\
&\quad+C\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[1-\psi_n\Big]\\
&\quad+2\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\sup_{\underline{\theta}_n^*\le s\le\overline{\theta}_n^*}|\widehat{Y}_s-\widehat{Y}_{\underline{\theta}_n^*}|\psi_n\mathbf{1}_E\Big]+C\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[h_\delta\big(d\big(\omega,(\Omega_n^*)^c\big)\big)\Big]\\
&\le Cn\mathcal{E}\Big[\rho_2\Big(\delta+\rho_n(\delta)+2\eta_n(\delta)\Big)\Big]+C\rho_n(\delta)+\varepsilon+C2^{-n}+C\mathcal{C}(\psi_n<1)\\
&\quad+2\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\sup_{\underline{\theta}_n^*\le s\le\overline{\theta}_n^*}|\widehat{Y}_s-\widehat{Y}_{\underline{\theta}_n^*}|\psi_n\mathbf{1}_E\Big]+C\mathcal{E}\Big[h_\delta\big(d\big(\omega,(\Omega_n^*)^c\big)\big)\Big].
\end{aligned}
\tag{5.39}
$$

Similarly we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[[\widehat{Y}_{t\vee\bar{\tau}_n} - \widehat{Y}_{t\vee\underline{\theta}_n^*}\psi_n]\mathbf{1}_E\Big] \\
&\quad\le C2^{-n} + C\mathcal{C}(\psi_n < 1) + \mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[[\widehat{Y}_{t\vee\bar{\tau}_n} - \widehat{Y}_{t\vee\underline{\theta}_n^*}]\mathbf{1}_{E\cap\Omega_n^*}\psi_n\Big] \\
&\quad\le C2^{-n} + C\mathcal{C}(\psi_n < 1) + 2\mathbb{E}^{\mathbb{P}^{N,\delta}}\Big[\sup_{\underline{\theta}_n^*\le s\le\overline{\theta}_n^*}|\widehat{Y}_s - \widehat{Y}_{\underline{\theta}_n^*}|\psi_n\mathbf{1}_E\Big].
\end{aligned}$$

This, together with (5.39), implies (5.36).

*Step* 3. Finally we prove the lemma for general stopping time $\tau$. We follow the arguments in Lemma 5.11. Let $\tau^n$ be a sequence of stopping times such that $\tau^n \downarrow \tau$ and each $\tau^n$ takes only finitely many values. By applying the dominated convergence theorem under $\mathbb{P}$, we may fix $n$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[|\widehat{Y}_{\tau^n\wedge\widehat{\tau}^*} - \widehat{Y}_\tau|\mathbf{1}_E\Big] \le \frac{\varepsilon}{2}.$$

Assume $\tau^n$ takes values $\{t_i, i = 1, \dots, m\}$, and for each $i$, denote $E_i := E \cap \{\tau^n = t_i < \widehat{\tau}^*\} \in \mathcal{F}_{t_i}$. Then $\{E_i, 1 \le i \le m\}$ form a partition of $\tilde{E} := E \cap \{\tau^n < \widehat{\tau}^*\}$. For each $i$, by Step 1 there exists $\mathbb{P}^i \in \mathcal{P}(\mathbb{P}, t_i, E_i)$ such that

$$\mathbb{E}^{\mathbb{P}}\left[\widehat{Y}_{t_i}\mathbf{1}_{E_i}\right] \le \mathbb{E}^{\mathbb{P}^i}\left[\widehat{Y}_{\widehat{\tau}^*-}\mathbf{1}_{E_i}\right] + \frac{\varepsilon}{2m}.$$

Now define $\mathbb{P}_\varepsilon := \sum_{i=1}^m \mathbb{P}^i\mathbf{1}_{E_i} + \mathbb{P}\mathbf{1}_{\tilde{E}^c} \in \mathcal{P}(\mathbb{P}, \tau^n, \tilde{E}) \subset \mathcal{P}(\mathbb{P}, \tau, E)$. Recall that $\tilde{E} \in \mathcal{F}_{\tau^n}$ and note that $\widehat{Y}_{\widehat{\tau}^*} \le \widehat{Y}_{\widehat{\tau}^*-}$, thanks to the supermartingale property of $\widehat{Y}$. Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\widehat{Y}_\tau\mathbf{1}_E\Big] - \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\widehat{Y}_{\widehat{\tau}^*-}\mathbf{1}_E\Big] &\le \frac{\varepsilon}{2} + \mathbb{E}^{\mathbb{P}}\Big[\widehat{Y}_{\tau^n\wedge\widehat{\tau}^*}\mathbf{1}_E\Big] - \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\widehat{Y}_{\widehat{\tau}^*-}\mathbf{1}_E\Big] \\
&\le \frac{\varepsilon}{2} + \mathbb{E}^{\mathbb{P}}\Big[\widehat{Y}_{\tau^n}\mathbf{1}_{\tilde{E}}\Big] - \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\widehat{Y}_{\widehat{\tau}^*-}\mathbf{1}_{\tilde{E}}\Big] \\
&= \frac{\varepsilon}{2} + \sum_{i=1}^m\Big(\mathbb{E}^{\mathbb{P}}\Big[\widehat{Y}_{t_i}\mathbf{1}_{E_i}\Big] - \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\widehat{Y}_{\widehat{\tau}^*-}\mathbf{1}_{E_i}\Big]\Big) \\
&\le \frac{\varepsilon}{2} + \sum_{i=1}^m \frac{\varepsilon}{2m} = \varepsilon.
\end{aligned}$$

The proof is complete now. ■

We need one more lemma.

**Lemma 5.13.** *Let $\mathbb{P} \in \mathcal{P}$, $\tau \in \mathcal{T}$, and $E \in \mathcal{F}_\tau$ such that $\tau \le \mathrm{H}$ on $E$. For any $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{P}(\mathbb{P}, \tau, E)$ such that*

$$\mathrm{H} \le \tau + \frac{1}{L}d(\omega_\tau, O^c) + 3\varepsilon + \sup_{\tau\le t\le\tau+\varepsilon}|\omega_t - \omega_\tau|, \quad \mathbb{P}_\varepsilon\text{-a.s. on } E.$$

**Proof.** First, there exists $\tilde{\tau} \in \mathcal{T}$ such that $\tau \le \tilde{\tau} \le \tau + \varepsilon$ and $\tilde{\tau}$ takes only finitely many values $0 \le t_1 < \cdots < t_n = t_0$. Denote $E_i := E \cap \{\tilde{\tau} = t_i < \mathrm{H}\} \in \mathcal{F}_{t_i}$. Then $\{E_i, 1 \le i \le n\}$ is a partition of $E \cap \{\tilde{\tau} < \mathrm{H}\}$ and

$$\mathrm{H} \le \tilde{\tau} \le \tau + \varepsilon \quad \text{on } E \cap \{\tilde{\tau} \ge \mathrm{H}\}. \tag{5.40}$$

For any $i$, there exists a partition $(E^i_j)_{j\ge 1}$ of $E_i$ such that $|\omega_{t_i} - \omega'_{t_i}| \le L\varepsilon$ for any $\omega, \omega' \in E^i_j$. For each $(i, j)$, fix an $\omega^{ij} \in E^i_j$ and a unit vector $\alpha^{ij}$ pointing to the direction from $\omega^{ij}_{t_i}$ to $O^c$. Now for any $\omega \in E^i_j$, define $\mathbb{P}^{i,j,\omega} \in \mathcal{P}_{t_i}$ as follows:

$$\beta = \mathbf{0}, \quad \alpha_t = \frac{1}{\varepsilon}[\omega^{ij}_{t_i} - \omega_{t_i}]\mathbf{1}_{[t_i, t_i+\varepsilon)}(t) + L\alpha^{ij}\mathbf{1}_{[t_i+\varepsilon, T]}(t).$$

We see that

$$\mathrm{H}^{t_i,\omega} = \left[t_i + \varepsilon + \frac{1}{L}d(\omega^{ij}_{t_i}, O^c)\right] \wedge t_0, \quad \mathbb{P}^{i,j,\omega}\text{-a.s. on } E^i_j.$$

Similar to the proof of (5.12), there exists $\mathbb{P}_\varepsilon \in \mathcal{P}(\mathbb{P}, \tilde\tau, E) \subset \mathcal{P}(\mathbb{P}, \tau, E)$ such that the regular conditional probability distribution $\mathbb{P}^{t_i,\omega}_\varepsilon = \mathbb{P}^{i,j,\omega}$ for $\mathbb{P}$-a.e. $\omega \in E^i_j$. Then

$$\begin{aligned}\mathrm{H} &\le \tau + 2\varepsilon + \frac{1}{L}[d(\omega_{t_i}, O^c) + L\varepsilon] \le \tau + 3\varepsilon + \frac{1}{L}\Big[d(\omega_\tau, O^c) + |\omega_\tau - \omega_{t_i}|\Big]\\ &\le \tau + 3\varepsilon + \frac{1}{L}\Big[d(\omega_\tau, O^c) + \sup_{\tau\le t\le\tau+\varepsilon}|\omega_t - \omega_\tau|\Big], \quad \mathbb{P}_\varepsilon\text{-a.s. on } E^i_j.\end{aligned}$$

This, together with (5.40), proves the lemma. ■

We are now ready to complete the

**Proof of Proposition 5.9.** The inequality $\mathcal{E}[\widehat{Y}_{\widehat\tau^*}] \le \mathcal{E}[\widehat{Y}_{\widehat\tau^*-}]$ is a direct consequence of the $\mathcal{E}$-supermartingale property of $\widehat{Y}$ established in Theorem 5.4. As for the reverse inequality, since $\widehat{Y}$ is continuous on $[0, \mathrm{H})$ and $\mathrm{H}_n \uparrow \mathrm{H}$ with $\mathrm{H}_n < \mathrm{H}$, it suffices to show that, for any $\mathbb{P} \in \mathcal{P}$ and any $\varepsilon > 0$

$$I_n := \mathbb{E}^{\mathbb{P}}[\widehat{Y}_{\widehat\tau^*\wedge \mathrm{H}_n}] - \mathcal{E}[\widehat{Y}_{\widehat\tau^*}] \le 5\varepsilon \quad \text{for sufficiently large } n. \tag{5.41}$$

Let $\delta > 0, n > \frac{1}{L\delta}$. Set $t_n := t_0 - \frac{1}{n}$, $\tau^0 := \widehat\tau^* \wedge \mathrm{H}_n$, and $\mathbb{P}^0 := \mathbb{P}$. We proceed in two steps.

*Step* 1. Apply Lemma 5.11 with $\mathbb{P}^0, \tau^0, \widehat\tau^*$, and $\Omega$, there exist $\mathbb{P}^{1,1} \in \mathcal{P}(\mathbb{P}^0, \tau^0, \Omega)$ and a stopping time $\tilde\tau^1$ taking values in $[\tau^0, \widehat\tau^*]$, such that

$$\mathbb{E}^{\mathbb{P}^0}[\widehat{Y}_{\tau^0}] \le \mathbb{E}^{\mathbb{P}^{1,1}}\Big[\widehat{X}_{\tilde\tau^1}\mathbf{1}_{\{\tilde\tau^1<\widehat\tau^*\}} + \widehat{Y}_{\widehat\tau^*}\mathbf{1}_{\{\tilde\tau^1=\widehat\tau^*\}}\Big] + \varepsilon.$$

Denote $E_1 := \{\tilde\tau^1 < t_n\} \in \mathcal{F}_{\tilde\tau^1}$. By (5.3) and following the same argument as for the estimate in (4.6), we have: $\mathbb{P}^{1,1}$-a.s. on $E^c_1 \cap \{\tilde\tau^1 < \widehat\tau^*\}$,

$$\begin{aligned}\widehat{X}_{\tilde\tau^1} &\le \widehat{X}_{\tilde\tau^1} - \mathbb{E}^{\mathbb{P}^{1,1}}_{\tilde\tau^1}[\widehat{X}_{\widehat\tau^*}] + \mathbb{E}^{\mathbb{P}^{1,1}}_{\tilde\tau^1}[\widehat{Y}_{\widehat\tau^*}]\\ &\le \mathbb{E}^{\mathbb{P}^{1,1}}_{\tilde\tau^1}\left[\rho_0\left(\frac{1}{n} + \|B^{\tilde\tau^1}\|_{\tilde\tau^1+\frac{1}{n}}\right)\right] + \mathbb{E}^{\mathbb{P}^{1,1}}_{\tilde\tau^1}[\widehat{Y}_{\widehat\tau^*}] \le C\bar\rho_0(n^{-1}) + \mathbb{E}^{\mathbb{P}^{1,1}}_{\tilde\tau^1}[\widehat{Y}_{\widehat\tau^*}].\end{aligned}$$

Then, denoting $E_2 := E_1 \cap \{\tilde\tau^1 < \widehat\tau^*\} \in \mathcal{F}_{\tilde\tau^1}$, we get:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^0}\Big[\widehat{Y}_{\tau^0}\Big] &\le \mathbb{E}^{\mathbb{P}^{1,1}}\Big[\widehat{X}_{\tilde\tau^1}\mathbf{1}_{E_2} + \widehat{X}_{\tilde\tau^1}\mathbf{1}_{E^c_1\cap\{\tilde\tau^1<\widehat\tau^*\}} + \widehat{Y}_{\widehat\tau^*}\mathbf{1}_{\{\tilde\tau^1=\widehat\tau^*\}}\Big] + \varepsilon\\ &\le \mathbb{E}^{\mathbb{P}^{1,1}}\Big[\widehat{X}_{\tilde\tau^1}\mathbf{1}_{E_2} + \widehat{Y}_{\widehat\tau^*}\mathbf{1}_{E^c_2}\Big] + C\bar\rho_0(n^{-1})\mathbb{P}^0[E^c_1] + \varepsilon.\end{aligned} \tag{5.42}$$

Next, set $\tilde\delta := [\delta^2\bar\rho_0(3\delta)] \wedge \frac{\delta}{3}$. Apply Lemma 5.13 on $\mathbb{P}^{1,1}$, $\tilde\tau^1$, $E_2$, and $\tilde\delta$, there exists $\mathbb{P}^{1,2} \in \mathcal{P}(\mathbb{P}^{1,1}, \tilde\tau^1, E_2)$ such that

$$\mathrm{H} \le \tilde\tau^1 + \frac{1}{L} d(\omega_{\tilde\tau^1}, O^c) + \delta + \|\omega_t^{\tilde\tau^1}\|_{\tilde\tau^1+\tilde\delta}, \quad \mathbb{P}^{1,2}\text{-a.s. on } E_2.$$

Since $\tilde\tau^1 \le \widehat\tau^* \le \mathrm{H}$, we have

$$\widehat\tau^* - \tilde\tau^1 \le 3\delta, \quad \mathbb{P}^{1,2}\text{-a.s. on } E_2 \cap \{d(\omega_{\tilde\tau^1}, O^c) \le L\delta\} \cap \{\|\omega^{\tilde\tau^1}\|_{\tilde\tau^1+\tilde\delta} \le \delta\}.$$

Then, by (5.3) and (4.6) again we have: $\mathbb{P}^{1,2}$-a.s. on $E_2 \cap \{d(\omega_{\tilde\tau^1}, O^c) \le L\delta\} \in \mathcal{F}_{\tilde\tau^1}$,

$$\begin{aligned}
\widehat X_{\tilde\tau^1} &\le \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\widehat X_{\widehat\tau^*}] + \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}\Big[\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau^1, B), (\widehat\tau^*, B)\big)\Big)\Big]\\
&= \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\widehat X_{\widehat\tau^*}] + \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}\Big[\rho_0\Big(\mathbf{d}_\infty\big((\tilde\tau^1, B), (\widehat\tau^*, B)\big)\Big)\big[\mathbf{1}_{\{\|B^{\tilde\tau^1}\|_{\tilde\tau_1+\tilde\delta}\le\delta\}} + \mathbf{1}_{\{\|B^{\tilde\tau^1}\|_{\tilde\tau_1+\tilde\delta}>\delta\}}\big]\Big]\\
&\le \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\widehat X_{\widehat\tau^*}] + \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}\Big[\rho_0\Big(3\delta + \|B^{\tilde\tau^1}\|_{\tilde\tau^1+3\delta}\Big)\Big] + C\delta^{-2}\mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\|B^{\tilde\tau^1}\|^2_{\tilde\tau_1+\tilde\delta}]\\
&\le \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\widehat X_{\widehat\tau^*}] + C\bar\rho_0(3\delta) + \frac{C\tilde\delta}{\delta^2} \le \mathbb{E}^{\mathbb{P}^{1,2}}_{\tilde\tau^1}[\widehat X_{\widehat\tau^*}] + C\bar\rho_0(3\delta).
\end{aligned}$$

Note that $n^{-1} \le L\delta \le 3\delta$. Thus, denoting $E_3 := E_2 \cap \{d(\omega_{\tilde\tau^1}, O^c) > L\delta\} \in \mathcal{F}_{\tilde\tau^1}$, (5.42) leads to:

$$\mathbb{E}^{\mathbb{P}^0}\Big[\widehat Y_{\tau^0}\Big] \le \mathbb{E}^{\mathbb{P}^{1,2}}\Big[\widehat X_{\tilde\tau^1}\mathbf{1}_{E_3} + \widehat Y_{\widehat\tau^*}\mathbf{1}_{E_3^c}\Big] + C\bar\rho_0(3\delta)\mathbb{P}^{1,2}(E_3^c) + \varepsilon. \tag{5.43}$$

Moreover, apply Lemma 5.12 with $\mathbb{P}^{1,2}$, $\tilde\tau^1$, $E_3$, and $\varepsilon$, there exists $\mathbb{P}^{1,3} \in \mathcal{P}(\mathbb{P}^{1,2}, \tilde\tau^1, E_3)$ such that

$$\mathbb{E}^{\mathbb{P}^{1,2}}\Big[\widehat X_{\tilde\tau^1}\mathbf{1}_{E_3}\Big] \le \mathbb{E}^{\mathbb{P}^{1,2}}\Big[\widehat Y_{\tilde\tau^1}\mathbf{1}_{E_3}\Big] \le \mathbb{E}^{\mathbb{P}^{1,3}}\Big[\widehat Y_{\widehat\tau^*-}\mathbf{1}_{E_3}\Big] + \varepsilon.$$

Define $\tau^1 := \inf\{t \ge \tilde\tau^1 : d(\omega_t, O^c) \le \frac{1}{n}\} \wedge \widehat\tau^*$. Note that $\tau^1 < \mathrm{H}$ on $E_3$ and $\widehat Y$ is a $\mathbb{P}^{1,3}$-supermartingale. Then

$$\mathbb{E}^{\mathbb{P}^{1,3}}\Big[\widehat Y_{\widehat\tau^*-}\mathbf{1}_{E_3}\Big] \le \mathbb{E}^{\mathbb{P}^{1,3}}\Big[\widehat Y_{\tau^1}\mathbf{1}_{E_3}\Big].$$

Thus

$$\mathbb{E}^{\mathbb{P}^{1,2}}\Big[\widehat X_{\tilde\tau^1}\mathbf{1}_{E_3}\Big] \le \mathbb{E}^{\mathbb{P}^{1,3}}\Big[\widehat Y_{\tau^1}\mathbf{1}_{E_3}\Big] + \varepsilon.$$

Plug this into (5.43), we obtain

$$\mathbb{E}^{\mathbb{P}^0}\Big[\widehat Y_{\tau^0}\Big] \le \mathbb{E}^{\mathbb{P}^{1,3}}\Big[\widehat Y_{\tau^1}\mathbf{1}_{E_3} + \widehat Y_{\widehat\tau^*}\mathbf{1}_{E_3^c}\Big] + C\bar\rho_0(3\delta)\mathbb{P}^{1,3}(E_3^c) + 2\varepsilon.$$

We now denote $\mathbb{P}^1 := \mathbb{P}^{1,3} \in \mathcal{P}(\mathbb{P}^0, \tau^0, \Omega)$, and

$$D_1 := E_3 \cap \{\tau^1 < \widehat\tau^*\} = \{\tilde\tau^1 < t_n \wedge \tilde\tau^*\} \cap \{d(\omega_{\tilde\tau^1}, O^c) > L\delta\} \cap \{\tau^1 < \widehat\tau^*\} \in \mathcal{F}_{\tau^1}. \tag{5.44}$$

Then

$$\mathbb{E}^{\mathbb{P}^0}\Big[\widehat Y_{\tau^0}\Big] \le \mathbb{E}^{\mathbb{P}^1}\Big[\widehat Y_{\tau^1}\mathbf{1}_{D_1} + \widehat Y_{\widehat\tau^*}\mathbf{1}_{D_1^c}\Big] + C\bar\rho_0(3\delta)\mathbb{P}^1(D_1^c) + 2\varepsilon. \tag{5.45}$$

*Step* 3: Iterating the arguments of Step 1, we may define $(\tilde{\tau}^m, \tau^m, \mathbb{P}^m, D_m)_{m\geq 1}$ such that:

$$\mathbb{P}^{m+1} \in \mathcal{P}(\mathbb{P}^m, \tau^m, D_m), \quad \tau^m \leq \tilde{\tau}^{m+1} \leq \widehat{\tau}^*;$$
$$\tau^{m+1} := \inf\Big\{t \geq \tilde{\tau}^{m+1} : d(\omega_t, O^c) \leq \frac{1}{n}\Big\} \wedge \widehat{\tau}^*$$
$$D_{m+1} := D_m \cap \{\tilde{\tau}^{m+1} < t_n \wedge \widehat{\tau}^*\} \cap \{d(\omega_{\tilde{\tau}^{m+1}}, O^c) > L\delta\} \cap \{\tau^{m+1} < \widehat{\tau}^*\};$$

and

$$\mathbb{E}^{\mathbb{P}^m}\Big[\widehat{Y}_{\tau^m}\mathbf{1}_{D_m}\Big] \leq \mathbb{E}^{\mathbb{P}^{m+1}}\Big[\widehat{Y}_{\tau^{m+1}}\mathbf{1}_{D_{m+1}} + \widehat{Y}_{\widehat{\tau}^*}\mathbf{1}_{D_m\cap D^c_{m+1}}\Big] + C\bar{\rho}_0(3\delta)\mathbb{P}^{m+1}(D_m \cap D^c_{m+1}) + 2^{1-m}\varepsilon.$$

By induction, for any $m \geq 1$ we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^0}\Big[\widehat{Y}_{\tau^0}\Big] &\leq \mathbb{E}^{\mathbb{P}^m}\Big[\widehat{Y}_{\tau^m}\mathbf{1}_{D_m} + \widehat{Y}_{\widehat{\tau}^*}\mathbf{1}_{D^c_m}\Big] + C\bar{\rho}_0(3\delta)\mathbb{P}^m(D^c_m) + 4\varepsilon \\ &\leq \mathbb{E}^{\mathbb{P}^m}[\widehat{Y}_{\widehat{\tau}^*}] + 2C_0\mathbb{P}^m[D_m] + C\bar{\rho}_0(3\delta) + 4\varepsilon. \end{aligned} \tag{5.46}$$

Note that

$$\begin{aligned}\mathbb{P}^m[D_m] &\leq \mathbb{P}^m\Big[\cap_{i=1}^m\Big\{|B_{\tilde{\tau}^i} - B_{\tau^{i-1}}| \geq L\delta - \frac{1}{n}\Big\} \cap \Big\{|B_{\tau^i} - B_{\tilde{\tau}^i}| \geq L\delta - \frac{1}{n}\Big\}\Big] \\ &\leq \mathbb{P}^m\left[\sum_{i=1}^m[|B_{\tilde{\tau}^i} - B_{\tau^{i-1}}|^2 + |B_{\tau^i} - B_{\tilde{\tau}^i}|^2] \geq 2m\left(L\delta - \frac{1}{n}\right)^2\right] \\ &\leq \frac{1}{2m\left(L\delta - \frac{1}{n}\right)^2}\mathbb{E}^{\mathbb{P}^m}\left[\sum_{i=1}^m[|B_{\tilde{\tau}^i} - B_{\tau^{i-1}}|^2 + |B_{\tau^i} - B_{\tilde{\tau}^i}|^2]\right] \leq \frac{C}{2m\left(L\delta - \frac{1}{n}\right)^2}.\end{aligned}$$

Then, (5.46) leads to

$$I_n \leq \frac{C}{2m\left(L\delta - \frac{1}{n}\right)^2} + C\bar{\rho}_0(3\delta) + 4\varepsilon,$$

which implies, by sending $m \to \infty$ that

$$I_n \leq C\bar{\rho}_0(3\delta) + 4\varepsilon.$$

Hence, by choosing $\delta$ small enough such that $\bar{\rho}_0(3\delta) \leq \varepsilon$, we see that (5.41) holds true for $n > \frac{1}{L\delta}$. ■

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## Appendix

### A.1. Regular conditional probability distribution

We first recall the definition of r.c.p.d. from Stroock–Varadhan [16]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ is a sub-$\sigma$-field. An r.c.p.d. $\{\mathbb{P}^\omega_{\mathcal{G}}, \omega \in \Omega\}$ is a family of probability measures on $\mathcal{F}$ satisfying the following requirements:

- For all $E \in \mathcal{F}$, the mapping $\omega \to \mathbb{P}^\omega_{\mathcal{G}}(E)$ is $\mathcal{G}$-measurable;
- For all $\xi \in \mathbb{L}^\infty(\mathcal{F})$, the conditional expectation $\mathbb{E}^{\mathbb{P}^\omega_{\mathcal{G}}}[\xi] = \mathbb{E}[\xi|\mathcal{G}](\omega)$, for $\mathbb{P}$-a.e. $\omega$;
- For any $\omega \in \Omega$, $\mathbb{P}^\omega_{\mathcal{G}}(\Omega^\omega_{\mathcal{G}}) = 1$, where $\Omega^\omega_{\mathcal{G}} := \cap\{E \in \mathcal{G} : \omega \in E\}$.

We note that en r.c.p.d. exists whenever $\mathcal{G}$ is countably generated.

In the special case that $\Omega := \{\omega \in C([0,T], \mathbb{R}^d) : \omega_0 = \mathbf{0}\}$ is our canonical space and $\mathcal{G} = \mathcal{F}_\tau$ for some $\tau \in \mathcal{T}$, it holds that

$$\Omega^\omega_{\mathcal{F}_\tau} := \left\{\omega' \in \Omega : \tau(\omega') = \tau(\omega) \text{ and } \omega'_{\wedge\tau} = \omega_{\wedge\tau}\right\} = \left\{\omega \otimes_{\tau(\omega)} \omega' : \omega' \in \Omega^{\tau(\omega)}\right\}. \tag{A.1}$$

Then, as in [14], we define for all $\omega \in \Omega$ a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}^{\tau(\omega)}_T$ by:

$$\mathbb{P}^{\tau,\omega}(E) := \mathbb{P}^\omega_{\mathcal{F}_\tau}\Big(\{\omega \otimes_{\tau(\omega)} \omega' : \omega' \in E\}\Big), \quad \forall E \in \mathcal{F}^{\tau(\omega)}_T, \tag{A.2}$$

and still call it an r.c.p.d. of $\mathbb{P}$ conditional on $\mathcal{F}_\tau$. One may easily check that $\omega \longmapsto \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]$ is $\mathcal{F}_\tau$-measurable, for all $\xi \in \mathbb{L}^\infty(\mathcal{F}_T)$, and $\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\big[\xi^{\tau,\omega}\big] = \mathbb{E}[\xi|\mathcal{F}_\tau](\omega)$, for $\mathbb{P}$-a.e. $\omega$ and for all $\xi \in \mathbb{L}^\infty(\mathcal{F})$.

### A.2. Proof of Lemma 2.3

Recall the notations in the beginning of Section 2.2. Let $\mathbb{F} := \mathbb{F}^B$ and $\tilde{\mathbb{F}} := \tilde{\mathbb{F}}^{\tilde{B}}$ be the natural filtrations on $\Omega$ and $\tilde{\Omega}$, respectively. Moreover, we may identify $\mathbb{F}$ with the filtration $\tilde{\mathbb{F}}^B$ on $\tilde{\Omega}$ generated by $B$: $\tilde{\mathcal{F}}^B_t = \{E \times \Omega^2 : E \in \mathcal{F}^B_t\}$.

(i) First, it follows from standard arguments, see e.g. Zheng [17, Theorem 3], that $\mathcal{P}^L_t$ is weakly compact. Then Property (P1) holds.

(ii) We next check without loss of generality Property (P2) only at $t = 0$. Let $\tau \in \mathcal{T}$ and $\mathbb{P} \in \mathcal{P}^L$ with corresponding $\mathbb{Q}$ as in (2.2). Define $\tilde{\tau}(\tilde{\omega}) := \tau(\omega)$ for $\tilde{\omega} := (\omega, a, m) \in \tilde{\Omega}$, then clearly $\tilde{\tau}$ is an $\tilde{\mathbb{F}}^B$-stopping time, hence also an $\tilde{\mathbb{F}}$-stopping time. By Stroock–Varadhan [16], the r.c.p.d. $\mathbb{Q}^{\tilde{\omega}}_{\tilde{\mathcal{F}}^B_{\tilde{\tau}}}$ exists. Note that $\tilde{\omega} \longmapsto \mathbb{Q}^{\tilde{\omega}}_{\tilde{\mathcal{F}}^B_{\tilde{\tau}}}(E)$ is $\tilde{\mathcal{F}}^B_{\tilde{\tau}}$-measurable for any $E \in \tilde{\mathcal{F}}_T$, it follows that $\mathbb{Q}^{\tilde{\omega}}_{\tilde{\mathcal{F}}^B_{\tilde{\tau}}}$ depends only on $\omega$ and thus we may denote it as $\mathbb{Q}^{\omega}_{\tilde{\mathcal{F}}^B_{\tilde{\tau}}}$.

Recall the shifted spaces $\Omega^t$, $\tilde{\Omega}^t$, $\mathbb{F}^t$, and $\tilde{\mathbb{F}}^t$. We now define the following probability measure on the shifted space $\Omega^{\tau(\omega)}$:

$$\begin{aligned} &\mathbb{Q}^{\tau,\omega}[\tilde{E}] := \mathbb{Q}^\omega_{\tilde{\mathcal{F}}^B_{\tilde{\tau}}}\Big(\tilde{\omega}^1 \otimes_{\tau(\omega)} \tilde{\omega}^2 : \tilde{\omega}^1 \in \tilde{\Omega}, \tilde{\omega}^2 \in \tilde{E}\Big), \quad \forall \tilde{E} \in \tilde{\mathcal{F}}^{\tau(\omega)}_T; \\ &\mathbb{P}^{\tau,\omega}[E] := \mathbb{Q}^{\tau,\omega}\big[E \times (\Omega^{\tau(\omega)})^2\big], \quad \forall E \in \mathcal{F}^{\tau(\omega)}_T. \end{aligned} \tag{A.3}$$

It is straightforward to check that $\mathbb{P}^{\tau,\omega}$ is an r.c.p.d. of $\mathbb{P}$ conditional on $\mathcal{F}^\tau$, and $\mathbb{Q}^{\tau,\omega}$ is the required extension on $\tilde{\Omega}^{\tau(\omega)}$ satisfying (2.2) for $\mathbb{P}$-a.e. $\omega$. This verifies (P2).

(iii) It remains to check Property (P3). Assume $\mathbb{Q}$ and $\mathbb{Q}^i$ are the corresponding extensions of $\mathbb{P}$ and $\mathbb{P}^i$. Define

$$\hat{\mathbb{Q}} := \mathbb{Q} \otimes_t \left[ \sum_{i=1}^{\infty} \mathbb{Q}^i \mathbf{1}_{E_i \times (\Omega^s)^2} + \mathbb{Q} \mathbf{1}_{\cap_{i=1}^{\infty} (E_i^c \times (\Omega^s)^2)} \right].$$

Following similar arguments as in (ii) one can show that $\hat{\mathbb{Q}}$ satisfies (2.2). It is clear that $\hat{\mathbb{P}}(E) = \hat{\mathbb{Q}}(E \times (\Omega^s)^2)$ for all $E \in \mathcal{F}_T^s$. Then $\hat{\mathbb{P}} \in \mathcal{P}_s^L$ and thus (P3) holds. ∎

### A.3. Some additional results

In this subsection we provide some results which are interesting for our discussion, although they are technically not used in the paper.

**Proof of Remark 3.2.** Fix $\omega \in \Omega$, and let $\{t_n\}$ and $\{s_n\}$ be two sequences such that $t_n \uparrow t$, $s_n \uparrow t$, and $X_{t_n}(\omega) \longrightarrow \overline{\lim}_{s\uparrow t} X_s(\omega)$, $X_{s_n}(\omega) \longrightarrow \underline{\lim}_{s\uparrow t} X_s(\omega)$. Here and in the sequel, in $\lim_{s\uparrow t}$ we take the notational convention that $s < t$. Without loss of generality, we may assume $t_n < s_n < t_{n+1}$ for $n = 1, 2, \ldots$. Then for the $\rho_0$ defined in (3.1) we have

$$\begin{aligned} 0 &\le \overline{\lim_{s\uparrow t}} X_s(\omega) - \underline{\lim_{s\uparrow t}} X_s(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) - \lim_{n\to\infty} X_{s_n}(\omega) \\ &\le \underline{\lim_{n\to\infty}} \rho_0\Big(\mathbf{d}_\infty\big((t_n, \omega), (s_n, \omega)\big)\Big) = 0. \end{aligned}$$

This implies the existence of $X_{t-}(\omega)$. Moreover,

$$X_{t-}(\omega) - X_t(\omega) = \lim_{s\uparrow t} X_s(\omega) - X_t(\omega) \le \underline{\lim_{s\uparrow t}} \rho\Big(\mathbf{d}_\infty\big((s, \omega), (t, \omega)\big)\Big) = 0,$$

completing the proof. ∎

**Lemma A.1.** *Let the nondegeneracy condition (3.7) hold and $X$ be bounded and uniformly continuous in $(t, \omega)$ under $d_\infty$. Then $\hat{Y}^{\mathrm{H}}$ defined in (3.5) is left continuous at* $\mathrm{H}$.

**Proof.** We first claim that, for any $\omega \in \Omega$ and $\varepsilon > 0$

$$\lim_{t\uparrow \mathrm{H}(\omega)} \mathcal{C}_t[\mathrm{H}^{t,\omega} \ge t + \varepsilon] = 0. \tag{A.4}$$

Indeed, let $\mathrm{H}$ correspond to $O$ and $t_0$ as in (3.3). If $\mathrm{H}(\omega) = t_0$, since $\mathrm{H}^{t,\omega} \le t_0$, (A.4) is obvious. We now assume $t_1 := \mathrm{H}(\omega) < t_0$ and thus $\omega_{t_1} \in O^c$. Note that $t < \mathrm{H}(\omega)$ implies $\omega_t \in O$. Denote $\delta := d(\omega_t, O^c)$, then $0 < \delta \le |\omega_t - \omega_{t_1}|$. Let $\eta$ be a unit vector pointing to the direction from $\omega_t$ to $O^c$. Since $O$ is convex, we see that

$$\text{for any } x \in \mathbb{R}^d, x \cdot \eta \ge \delta \quad \text{implies } x + \omega_t \in O^c. \tag{A.5}$$

Since we will send $t \uparrow t_1$, we may assume $\delta \le \varepsilon$. Then, for any $\mathbb{P} \in \mathcal{P}_t$ with corresponding $\alpha$, $\beta$, and $W$, we have

$$\mathbb{P}\Big(\mathrm{H}^{t,\omega} \ge t + \varepsilon\Big) \le \mathbb{P}\Big(\mathrm{H}^{t,\omega} \ge t + \delta\Big) \le \mathbb{P}\Big(\sup_{t\le s\le t+\delta} (B_s^t \cdot \eta) < \delta\Big) \le \mathbb{P}\Big(\sup_{t\le s\le t+\delta} M_s \le C\delta\Big)$$

where $M_s := \int_t^s \beta_r dW_r \cdot \eta$ is a scalar $\mathbb{P}$-martingale. Denote $A_s := \int_t^s |\beta_r \eta|^2 dr$ and introduce the time change: $\tau_r := \inf\{s \ge t : A_s \ge r - t\}$ and $N_r := M_{\tau_r}$. Then $N$ is a $\mathbb{P}$-Brownian motion.

Since $\beta \geq cI_d$, then $c^2(\tau_r - t) \leq r - t$, and thus

$$\mathbb{P}\Big(\mathrm{H}^{t,\omega} \geq t+\varepsilon\Big) \leq \mathbb{P}\Big(\sup_{t\leq s\leq t+\delta} M_s \leq C\delta\Big) \leq \mathbb{P}\Big(\sup_{t\leq r\leq t+c^2\delta} N_r \leq C\delta\Big) = C\sqrt{\delta},$$

where $C$ is independent of $\mathbb{P}$. Then $\mathcal{C}_t[\mathrm{H}^{t,\omega} \geq t+\varepsilon] \leq C\sqrt{\delta}$ for $\delta \leq \varepsilon$. Now send $t \uparrow \mathrm{H}(\omega)$, we have $\delta \to 0$ and thus (A.4) holds.

We now prove the lemma. Let $\rho$ denote the modulus of continuity function of $X$. Note that in this case $\hat{X}^{\mathrm{H}} = X_{\mathrm{H}\wedge\cdot}$. Fix $\omega \in \Omega$. For $t < t_1 := \mathrm{H}(\omega)$ and $\varepsilon > 0$, denoting $E := \{\mathrm{H}^{t,\omega} \leq t+\varepsilon\} \cap \{\|B^t\|_{t+\varepsilon} \leq \varepsilon^{\frac{1}{3}}\}$, we have

$$\begin{aligned}
|\hat{Y}_t^{\mathrm{H}}(\omega) - \hat{Y}_{\mathrm{H}}^{\mathrm{H}}(\omega)| &\leq \sup_{\tau\in\mathcal{T}^t} \mathcal{E}_t\Big[|X^{t,\omega}_{\tau\wedge\mathrm{H}^{t,\omega}} - X_{\mathrm{H}}(\omega)|\Big] \\
&\leq C\mathcal{C}_t[E^c] + \sup_{\tau\in\mathcal{T}^t} \mathcal{E}_t\Big[\rho\big(d_\infty((\tau\wedge\mathrm{H}^{t,\omega}, \omega\otimes_t B^t), (t_1,\omega))\big)\mathbf{1}_E\Big] \\
&\leq C\mathcal{C}_t[\mathrm{H}^{t,\omega} \geq t+\varepsilon] + C\mathcal{C}_t[\|B^t\|_{t+\varepsilon} \geq \varepsilon^{\frac{1}{3}}] \\
&\quad + \mathcal{E}_t\Big[\rho\Big(d_\infty((\mathrm{H}^{t,\omega}, \omega\otimes_t B^t), (t,\omega)) + d_\infty((t,\omega),(t_1,\omega))\Big)\mathbf{1}_E\Big] \\
&\leq C\mathcal{C}_t[\mathrm{H}^{t,\omega} \geq t+\varepsilon] + C\varepsilon^{\frac{1}{3}} + \rho\Big(\varepsilon + \varepsilon^{\frac{1}{3}} + d_\infty((t,\omega),(t_1,\omega))\Big).
\end{aligned}$$

Then, by (A.4) we have

$$\overline{\lim_{t\uparrow\mathrm{H}(\omega)}} |\hat{Y}_t^{\mathrm{H}}(\omega) - \hat{Y}_{\mathrm{H}}^{\mathrm{H}}(\omega)| \leq C\varepsilon^{\frac{1}{3}} + \rho\Big(\varepsilon + \varepsilon^{\frac{1}{3}}\Big).$$

Since $\varepsilon$ is arbitrary, we prove the result. ■

However, in the degenerate case in general $\hat{Y}^{\mathrm{H}}$ may be discontinuous at H.

**Example A.2.** Set $X_t(\omega) := t$ and let H correspond to $O$ and $t_0$. Clearly $\widehat{X}^{\mathrm{H}} = X$, $\widehat{Y}^{\mathrm{H}}_{\mathrm{H}} = \mathrm{H}$ and $\widehat{Y}_t^{\mathrm{H}}(\omega) \leq t_0$. However, for any $t < \mathrm{H}(\omega)$, set $\tau := t_0$ and $\mathbb{P} \in \mathcal{P}_t$ such that $\alpha^{\mathbb{P}} = 0$, $\beta^{\mathbb{P}} = 0$, we see that $\widehat{Y}_t^{\mathrm{H}}(\omega) \geq \mathbb{E}^{\mathbb{P}}\Big[X(\mathrm{H}(\omega\otimes_t B^t), \omega\otimes_t B^t)\Big] = X(\mathrm{H}(\omega_{\cdot\wedge t}), \omega_{\cdot\wedge t}) = \mathrm{H}(\omega_{\cdot\wedge t}) = t_0$. That is, $\widehat{Y}_t^{\mathrm{H}}(\omega) = t_0$. Thus $\widehat{Y}^{\mathrm{H}}$ is discontinuous at H whenever $\mathrm{H}(\omega) < t_0$. ■

## References


[1] E. Bayraktar, I. Karatzas, S. Yao, Optimal stopping for dynamic convex risk measures, Illinois J. Math. 54 (3) (2010) 1025–1067.

[2] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992) 1–67.

[3] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths, Potential Anal. 34 (2) (2011) 139–161.

[4] L. Denis, C. Martini, A theorectical framework for the pricing of contingent claims in the presence of model uncertainty, Ann. Appl. Probab. 16 (2) (2006) 827–852.

[5] I. Ekren, C. Keller, N. Touzi, J. Zhang, On viscosity solutions of path dependent PDEs, Ann. Probab. 42 (2014) 204–236.

[6] I. Ekren, N. Touzi, J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I, Preprint, arXiv:1210.0006.

[7] I. Ekren, N. Touzi, J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II, Preprint arXiv:1210.0007.

[8] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, M.C. Quenez, Reflected solutions of backward SDE's, and related obstacle problems for PDE's, Ann. Probab. 25 (2) (1997) 702–737.

[9] W. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, second ed., Springer, New York, 2006.
[10] N.V. Krylov, Controlled Diffusion Proceses, Springer-Verlag, 1980.
[11] M. Nutz, R. van Handel, Constructing sublinear expectations on path space, Stochastic Process. Appl. 123 (2013) 3100–3121.
[12] S. Peng, $G$-Brownian motion and dynamic risk measure under volatility uncertainty, 2007, Preprint, arXiv:0711.2834.
[13] M. Soner, N. Touzi, J. Zhang, Wellposedness of second order BSDEs, Probab. Theory Related Fields 153 (2012) 149–190.
[14] M. Soner, N. Touzi, J. Zhang, Dual formulation of second order target problems, Ann. Appl. Probab. 23 (2013) 308–347.
[15] Y. Song, Properties of hitting times for $G$-martingales and their applications, Stochastic Process. Appl. 121 (2011) 1770–1784.
[16] D.W. Stroock, S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
[17] W. Zheng, Tightness results for laws of diffusion processes application to stochastic mechanics, Ann. Inst. Henri Poincaré Probab. Stat. 21 (2) (1985) 103–124.