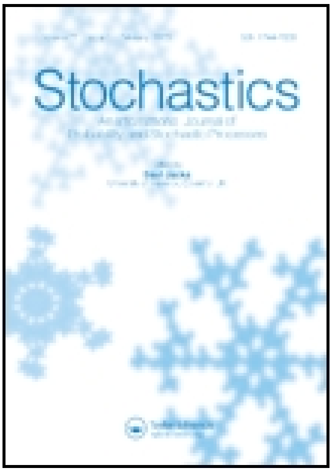

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A complete representation theorem for G-martingales

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In this article, we establish a complete representation theorem for G-martingales. Unlike the existing results in the literature, we provide the existence and uniqueness of the second-order term, which corresponds to the second-order derivative in Markovian case. The main ingredient of the article is a new norm for that second-order term, which is based on an operator introduced by Song.

Keywords: G-expectations; G-martingales; martingale representation theorem; nonlinear expectations

AMS Subject Classification: 60H10; 60H30

1. Introduction

The notion of G-expectation is a type of nonlinear expectation proposed by Peng [18,19]. In the Markovian case, it corresponds to a fully nonlinear partial differential equation (PDE). We also refer to Cheridito et al. [1] and Soner et al. [23,24] for the closely related theory of second-order backward stochastic differential equations (SDEs). The theory has received very strong attention in the literature in recent years, we refer to the survey paper [20] and the references therein as well as some more recent developments: [4,5,10,12,13,15,16,26], to mention a few. Their typical applications include, among others, stochastic optimization with diffusion control and economic/financial models with volatility uncertainty (see, e.g. [3,6,14]) and numerical methods for high-dimensional fully nonlinear PDEs (see e.g. [7,27,8]).

G-expectation is a typical nonlinear expectation. It can be regarded as a nonlinear generalization of Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, where $\Omega = C([0,\infty), \mathbb{R}^d)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and $\mathbb{P}_0$ is a Wiener probability measure defined on $(\Omega, \mathcal{F})$. Recall that the Wiener measure is defined such that the canonical process $B_t(\omega) := \omega_t$, $t \geq 0$, is a continuous process with stationary and independent increments, namely $(B_t)_{t\geq 0}$ is a Brownian motion. G-expectation $\mathbb{E}^G$ is a sublinear expectation on the same canonical space Ω, such that the same canonical process B is a G-Brownian motion, i.e. it is a continuous process with stationary and independent increments. One important feature of this notion is its time consistency. To be precise, let ξ be a random variable and $Y_t := \mathbb{E}_t^G[\xi]$ denote the conditional G-expectation, then one has $\mathbb{E}_s^G[\xi] = \mathbb{E}_s^G[\mathbb{E}_t^G(\xi)]$ for any $s < t$. For this reason, we call the conditional G-expectation a G-martingale, or a martingale under G-expectation. It is well known that a martingale under Wiener measure can be written as a stochastic integral against the Brownian motion. Then a very natural and fundamental

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question in this nonlinear G-framework is:

$$\text{What is the structure of a } G\text{-martingale } \ Y? \tag{1.1}$$

Peng [18] has observed that, for $Z \in \mathcal{H}_G^2$ and $\eta \in \mathcal{M}_G^1$ (see (2.12) and (2.17)), the following process Y is always a G-martingale:

$$\mathrm{d}Y_t = Z_t \,\mathrm{d}B_t - G(\eta_t)\,\mathrm{d}t + \frac{1}{2}\eta_t \,\mathrm{d}\langle B\rangle_t. \tag{1.2}$$

Here, G is the deterministic function Peng [18] used to define G-expectations and $\langle B\rangle$ is the quadratic variation of the G-Brownian motion B. We remark that, in a Markovian framework, we have $Y_t = u(t, B_t)$, where u is a smooth function satisfying the following fully nonlinear PDE:

$$\partial_t u + G(\partial_{xx} u) = 0. \tag{1.3}$$

Then $Z_t = \partial_x u(t, B_t)$ and $\eta_t = \partial_{xx} u(t, B_t)$. In particular, if $\xi = g(B_T)$, then by PDE arguments we see immediately that $Y_t := \mathbb{E}_t^G[\xi]$ has a representation (1.2). Peng proved further that this (Z, η)-representation holds if ξ is in a dense subspace $\mathcal{L}_{ip}$ of $\mathcal{L}_G^p$ (see (2.5)). But observing that $\mathcal{L}_{ip}$ is not a complete space, a very interesting question was then raised to give a complete (Z, η)-representation theorem for $\mathbb{E}_t^G[\xi]$.

The first partial answer was provided by Xu and Zhang [28]: if Y is a symmetric G-martingale, that is both Y and $-Y$ are G-martingales, then

$$\mathrm{d}Y_t = Z_t \,\mathrm{d}B_t \quad \text{for some process } \ Z. \tag{1.4}$$

However, symmetric G-martingales capture only the linear part in this nonlinear framework, and it is important to understand the structure of non-symmetric G-martingales.

By introducing a new norm $\|\cdot\|_{\mathbb{L}_G^2}$ (see (2.22)), Soner et al. [22] proved a more general representation theorem: for $\xi \in \mathbb{L}_G^2$,

$$\mathrm{d}Y_t = Z_t \,\mathrm{d}B_t - \mathrm{d}K_t, \tag{1.5}$$

where K is an increasing process such that $-K$ is a G-martingale. It has been proved independently in [22] and Song [25] that $\mathbb{L}_G^p \supset \cap_{q>p} \mathcal{L}_G^q$, where $\|\cdot\|_{\mathcal{L}_G^q}$ is the norm introduced in [18] (see (2.5)). Moreover, [25] extended representation (1.5) to the case $p > 1$.

Now the question is: when does the process K in (1.5) have the structure

$$\mathrm{d}K_t = G(\eta_t)\,\mathrm{d}t - \frac{1}{2}\eta_t \,\mathrm{d}\langle B\rangle_t? \tag{1.6}$$

Several efforts have been made in this direction. Hu and Peng [11] and Pham and Zhang [21] made some progresses on the existence of η. However, there is no characterization of the process η, and in particular, they do not provide an appropriate norm for η. On the other hand, Song [26] proved the uniqueness of η in the space $\mathcal{M}_G^1$. A clever operator was introduced in this work, which successfully isolates the term $(1/2)\eta_t \,\mathrm{d}\langle B\rangle_t$ from $\mathrm{d}K_t$, and thus, essentially captures the uncertainty of the underlying distributions. This idea turns out to be the building block of this article.

Our main contribution of this article is to introduce a norm for the process η, based on the work [26]. We shall prove the existence and uniqueness of the component η, which provides an essentially complete answer to Peng's question (1.1). Moreover, we shall provide *a priori* norm estimates. In particular, given ξ_1 and ξ_2 in appropriate space, let (Y^i, Z^i, η^i), $i = 1, 2$, be the corresponding terms, we shall estimate the norms of $Z^1 - Z^2$ and $\eta^1 - \eta^2$ in terms of that of $Y^1 - Y^2$, where the latter one is more tractable due to the representation formula $Y_t = \mathbb{E}_t^G[\xi]$. Unlike [26], we prove the estimates via PDE arguments.

The rest of the paper is organized as follows. In Section 2, we introduce G-martingales and the involved spaces. In Section 3, we propose the new norm for η and provide the crucial estimates. In Section 4, we establish the complete representation theorem for G-martingales.

2. Preliminaries

In this section, we introduce G-expectations and G-martingales. We shall focus on a simple setting in which we will establish the martingale representation theorem. However, these notions can be extended to much more general framework, as in many works in the literature.

We start with some notations in multiple dimensional setting. Fix a dimension d. Let $\mathbb{R}^d$ and $\mathbb{S}^d$ denote the sets of d-dimensional column vectors and $d \times d$-symmetric matrices, respectively. For $\sigma_1, \sigma_2 \in \mathbb{S}^d$, $\sigma_1 \leq \sigma_2$ (respectively, $\sigma_1 < \sigma_2$) means that $\sigma_2 - \sigma_1$ is non-negative (respectively, positive) definite, and we denote by $[\sigma_1, \sigma_2]$ the set of $\sigma \in \mathbb{S}^d$ satisfying $\sigma_1 \leq \sigma \leq \sigma_2$. Throughout the article, we use 0 to denote the d-dimensional zero vector or zero matrix, and I_d the $d \times d$ identity matrix. For $x, \tilde{x} \in \mathbb{R}^d$, $\gamma, \tilde{\gamma} \in \mathbb{S}^d$, define

$$x \cdot \tilde{x} := x^{\mathrm{T}} \tilde{x}, \quad |x| := \sqrt{x \cdot x}, \quad \text{and} \quad \gamma : \tilde{\gamma} := \mathrm{tr}(\gamma \tilde{\gamma}), \quad |\gamma| := \sqrt{\gamma : \gamma}, \tag{2.1}$$

where x^{T} denotes the transpose of x. One can easily check that

$$|\gamma : \tilde{\gamma}| \leq |\gamma||\tilde{\gamma}|, \quad \text{and} \quad -\gamma \leq \tilde{\gamma} \leq \gamma \text{ implies that } |\tilde{\gamma}| \leq |\gamma|. \tag{2.2}$$

2.1 Conditional G-expectations

We fix a finite time interval $[0, T]$ and two constant matrices $0 < \underline{\sigma} < \bar{\sigma}$ in $\mathbb{S}^d$. Define

$$G(\gamma) := \frac{1}{2} \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} (\sigma^2 : \gamma), \quad \text{for all } \gamma \in \mathbb{S}^d. \tag{2.3}$$

Let $\Omega := \omega \in C([0, T], \mathbb{R}^d) : \omega_0 = \mathbf{0}$ be the canonical space, B the canonical process and $\mathbb{F} := \mathbb{F}^B$ the filtration generated by B. For $\xi = \varphi(B_T)$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a bounded and Lipschitz continuous function, following Peng [18] we define the conditional G-expectation $\mathbb{E}_t^G[\xi] := u(t, B_t)$, where u is the (unique) classical solution of the following PDE on $[0, T]$:

$$\partial_t u + G(\partial_{xx} u) = 0, \quad u(T, x) = \varphi(x). \tag{2.4}$$

Let $\mathcal{L}_{ip}$ denote the set of random variables $\xi = \varphi(B_{t_1}, \ldots, B_{t_n})$ for some $0 \leq t_1 < \cdots < t_n \leq T$ and some Lipschitz continuous function φ. In the same spirit, one may define $\mathbb{E}_t^G[\xi]$ backwardly over each interval $[t_i, t_{i+1}]$. In particular, when $t = 0$ we define $\mathbb{E}^G[\xi] := \mathbb{E}_0^G[\xi]$.

For any $p \geq 1$, define

$$\|\xi\|^p_{\mathcal{L}^p_G} := \mathbb{E}^G[|\xi|^p], \quad \xi \in \mathcal{L}_{ip}. \tag{2.5}$$

Clearly, this defines a norm in $\mathcal{L}_{ip}$. Let $\mathcal{L}^p_G$ denote the closure of $\mathcal{L}_{ip}$ under the norm $\|\cdot\|_{\mathcal{L}^p_G}$, taking the quotient as in the standard literature (i.e. we do not distinguish random variables ξ_1 and ξ_2 if $\|\xi_1 - \xi_2\|_{\mathcal{L}^p_G} = 0$). As a mapping on the space $\mathcal{L}_{ip}$, the conditional G-expectation is continuous w.r.t. the norm $\|\cdot\|_{\mathcal{L}^1_G}$. So, one can easily extend it to all $\xi \in \mathcal{L}^1_G$.

We next provide a representation of conditional G-expectations by using the quasi-sure stochastic analysis, initiated by Denis and Martini [3] for superhedging problem under volatility uncertainty. Let $\mathcal{A}$ denote the space of $\mathbb{F}$-progressively measurable processes taking values in $[\underline{\sigma}, \overline{\sigma}]$. Denoting by $\mathbb{P}_0$ the Wiener measure, we define

$$\mathcal{P} := \left\{\mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\sigma)^{-1} : \sigma \in \mathcal{A}\right\} \quad \text{where } X^\sigma_t := \int_0^t \sigma_s \, \mathrm{d}B_s, \ \ \mathbb{P}_0\text{-a.s.} \tag{2.6}$$

Then B is a $\mathbb{P}$-martingale for each $\mathbb{P} \in \mathcal{P}$. Following [3], we say

$$\begin{aligned} &\text{a property holds } \mathcal{P}\text{-quasi surely, abbreviated as } \mathcal{P}\text{-q.s.,} \\ &\quad \text{if it holds } \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}. \end{aligned} \tag{2.7}$$

We note that $\|\xi\|_{\mathcal{L}^1_G} = 0$ if and only if $\xi = 0$, $\mathcal{P}$-q.s. Throughout this article, random variables are considered the same if they are equal $\mathcal{P}$-q.s. Then elements in $\mathcal{L}^1_G$ can be viewed as standard random variables, but in $\mathcal{P}$-q.s. sense. In particular, for any $\xi \in \mathcal{L}^1_G$, conditional G-expectation $\mathbb{E}^G_t[\xi]$ is defined $\mathcal{P}$-q.s.

It was proved in Denis et al. [2] that:

$$\mathbb{E}^G[\xi] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi], \quad \xi \in \mathcal{L}^1_G. \tag{2.8}$$

This result was extended by Soner et al. [22] to conditional G-expectations: for any $\xi \in \mathcal{L}^1_G$, $t \in [0, T]$ and $\mathbb{P} \in \mathcal{P}$,

$$\mathbb{E}^G_t[\xi] = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[\xi], \ \ \mathbb{P}\text{-a.s., where } \mathcal{P}(t,\mathbb{P}) := \left\{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\right\}. \tag{2.9}$$

We remark that Peng [17] had similar ideas in the contexts of strong formulation.

We finally note that $\mathbb{E}^G_t$ is obviously a sublinear expectation (again, all the equalities and inequalities are viewed in $\mathcal{P}$-q.s. sense): for any $\xi, \xi_1, \xi_2 \in \mathcal{L}^1_G$,

$$\begin{aligned} &\mathbb{E}^G_t[\xi] = \xi, \quad \text{if } \xi \text{ is } \mathcal{F}_t\text{-measurable}; \quad && \mathbb{E}^G_t[\lambda\xi] = \lambda\xi, \quad \text{for all } \lambda \geq 0; \\ &\mathbb{E}^G_t[\xi_1] \leq \mathbb{E}^G_t[\xi_2], \quad \text{if } \xi_1 \leq \xi_2; && \mathbb{E}^G_t[\xi_1 + \xi_2] \leq \mathbb{E}^G_t[\xi_1] + \mathbb{E}^G_t[\xi_2]. \end{aligned} \tag{2.10}$$

2.2 *Stochastic integrals*

First, notice that there exists a unique ($\mathcal{P}$-q.s.) $\mathbb{S}^d$-valued process $\langle B \rangle$ such that $B_t B_t^T - \langle B \rangle_t$ is a symmetric G-martingale. In fact, under each $\mathbb{P} \in \mathcal{P}$, $\langle B \rangle$ is the same as the quadratic variation of the $\mathbb{P}$-martingale B, and consequently,

$$\underline{\sigma}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\langle B\rangle_t \leq \bar{\sigma}^2, \quad \mathcal{P}\text{-q.s.} \tag{2.11}$$

Naturally we call $\langle B\rangle$ the quadratic variation of B. Next, we call that an $\mathbb{F}$-progressively measurable process Z with appropriate dimension is an elementary process if it takes the form $Z = \sum_{i=0}^{n-1} Z_{t_i} 1_{[t_i,t_{i+1})}$ for some $0 = t_0 < \cdots < t_n \leq T$ and each component of Z_{t_i} is in $\mathcal{L}_{ip}$. Let $\mathcal{H}_G^0$ denote the space of $\mathbb{R}^d$-valued elementary processes. For any $p \geq 1$, define

$$\|Z\|^p_{\mathcal{H}_G^p} := \mathbb{E}^G\left[\left(\int_0^T (Z_t Z_t^T) : \mathrm{d}\langle B\rangle_t\right)^{p/2}\right], \quad Z \in \mathcal{H}_G^0; \tag{2.12}$$

and let $\mathcal{H}_G^p$ denote the closure of $\mathcal{H}_G^0$ under the norm $\|\cdot\|_{\mathcal{H}_G^p}$.

Now for each $Z \in \mathcal{H}_G^0$, we define its stochastic integral:

$$\int_0^t Z_s \cdot \mathrm{d}B_s := \sum_{i=0}^{n-1} Z_{t_i}\cdot[B_{t_{i+1}\wedge t} - B_{t_i\wedge t}], \tag{2.13}$$

One can easily prove the Burkholder–Davis–Gundy Inequality (see, e.g. Song [25] Proposition 4.3): for any $p > 0$, there exist constants $0 < c_p < C_p < \infty$ such that

$$c_p\|Z\|^p_{\mathcal{H}_G^p} \leq \mathbb{E}^G\left[\sup_{0\leq t\leq T}|\int_0^t Z_s \cdot \mathrm{d}B_s|^p\right] \leq C_p\|Z\|^p_{\mathcal{H}_G^p}. \tag{2.14}$$

Then one can extend the stochastic integral to all $Z \in \mathcal{H}_G^p$.

2.3 G-martingales

One important feature of conditional G-expectations is the time consistency, which can also be viewed as dynamic programming principle:

$$\mathbb{E}_s^G\left[\mathbb{E}_t^G(\xi)\right] = \mathbb{E}_s^G[\xi], \quad \text{for all} \quad \xi \in \mathcal{L}_G^1 \quad \text{and} \quad 0 \leq s < t \leq T. \tag{2.15}$$

We recall that

$$\text{a process } Y \text{ is called a } G\text{-martingale if } \mathbb{E}_s^G[Y_t] = Y_s \text{ for all } 0 \leq s < t \leq T. \tag{2.16}$$

Therefore, Y is a G-martingale if and only if $Y_t = \mathbb{E}_t^G[\xi]$ for $\xi = Y_T$.

Let X, Y be two G-martingales. In general, neither $-X$ nor $X + Y$ is a G-martingale since the conditional G-expectation is only sublinear. If $-X$ is also a G-martingale, then we call X a symmetric G-martingale, and in this case, one can easily check that $X + Y$ is still a G-martingale.

It is clear that $\int_0^t Z_s \cdot \mathrm{d}B_s$ is a symmetric G-martingale for all $Z \in \mathcal{H}_G^1$. In particular, the canonical process B is a symmetric G-martingale and is called a G-Brownian motion. However, G-martingales have a much richer structure. Let $\mathcal{M}_G^0$ be the space of $\mathbb{S}^d$-valued elementary processes. Define

$$\|\eta\|^p_{\mathcal{M}_G^p} := \mathbb{E}^G\left[\left(\int_0^T |\eta_t|\,\mathrm{d}t\right)^p\right], \quad \eta \in \mathcal{M}_G^0; \tag{2.17}$$

and let $\mathcal{M}_G^p$ denote the closure of $\mathcal{M}_G^0$ under the norm $\|\cdot\|_{\mathcal{M}_G^p}$. An interesting fact observed by Peng [18] is that the following decreasing process is also a G-martingale:

$$-K_t := \frac{1}{2}\int_0^t \eta_s : \mathrm{d}\langle B\rangle_s - \int_0^t G(\eta_s)\,\mathrm{d}s, \quad \eta \in \mathcal{M}_G^1. \tag{2.18}$$

Consequently, the following process Y is always a G-martingale:

$$Y_t = Y_0 + \int_0^t Z_s \cdot \mathrm{d}B_s - \left[\int_0^t G(\eta_s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \eta_s : \mathrm{d}\langle B\rangle_s\right], \quad Z \in \mathcal{H}_G^1, \ \ \eta \in \mathcal{M}_G^1. \tag{2.19}$$

On the other hand, for any $\xi \in \mathcal{L}_{ip}$, by Peng [19] there exist $Z \in \mathcal{H}_G^1$ and $\eta \in \mathcal{M}_G^1$ such that $Y_t := \mathbb{E}_t^G[\xi]$ satisfies (2.19). In particular, when $\xi = \varphi(B_T)$, for the classical solution u of PDE (2.4), we have:

$$Y_t = u(t, B_t), \quad Z_t = \partial_x u(t, B_t), \quad \eta_t = \partial_{xx} u(t, B_t). \tag{2.20}$$

Our goal of this article is to answer the following natural question proposed by Peng [19].

$$\text{For what } \xi \text{ do there exist unique } Z \in \mathcal{H}_G^1 \text{ and } \eta \in \mathcal{M}_G^1 \text{ satisfying (2.19)?} \tag{2.21}$$

The problem was partially solved by Soner et al. [22], in which the following norm was introduced:

$$\|\xi\|_{\mathbb{L}_G^p}^p := \mathbb{E}^G\left[\sup_{0\le t\le T} \left(\mathbb{E}_t^G[|\xi|]\right)^p\right], \quad \xi \in \mathcal{L}_{ip}. \tag{2.22}$$

Let $\mathbb{L}_G^p$ denote the closure of $\mathcal{L}_{ip}$ under the norm $\|\cdot\|_{\mathbb{L}_G^p}$. Then for any $\xi \in \mathbb{L}_G^2$, there exist unique $Z \in \mathcal{H}_G^2$ and an increasing process K with $K_0 = 0$ such that

$$Y_t := \mathbb{E}_t^G[\xi] = Y_0 + \int_0^t Z_s \cdot \mathrm{d}B_s - K_t \quad \text{and} \quad \|Z\|_{\mathcal{H}_G^2} + \|K_T\|_{\mathcal{L}_G^2} \le C\|\xi\|_{\mathbb{L}_G^2}. \tag{2.23}$$

It was proved independently by [22] and Song [25] that $\|\xi\|_{\mathbb{L}_G^p} \le C_{p,q}\|\xi\|_{\mathcal{L}_G^q}$ for any $1 \le p < q$. Moreover, the above representation was extended by [25] to the case $p > 1$.

2.4 Summary of notations

For readers' convenience, we collect here some notations used in the article (some of them will be defined later):

- The inner product $\cdot$, the trace operator $:$ and the norms $|x|$, $|\gamma|$ are defined by (2.1).
- The function G, G^α and G_ε are defined by (2.3), (3.1) and (3.5), respectively.
- The class of probability measures $\mathcal{P}$, the G-expectation $\mathbb{E}^G$ and the conditional G-expectation $\mathbb{E}_t^G$ are defined by (2.6), (2.8) and (2.9), respectively.
- The norms $\|\xi\|_{\mathcal{L}_G^p}$ and $\|\xi\|_{\mathbb{L}_G^p}$ for ξ are defined by (2.5) and (2.22), respectively.
- The norms $\|Z\|_{\mathcal{H}_G^p}$ for Z and $\|\eta\|_{\mathcal{M}_G^p}$ for η are defined by (2.12) and (2.17), respectively.

- The norm $\|Y\|_{\mathbb{D}_G^p}$ for càdlàg processes Y, see also (2.22), is defined by:

$$\|Y\|_{\mathbb{D}_G^p}^p := \mathbb{E}^G\left[\sup_{0\le t\le T}|Y_t|^p\right]. \tag{2.24}$$

- The operator $\mathcal{E}^\alpha_{t_1,t_2}$ is defined by (3.2).
- The constants c_0, C_0 are defined by (3.4).
- The function δ_n is defined by (3.7).
- The norms $\|\eta\|_{\mathbb{M}_G}$ and $\|\eta\|_{\mathbb{M}_G^*}$ for η are defined by (3.11) and (3.18), respectively.
- The space $\mathcal{M}^1_{G_0}$ and class $\mathcal{P}_0$ are defined by (3.19) and (3.16), respectively.
- The metric $d_{G,p}(\xi_1,\xi_2)$ for ξ is defined by (4.3) and $\mathbb{L}_G^{*p}$ is the corresponding closure space.
- For $0\le s\le t\le T$, the shifted canonical process B_t^s is defined by:

$$B_t^s := B_t - B_s. \tag{2.25}$$

3. A new norm for η

Our main contribution of the article is to introduce a norm for η. For that purpose, we shall introduce two nonlinear operators, one via PDE arguments and the other via probabilistic arguments. The latter one is strongly motivated by the work Song [26], and the connection between the two operators is established in Lemma 3.4.

3.1 The nonlinear operator via PDE arguments

We first introduce a new nonlinear operator $\mathcal{E}^\alpha$ on Lipschitz continuous functions, with a parameter $\alpha\in\mathbb{S}^d$. Define

$$G^\alpha(\gamma) = \frac{1}{2}[G(\gamma+2\alpha)+G(\gamma-2\alpha)], \quad \gamma\in\mathbb{S}^d. \tag{3.1}$$

Given $0\le t_1<t_2\le T$ and a Lipschitz continuous function φ, define $\mathcal{E}^\alpha_{t_1,t_2}(\varphi) := u^\alpha(t_1,\cdot)$, where u^α is the unique viscosity solution of the following PDE on $[t_1,t_2]$:

$$\partial_t u^\alpha + G^\alpha(\partial_{xx}u^\alpha) = 0, \quad u^\alpha(t_2,x) = \varphi(x). \tag{3.2}$$

Clearly, G^α is strictly increasing and convex in γ, then PDE (3.2) is parabolic and is well posed. We collect below some obvious properties of G^α and $\mathcal{E}^\alpha$, whose proofs are omitted.

Lemma 1. For any $\alpha\in\mathbb{S}^d$,

(i) $\mathcal{E}^\alpha$ satisfies the semi-group property:

$$\mathcal{E}^\alpha_{t_1,t_2}\left(\mathcal{E}^\alpha_{t_2,t_3}(\varphi)\right) = \mathcal{E}^\alpha_{t_1,t_3}(\varphi), \quad \text{for any } 0\le t_1<t_2<t_3\le T. \tag{3.3}$$

(ii) $G^{-\alpha} = G^\alpha \ge G = G^0$.

(iii) If $\varphi = c$ is a constant, then $\mathcal{E}^\alpha_{t_1,t_2}(c) = c + G^\alpha(\mathbf{0})(t_2-t_1)$.

The next property will be crucial for our estimates. Let

$$c_0 := \text{the smallest eigenvalue of } \frac{1}{2}[\bar{\sigma}^2 - \underline{\sigma}^2], \quad \text{and} \quad C_0 := \frac{1}{2}|\bar{\sigma}^2 - \underline{\sigma}^2|. \tag{3.4}$$

Then clearly $C_0 \geq c_0 > 0$ and $\underline{\sigma}^2 + c_0 I_d \leq \bar{\sigma}^2 - c_0 I_d$. Denote, for $\varepsilon \leq c_0$,

$$G_\varepsilon(\gamma) := \frac{1}{2} \sup_{\sigma \in [\underline{\sigma}_\varepsilon, \bar{\sigma}_\varepsilon]} (\sigma^2 : \gamma), \quad \text{where } \underline{\sigma}_\varepsilon^2 := \underline{\sigma}^2 + \varepsilon I_d, \ \bar{\sigma}_\varepsilon^2 := \bar{\sigma}^2 - \varepsilon I_d. \tag{3.5}$$

LEMMA 3.2.

(i) For any $0 < \varepsilon \leq c_0$ and $\alpha, \gamma \in \mathbb{S}^d$, it holds that

$$G_\varepsilon(\gamma) + \varepsilon|\alpha| \leq G^\alpha(\gamma) \leq G(\gamma) + C_0|\alpha|. \tag{3.6}$$

(ii) Assume $\underline{\varphi} \leq \varphi \leq \bar{\varphi}$ are Lipschitz continuous functions and $0 \leq t_1 < t_2 \leq T$. Then

$$\mathbb{E}^{G_\varepsilon}\left[\underline{\varphi}(x + B_{t_2}^{t_1})\right] + \varepsilon|\alpha|(t_2 - t_1) \leq \mathcal{E}_{t_1,t_2}^\alpha(\varphi)(x) \leq \mathbb{E}^G\left[\bar{\varphi}(x + B_{t_2}^{t_1})\right] + C_0|\alpha|(t_2 - t_1).$$

Proof.

(i) We first prove the left inequality. Let $\alpha_1, \ldots, \alpha_d$ denote the eigenvalues of α and $\hat{\alpha}$ the diagonal matrix with components $\alpha_1, \ldots, \alpha_d$. Then $|\alpha| = (\alpha_1^2 + \cdots + \alpha_d^2)^{1/2}$, and there exists an orthogonal matrix Q such that $Q^T\alpha Q = \hat{\alpha}$. Let $\hat{c}_\varepsilon$ denote a diagonal matrix whose diagonal components take values ε or $-\varepsilon$. Now for any $\sigma_\varepsilon \in [\underline{\sigma}_\varepsilon, \bar{\sigma}_\varepsilon]$, by (3.5) we have

$$\sigma_\varepsilon^2 + Q\hat{c}_\varepsilon Q^T \in [\underline{\sigma}^2, \bar{\sigma}^2] \quad \text{and} \quad \sigma_\varepsilon^2 - Q\hat{c}_\varepsilon Q^T \in [\underline{\sigma}^2, \bar{\sigma}^2].$$

Then

$$\begin{aligned} 2G^\alpha(\gamma) &= G(\gamma + 2\alpha) + G(\gamma - 2\alpha) \\ &\geq \frac{1}{2}\left[(\sigma_\varepsilon^2 + Q\hat{c}_\varepsilon Q^T) : (\gamma + 2\alpha) + (\sigma_\varepsilon^2 - Q\hat{c}_\varepsilon Q^T) : (\gamma - 2\alpha)\right] \\ &= \sigma_\varepsilon^2 : \gamma + 2(Q\hat{c}_\varepsilon Q^T) : \alpha = \sigma_\varepsilon^2 : \gamma + 2\hat{c}_\varepsilon : (Q^T\alpha Q) = \sigma_\varepsilon^2 : \gamma + 2\hat{c}_\varepsilon : \hat{\alpha}. \end{aligned}$$

By the arbitrariness of σ_ε and $\hat{c}_\varepsilon$, we get

$$G^\alpha(\gamma) \geq G_\varepsilon(\gamma) + \varepsilon\sum_{i=1}^d |\alpha_i| \geq G_\varepsilon(\gamma) + \varepsilon|\alpha|.$$

We now prove the right inequality of (3.6). For any $\sigma_1, \sigma_2 \in [\underline{\sigma}, \bar{\sigma}]$, we have

$$\sigma_1^2 : (\gamma + 2\alpha) + \sigma_2^2 : (\gamma - 2\alpha) = (\sigma_1^2 + \sigma_2^2) : \gamma + 2(\sigma_1^2 - \sigma_2^2) : \alpha.$$

Note that

$$\underline{\sigma}^2 \le \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \le \bar{\sigma}^2, \quad -[\bar{\sigma}^2 - \underline{\sigma}^2] \le \sigma_1^2 - \sigma_2^2 \le \bar{\sigma}^2 - \underline{\sigma}^2.$$

Then, by (2.2),

$$\sigma_1^2 : (\gamma + 2\alpha) + \sigma_2^2 : (\gamma - 2\alpha) \le 4G(\gamma) + 4C_0|\alpha|.$$

Since σ_1, σ_2 are arbitrary, we prove the right inequality of (3.6), and hence (3.6).

(ii) One can easily check that

$$\mathbb{E}^{G_\varepsilon}\left[\underline{\varphi}(x + B_{t_2}^{t_1})\right] + \varepsilon|\alpha|(t_2 - t_1) = \underline{v}^\alpha(t_1, x),$$

$$\mathbb{E}^{G}\left[\bar{\varphi}(x + B_{t_2}^{t_1})\right] + C_0|\alpha|(t_2 - t_1) = \bar{v}^\alpha(t_1, x),$$

where $\underline{v}^\alpha, \bar{v}^\alpha$ are the unique viscosity solution of the following PDEs on $[t_1, t_2]$:

$$\partial_t \underline{v}^\alpha + G_\varepsilon(\partial_{xx}\underline{v}^\alpha) + \varepsilon|\alpha| = 0, \quad \underline{v}^\alpha(t_2, x) = \underline{\varphi}(x);$$

$$\partial_t \bar{v}^\alpha + G(\partial_{xx}\bar{v}^\alpha) + C_0|\alpha| = 0, \quad \bar{v}^\alpha(t_2, x) = \bar{\varphi}(x).$$

Then the statement follows directly from (3.6) and the comparison principle of PDEs. □

3.2 *The nonlinear operator via probabilistic arguments*

For any $n \ge 1$, denote $t_i^n := (i/n)T$, $i = 0, \ldots, n$, and define

$$\delta_n(t) = \sum_{i=0}^{n-1} (-1)^i 1_{[t_i^n, t_{i+1}^n)}, \quad t \in [0, T]. \tag{3.7}$$

This function was introduced in [26] which plays a key role for constructing a new norm for process η. According to [26], we have

Lemma 3.3. For any $\eta \in \mathcal{M}_G^1$, it holds that $\lim_{n\to\infty} \mathbb{E}^G\left[\int_0^T G(\eta_t)\delta_n(t)\,\mathrm{d}t\right] = 0$.

The next lemma establishes the connection between δ_n and $(G^\alpha, \mathcal{E}^\alpha)$.

Lemma 3.4. Let $0 \le s < t \le T$ and $\alpha \in \mathbb{S}^d$.

(i) For any $\gamma \in \mathbb{S}^d$, we have

$$\lim_{n\to\infty} \mathbb{E}_s^G\left[\int_s^t \left[\alpha\delta_n(r) + \frac{1}{2}\gamma\right] : \mathrm{d}\langle B\rangle_r\right] = G^\alpha(\gamma)(t - s). \tag{3.8}$$

(ii) For any $x \in \mathbb{R}^d$ and any Lipschitz continuous function φ, we have

$$\lim_{n\to\infty} \mathbb{E}_s^G\left[\int_s^t \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \varphi\left(x + B_t^s\right)\right] = \mathcal{E}_{s,t}^\alpha(\varphi)(x). \tag{3.9}$$

Proof.

(i) Fix n such that $(2T/n) < t - s$. Note that

$$
\begin{aligned}
&\mathbb{E}^G_{t^n_{2i}}\left[\int_{t^n_{2i}}^{t^n_{2i+2}}\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right]\\
&=\mathbb{E}^G_{t^n_{2i}}\left[\left(\frac{1}{2}\gamma+\alpha\right):\left[\langle B\rangle_{t^n_{2i+1}}-\langle B\rangle_{t^n_{2i}}\right]+\left(\frac{1}{2}\gamma-\alpha\right):\left[\langle B\rangle_{t^n_{2i+2}}-\langle B\rangle_{t^n_{2i+1}}\right]\right]\\
&=\mathbb{E}^G_{t^n_{2i}}\left[\left(\frac{1}{2}\gamma+\alpha\right):\left[\langle B\rangle_{t^n_{2i+1}}-\langle B\rangle_{t^n_{2i}}\right]+\mathbb{E}^G_{t^n_{2i+1}}\left[\left(\frac{1}{2}\gamma-\alpha\right):\left[\langle B\rangle_{t^n_{2i+2}}-\langle B\rangle_{t^n_{2i+1}}\right]\right]\right]\\
&=\mathbb{E}^G_{t^n_{2i}}\left[\left(\frac{1}{2}\gamma+\alpha\right):\left[\langle B\rangle_{t^n_{2i+1}}-\langle B\rangle_{t^n_{2i}}\right]+G(\gamma-2\alpha)\frac{T}{n}\right]\\
&=\mathbb{E}^G_{t^n_{2i}}\left[\left(\frac{1}{2}\gamma+\alpha\right):\left[\langle B\rangle_{t^n_{2i+1}}-\langle B\rangle_{t^n_{2i}}\right]\right]+G(\gamma-2\alpha)\frac{T}{n}\\
&=G(\gamma+2\alpha)\frac{T}{n}+G(\gamma-2\alpha)\frac{T}{n}=G^\alpha(\gamma)\big(t^n_{2i+2}-t^n_{2i}\big).
\end{aligned}
$$

Similarly, for any $i < j$,

$$
\mathbb{E}^G_{t^n_{2i}}\left[\int_{t^n_{2i}}^{t^n_{2j}}\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right]=G^\alpha(\gamma)\Big(t^n_{2j}-t^n_{2i}\Big).
$$

Now assume $t^n_{2i} \le s < t^n_{2i+1} \le t^n_{2j} \le t < t^n_{2j+2}$. Then

$$
\begin{aligned}
&\left|\mathbb{E}^G_s\left[\int_s^t\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right]-G^\alpha(\gamma)(t-s)\right|\\
&\quad\le\left|\mathbb{E}^G_s\left[\int_s^t\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right]-\mathbb{E}^G_s\left[\int_{t^n_{2i+2}}^{t^n_{2j}}\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right]\right|\\
&\qquad+\left|G^\alpha(\gamma)\Big(t^n_{2j}-t^n_{2i+2}\Big)-G^\alpha(\gamma)(t-s)\right|\\
&\quad\le\mathbb{E}^G_s\left[\left|\left[\int_s^{t^n_{2i+2}}+\int_{t^n_{2j}}^t\right]\left[\alpha\delta_n(r)+\frac{1}{2}\gamma\right]:\mathrm{d}\langle B\rangle_r\right|\right]+\frac{2T}{n}|G^\alpha(\gamma)|\\
&\quad\le\frac{2T}{n}|\bar\sigma^2|\left[|\alpha|+\frac{1}{2}|\gamma|\right]+\frac{2T}{n}|G^\alpha(\gamma)|\to 0,\quad\text{as } n\to\infty,
\end{aligned}
$$

where the last inequality holds thanks to (2.2). This proves the result.

(ii) Without loss of generality, assume $t = T$. Define

$$
\bar u(t,x):=\overline{\lim_{n\to\infty}}\,\bar u^n(t,x):=\overline{\lim_{n\to\infty}}\,\mathbb{E}^G_t\left[\int_t^T\delta_n(r)\alpha:\mathrm{d}\langle B\rangle_r+\varphi\big(x+B^t_T\big)\right],
$$

$$
\underline{u}(t,x):=\overline{\lim_{n\to\infty}}\,\underline{u}^n(t,x):=\underline{\lim}_{n\to\infty}\,\mathbb{E}^G_t\left[\int_t^T\delta_n(r)\alpha:\mathrm{d}\langle B\rangle_r+\varphi\big(x+B^t_T\big)\right].
$$

By the structure of G-framework it is clear that $\underline{u}$ and $\bar u$ are deterministic. Obviously $\underline{u}\le\bar u$. We claim that $\bar u$ and $\underline{u}$ are viscosity subsolution and supersolution, respectively, of PDE (3.2) with $t_1 = 0, t_2 = T$. Note that (3.2) satisfies the comparison principle for viscosity solutions. Then $\bar{u} \le \underline{u}$, and thus, $\bar{u}(t,x) = \underline{u}(t,x) = \mathcal{E}^{\alpha}_{t,T}(\varphi)(x)$. This proves the result.

We now prove that $\bar{u}$ is a viscosity subsolution, and the viscosity supersolution property of $\underline{u}$ can be proved similarly. As usual, we start from the partial dynamic programming principle: for $0 \le t < t+h \le T$,

$$\bar{u}(t,x) \le \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \bar{u}(t+h, x+B^t_{t+h})\right], \tag{3.10}$$

Indeed, by the time homogeneity of the problem, we have?

$$\begin{aligned}
\bar{u}^n(t,x) &= \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \mathbb{E}^G_{t+h}\left[\int_{t+h}^{T} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \varphi\big(x + B^t_T\big)\right]\right] \\
&= \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \bar{u}^n\big(t+h, x+B^t_{t+h}\big)\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\bar{u}(t,x) - \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \bar{u}\big(t+h, x+B^t_{t+h}\big)\right] \\
&\quad = \overline{\lim_{n\to\infty}} \, \bar{u}^n(t,x) - \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \bar{u}\big(t+h, x+B^t_{t+h}\big)\right] \\
&\quad \le \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[(\bar{u}^n - \bar{u})\big(t+h, x+B^t_{t+h}\big)\right].
\end{aligned}$$

Following standard arguments, it is obvious that $\bar{u}$ is uniformly Lipschitz continuous in x. Moreover, $\overline{\lim}_{n\to\infty} (\bar{u}^n - \bar{u})(t+h,x) = 0$ for any $x \in \mathbb{R}$. Then (3.10) follows directly from the simple Lemma 3.5.

We next derive the viscosity subsolution property from (3.10). Let $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\varphi \in C^{1,2}([t,T) \times \mathbb{R}^d)$ such that $0 = [\bar{u} - \varphi](t,x) = \max_{(s,y)\in[t,T]\times\mathbb{R}^d} [\bar{u} - \varphi](s,y)$. Denote $X_s := x + B^t_s$. For any $0 < h \le T - t$, by (3.10) and then applying Itô's formula we have

$$\begin{aligned}
\varphi(t,x) = \bar{u}(t,x) &\le \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \bar{u}(t+h, X_{t+h})\right] \\
&\le \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \varphi(t+h, X_{t+h})\right] \\
&= \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h} \delta_n(r)\alpha : \mathrm{d}\langle B\rangle_r + \varphi(t,x) + \int_t^{t+h}\left[\partial_t\varphi(r,X_r)\,\mathrm{d}r + \frac{1}{2}\partial_{xx}\varphi(r,X_r) : \mathrm{d}\langle B\rangle_r\right]\right] \\
&\le \overline{\lim_{n\to\infty}} \, \mathbb{E}^G\left[\int_t^{t+h}\left[\alpha\delta_n(r) + \frac{1}{2}\partial_{xx}\varphi(t,x)\right] : \mathrm{d}\langle B\rangle_r\right] + \varphi(t,x) + \partial_t\varphi(t,x)h \\
&\quad + \mathbb{E}^G\left[\int_t^{t+h}[\partial_t\varphi(r,X_r) - \partial_t\varphi(t,x)]\,\mathrm{d}r + \frac{1}{2}\int_t^{t+h}[\partial_{xx}\varphi(r,X_r) - \partial_{xx}\varphi(t,x)] : \mathrm{d}\langle B\rangle_r\right] \\
&\le G^{\alpha}(\partial_{xx}\varphi(t,x))h + \varphi(t,x) + \partial_t\varphi(t,x)h \\
&\quad + \mathbb{E}^G\left[\sup_{t\le r\le t+h}[|\partial_t\varphi(r,X_r) - \partial_t\varphi(t,x)| + \frac{|\overline{\sigma}^2|}{2}|\partial_{xx}\varphi(r,X_r) - \partial_{xx}\varphi(t,x)|\right]h,
\end{aligned}$$

thanks to (3.8). By standard arguments $\bar{u}$ is uniformly Lipschitz continuous in x, and note that viscosity property is a local property. Then, without loss of generality we may assume that $\partial_t \varphi$ and ∂_{xx} are bounded and uniformly continuous in (t,x) with a modulus of continuity function ρ. Thus,

$$0 \leq \partial_t \varphi(t,x) + G^{\alpha}(\partial_{xx}\varphi(t,x)) + C\mathbb{E}^G\left[\rho\left(C\left[h + \sup_{t \leq r \leq t+h} |B_r^t|\right]\right)\right].$$

Let $h \longrightarrow 0$ we can easily get

$$\partial_t \varphi(t,x) + G^{\alpha}(\partial_{xx}\varphi(t,x)) \geq 0.$$

Clearly $\bar{u}(T,x) = \varphi$. Therefore, $\bar{u}$ is a viscosity subsolution of PDE (3.2). □

LEMMA 3.5. Assume $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ are uniformly Lipschitz continuous functions, uniformly in n, and $\overline{\lim}_{n\to\infty} \varphi_n(x) \leq 0$ for all x. Then $\overline{\lim}_{n\to\infty} \mathbb{E}^G[\varphi_n(B_t)] \leq 0$ for any t.

Proof. Let L denote the uniform Lipschitz constant of φ_n. For any $\varepsilon > 0$ and $R > 0$, there exist finitely many x_i, $i = 1, \ldots, M$ and a partition $\cup_{i=1}^M O_i = O_R(\mathbf{0}) := \{x \in \mathbb{R}^d : |x| \leq R\}$ such that $|x - x_i| \leq \varepsilon$ for all $x \in O_i$. Denote $O_0 := \mathbb{R}^d \setminus O_R(\mathbf{0})$ and $x_0 := \mathbf{0}$. Then

$$\begin{aligned}
\varphi_n(B_t) &= \sum_{i=0}^M \varphi_n(B_t)\mathbf{1}_{O_i}(B_t) = \sum_{i=0}^M \varphi_n(x_i)\mathbf{1}_{O_i}(B_t) + \sum_{i=0}^M [\varphi_n(B_t) - \varphi_n(x_i)]\mathbf{1}_{O_i}(B_t) \\
&\leq \sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t) + L|B_t|\mathbf{1}_{O_0}(B_t) + L\varepsilon \sum_{i=1}^M \mathbf{1}_{O_i}(B_t) \\
&\leq \sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t) + \frac{L}{R}|B_t|^2 + L\varepsilon.
\end{aligned}$$

Thus, noting that our condition implies $\overline{\lim}_{n\to\infty} \varphi_n^+(x) = 0$,

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \mathbb{E}^G[\varphi_n(B_t)] &\leq \overline{\lim_{n\to\infty}} \mathbb{E}^G\left[\sum_{i=0}^M \varphi_n^+(x_i)\mathbf{1}_{O_i}(B_t) + \frac{L}{R}|B_t|^2 + L\varepsilon\right] \\
&\leq \sum_{i=0}^M \overline{\lim_{n\to\infty}} \varphi_n^+(x_i)\mathbb{E}^G[\mathbf{1}_{O_i}(B_t)] + \frac{L}{R}\mathbb{E}^G[|B_t|^2] + L\varepsilon = \frac{L}{R}\mathbb{E}^G[|B_t|^2] + L\varepsilon.
\end{aligned}$$

Let $R \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$, we prove the result. □

3.3 An intermediate norm for $\boldsymbol{\eta} \in \mathcal{M}_{\mathbf{G}}^{\boldsymbol{1}}$

We now use $\delta_n(t)$ to introduce the following norm for a process η.

THEOREM 3.6. For any $\eta \in \mathcal{M}_G^1$, the following limit exists:

$$\|\eta\|_{\mathbb{M}_G} := \lim_{n\to\infty} \mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t : \mathrm{d}\langle B\rangle_t\right]. \tag{3.11}$$

Proof. We first assume $\eta \in \mathcal{M}_G^0$. By otherwise considering a finer partition of $[0, T]$, without loss of generality we assume, for $0 = t_0 < \cdots < t_m = T$,

$$\eta = \sum_{i=0}^{m-1} \eta_{t_i} \mathbf{1}_{[t_i, t_{i+1})}, \quad \text{where } \eta_{t_i} = \varphi_i(B_{t_1}, \ldots, B_{t_i}) \tag{3.12}$$

and φ_i is uniformly Lipschitz continuous. Denote

$$\psi_i^n(B_{t_1}, \ldots, B_{t_i}) := \mathbb{E}_{t_i}^G \left[\int_{t_i}^T \delta_n(t) \eta_t : \mathrm{d}\langle B \rangle_t \right].$$

We prove by backward induction that

$$\lim_n \psi_i^n = \psi_i, \tag{3.13}$$

where, $\psi_m := 0$ and, for $i = m-1, \ldots, 0$,

$$\psi_i(x_1, \ldots, x_i) := \mathcal{E}_{t_i, t_{i+1}}^{\varphi_i(x_1, \ldots, x_i)}(\psi_{i+1}(x_1, \ldots, x_i, \cdot))(x_i). \tag{3.14}$$

Indeed, when $i = m$, (3.13) holds obviously. Assume (3.13) holds for $i+1$. Then by (3.9) we have

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \psi_i^n(B_{t_1}, \ldots, B_{t_i}) - \psi_i(B_{t_1}, \ldots, B_{t_i}) \\ &= \overline{\lim_{n\to\infty}} \mathbb{E}_{t_i}^G \left[\int_{t_i}^{t_{i+1}} \delta_n(t) \eta_{t_i} : \mathrm{d}\langle B \rangle_t + \psi_{i+1}^n(B_{t_1}, \ldots, B_{t_{i+1}}) \right] \\ &\quad - \lim_{n\to\infty} \mathbb{E}_{t_i}^G \left[\int_{t_i}^{t_{i+1}} \delta_n(t) \eta_{t_i} : \mathrm{d}\langle B \rangle_t + \psi_{i+1}(B_{t_1}, \ldots, B_{t_{i+1}}) \right] \Big| \\ &\leq \overline{\lim_{n\to\infty}} \mathbb{E}_{t_i}^G \left[\psi_{i+1}^n(B_{t_1}, \ldots, B_{t_{i+1}}) - \psi_{i+1}(B_{t_1}, \ldots, B_{t_{i+1}}) \right]. \end{aligned}$$

By induction assumption, $\lim_{n\to\infty} \psi_{i+1}^n = \psi_{i+1}$. Moreover, one can easily check that ψ_{i+1}^n is uniformly continuous in x_{i+1}, uniformly in n. Then by Lemma 3.5, we obtain

$$\overline{\lim_{n\to\infty}} \psi_i^n(B_{t_1}, \ldots, B_{t_i}) - \psi_i(B_{t_1}, \ldots, B_{t_i}) \leq 0.$$

Similarly, we can show that

$$\psi_i(B_{t_1}, \ldots, B_{t_i}) - \underline{\lim_{n\to\infty}} \psi_i^n(B_{t_1}, \ldots, B_{t_i}) \leq 0.$$

Thus, (3.13) holds for i. This completes the induction and hence proves that the limit in (3.11) for $\eta \in \mathcal{M}_G^0$.

We now consider general $\eta \in \mathcal{M}_G^1$. Let $\eta^m \in \mathcal{M}_G^0$ such that $\lim_{m\to\infty} \|\eta^m - \eta\|_{\mathcal{M}_G^1} = 0$. For each m, by previous arguments we see that

$\lim_{n\to\infty} \mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t^m : \mathrm{d}\langle B\rangle_t\right]$ exists. By (2.2), one can easily check that

$$
\begin{aligned}
&\left|\mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t^m : \mathrm{d}\langle B\rangle_t\right] - \mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t : \mathrm{d}\langle B\rangle_t\right]\right| \\
&\quad \le \mathbb{E}^G\left[\left|\int_0^T \delta_n(t)[\eta_t^m - \eta_t] : \mathrm{d}\langle B\rangle_t\right|\right] \le \mathbb{E}^G\left[\int_0^T \left|\eta_t^m - \eta_t\right|\left|\bar{\sigma}^2\right| \mathrm{d}t\right] \\
&\quad = |\bar{\sigma}^2| \|\eta^m - \eta\|_{\mathcal{M}_G^1} \to 0, \quad \text{as } m \to \infty.
\end{aligned}
$$

This clearly leads to the existence of $\lim_{n\to\infty} \mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t : \mathrm{d}\langle B\rangle_t\right]$. □

We now collect some basic properties of $\|\cdot\|_{\mathbb{M}_G}$. The left inequality of (3.15) is crucial for our purpose. We remark that the norm $\|\cdot\|_{\mathcal{M}^1_{G_\varepsilon}}$ was introduced by Hu and Peng [11] and a similar estimate was obtained by Song [26] by using different arguments. Recall the c_0 defined by (3.4).

THEOREM 3.7. $\|\cdot\|_{\mathbb{M}_G}$ defines a norm on $\mathcal{M}_G^1$, and for any $0 < \varepsilon \le c_0$, it holds that,

$$
\varepsilon \|\eta\|_{\mathcal{M}^1_{G_\varepsilon}} \le \|\eta\|_{\mathbb{M}_G} \le C_0 \|\eta\|_{\mathcal{M}^1_G}. \tag{3.15}
$$

To prove the theorem, we introduce some additional notations. Recall (3.5) and set

$$
\begin{aligned}
&\mathcal{A}_\varepsilon := \left\{\sigma \in \mathcal{A} : \underline{\sigma}_\varepsilon^2 \le \sigma^2 \le \bar{\sigma}_\varepsilon^2\right\}, \quad \mathcal{P}_\varepsilon := \{\mathbb{P}^\sigma : \sigma \in \mathcal{A}_\varepsilon\}, \\
&\mathcal{P}_0 := \lim_{\varepsilon\to 0} \mathcal{P}_\varepsilon = \bigcup_{0<\varepsilon\le c_0} \mathcal{P}_\varepsilon.
\end{aligned} \tag{3.16}
$$

That is, each element of $\mathcal{P}_0$ has a diffusion coefficient σ staying away uniformly from the boundaries $\underline{\sigma}$ and $\bar{\sigma}$. We remark that the following inclusions are strict:

$$
\begin{aligned}
&\mathcal{P}_0 \subset \{\mathbb{P}^\sigma : \sigma \in \mathcal{A}, \underline{\sigma} < \sigma < \bar{\sigma}\} \subset \mathcal{P}, \text{ but } \mathcal{P}_0 \\
&\qquad \subset \mathcal{P} \text{ is dense under the weak topology.}
\end{aligned} \tag{3.17}
$$

Proof.

(i) We first prove the estimates (3.15). Note that $\|\cdot\|_{\mathcal{M}^1_{G_\varepsilon}} \le \|\cdot\|_{\mathcal{M}^1_G}$. By using standard approximation arguments, it suffices to prove the statements for $\eta \in \mathcal{M}_G^0$. We now assume that η takes the form (3.12) and we shall use the notations in the proof of Theorem 3.6. In particular, by (3.13) we have $\|\eta\|_{\mathbb{M}_G} = \psi_0$. Define $\underline{\psi}_i^\varepsilon$ and $\bar{\psi}_i^\varepsilon$ by:

$$
\underline{\psi}_i^\varepsilon(B_{t_1}, \ldots, B_{t_i}) := \varepsilon \mathbb{E}_{t_i}^{G_\varepsilon}\left[\int_{t_i}^T |\eta_t| \, \mathrm{d}t\right], \quad \bar{\psi}_i^\varepsilon(B_{t_1}, \ldots, B_{t_i}) := C_0 \mathbb{E}_{t_i}^G\left[\int_{t_i}^T |\eta_t| \, \mathrm{d}t\right].
$$

Then $\underline{\psi}_m^\varepsilon = \bar{\psi}_m^\varepsilon = 0$, and

$$\underline{\psi}_i^\varepsilon(x_1, \ldots, x_i) = \mathbb{E}^{G_\varepsilon}\left[\underline{\psi}_{i+1}^\varepsilon\left(x_1, \ldots, x_i, x_i + B_{t_{i+1}}^{t_i}\right)\right] + \varepsilon|\varphi_i(x_1, \ldots, x_i)|(t_{i+1} - t_i);$$
$$\bar{\psi}_i^\varepsilon(x_1, \ldots, x_i) = \mathbb{E}^{G_\varepsilon}\left[\bar{\psi}_{i+1}^\varepsilon\left(x_1, \ldots, x_i, x_i + B_{t_{i+1}}^{t_i}\right)\right] + C_0|\varphi_i(x_1, \ldots, x_i)|(t_{i+1} - t_i).$$

Applying Lemma 3.2 (ii) and recalling (3.14), by induction one proves (3.15) immediately.

(ii) We now prove that $\|\cdot\|_{\mathbb{M}_G}$ defines a norm. Let $\eta \in \mathcal{M}_G^1$. First, by (3.15) we have $\|\eta\|_{\mathbb{M}_G} \geq 0$ and equality holds when $\eta = 0$, $\mathcal{P}$-q.s. On the other hand, assume $\|\eta\|_{\mathbb{M}_G} = 0$, then by the left inequality of (3.15) again we see that $\eta = 0$, $\mathcal{P}_0$-q.s. Now for any $\mathbb{P} \in \mathcal{P}$, by (3.17) there exists $\mathbb{P}_n \in \mathcal{P}_0$ such that $\mathbb{P}_n$ converges to $\mathbb{P}$ weakly. Since $\eta \in \mathcal{M}_G^1$, then $|\eta|$ is $\mathcal{P}$-q.s. continuous and it follows from [2] Lemma 27 that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\eta_t|\,\mathrm{d}t\right] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_n}\left[\int_0^T |\eta_t|\,\mathrm{d}t\right] = 0.$$

That is, $\eta = 0$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$. Therefore, $\|\eta\|_{\mathbb{M}_G} = 0$ if and only if $\eta = 0$, $\mathcal{P}$-q.s.

Next, for any $\lambda \in \mathbb{R}$, noting that $G^{-\alpha} = G^\alpha$ by Lemma 3.1 (ii), then (3.14) leads to

$$\|\lambda\eta\|_{\mathbb{M}_G} = \||\lambda|\eta\|_{\mathbb{M}_G} = \lim_{n\to\infty} \mathbb{E}^G\left[\int_0^T \delta_n(t)|\lambda|\eta_t : \mathrm{d}\langle B\rangle_t\right]$$
$$= \lim_{n\to\infty} |\lambda|\mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t : \mathrm{d}\langle B\rangle_t\right] = |\lambda|\|\eta\|_{\mathbb{M}_G}.$$

Finally, for any $\eta, \tilde{\eta} \in \mathcal{M}_G^0$, by the sublinearity of $\mathbb{E}^G$, we have

$$\mathbb{E}^G\left[\int_0^T \delta_n(t)[\eta_t + \tilde{\eta}_t] : \mathrm{d}\langle B\rangle_t\right] \leq \mathbb{E}^G\left[\int_0^T \delta_n(t)\eta_t : \mathrm{d}\langle B\rangle_t\right] + \mathbb{E}^G\left[\int_0^T \delta_n(t)\tilde{\eta}_t : \mathrm{d}\langle B\rangle_t\right].$$

Let $n \to \infty$ we obtain the triangle inequality: $\|\eta + \tilde{\eta}\|_{\mathbb{M}_G} \leq \|\eta\|_{\mathbb{M}_G} + \|\tilde{\eta}\|_{\mathbb{M}_G}$. That is, $\|\cdot\|_{\mathbb{M}_G}$ defines a norm on $\mathcal{M}_G^1$. □

3.4 *The new norm for* η

One drawback of the above norm $\|\cdot\|_{\mathbb{M}_G}$ is that we have to use different norms in the left and right sides of (3.15). Consequently, we are not able to prove the completeness of $\mathcal{M}_G^1$ under $\|\cdot\|_{\mathbb{M}_G}$. To be precise, given a Cauchy sequence $\eta^n \in \mathcal{M}_G^1$ under $\|\cdot\|_{\mathbb{M}_G}$, we are not able to obtain a process η such that $\lim_{n\to\infty}\|\eta^n - \eta\|_{\mathbb{M}_G} = 0$. For this reason, we shall modify $\|\cdot\|_{\mathbb{M}_G}$ slightly by using heavily the estimate (3.15). Set $\varepsilon_k := (1/(1+k))c_0$, $k \geq 1$ and define

$$\|\eta\|_{\mathbb{M}_G^*} := \sum_{k=1}^\infty 2^{-k}\|\eta\|_{\mathbb{M}_{G_{\varepsilon_k}}}, \quad \eta \in \mathcal{M}_G^1. \tag{3.18}$$

Then clearly $\|\cdot\|_{\mathbb{M}_G^*}$ defines a norm on $\mathcal{M}_G^1$, and we denote by $\mathbb{M}_G^*$ the closure of $\mathcal{M}_G^1$ under $\|\cdot\|_{\mathbb{M}_G^*}$. To understand the space $\mathbb{M}_G^*$, we note that $\mathcal{M}_{G_\varepsilon}^1$ is decreasing as $\varepsilon \to 0$. Set

$$\mathcal{M}_{G_0}^1 := \lim_{\varepsilon \to 0} \mathcal{M}_{G_\varepsilon}^1 = \bigcap_{0<\varepsilon\le c_0} \mathcal{M}_{G_\varepsilon}^1. \tag{3.19}$$

Remark 3.8.

(i) As mentioned earlier, elements in $\mathcal{M}_G^1$ (respectively $\mathcal{M}_{G_\varepsilon}^1$) are considered identical if they are equal $\mathcal{P}$-q.s. (respectively $\mathcal{P}_\varepsilon$-q.s.). Similarly, elements in $\mathbb{M}_G^*$ are considered identical if they are equal $\mathcal{P}_0$-q.s.

(ii) Obviously $\mathcal{M}_{G_\varepsilon}^1 \downarrow \mathcal{M}_{G_0}^1$ as $\varepsilon \downarrow 0$. Thus, the space $\mathcal{M}_{G_0}^1$ is independent of c_0.

(iii) By (3.15), it is obvious that

$$\mathcal{M}_G^1 \subset \mathbb{M}_G^* \subset \mathcal{M}_{G_0}^1. \tag{3.20}$$

Moreover, the above inclusions are strict. Indeed, consider the case $d = 1$ for simplicity. One may easily see that $\eta_t := \mathbf{1}_{\{\langle B\rangle_t = \underline{\sigma}^2\}}$ is in $\mathbb{M}_G^* \backslash \mathcal{M}_G^1$, and

$$\eta_t := \sum_{n=1}^{\infty} 2^n \varphi_n\left(\frac{\langle B\rangle_t - \underline{\sigma}^2 t}{(\bar{\sigma}^2 - \underline{\sigma}^2)t}\right)$$

is in $\mathcal{M}_{G_0}^1 \backslash \mathbb{M}_G^*$, where φ_n is the linear interpolation such that $\varphi_n(\gamma) = 0$ when $\gamma \le (1/(n+1))$ or $\gamma \ge (1/n)$ and $\varphi_n(\gamma) = 1$ when $\gamma = 1/2[(1/n) + (1/(n+1))]$.

We now have.

THEOREM 3.9. The space $\mathbb{M}_G^*$ is complete under the norm $\|\cdot\|_{\mathbb{M}_G^*}$.

Proof. First, it is clear that $\|\cdot\|_{\mathbb{M}_G^*}$ is a semi-norm on $\mathbb{M}_G^*$. Now assume $\eta \in \mathbb{M}_G^*$ such that $\|\eta\|_{\mathbb{M}_G^*} = 0$. By (3.15), $\|\eta\|_{\mathcal{M}_{G_\varepsilon}^1} = 0$ for all $\varepsilon \le c_0$. Then $\eta = 0$, $\mathcal{P}_\varepsilon$-q.s. for all $0 < \varepsilon \le c_0$ and thus $\eta = 0$, $\mathcal{P}_0$-q.s. That is, $\|\cdot\|_{\mathbb{M}_G^*}$ is a norm on $\mathbb{M}_G^*$ (again, in the $\mathcal{P}_0$-q.s. sense).

It remains to prove the completeness of the space. Let $\eta^n \in \mathbb{M}_G^*$ be a Cauchy sequence under $\|\cdot\|_{\mathbb{M}_G^*}$. For any $0 < \varepsilon \le c_0$, there exists k large enough such that $\varepsilon_k < \varepsilon$. By the left inequality of (3.15) we see that

$$\|\eta^n - \eta^m\|_{\mathcal{M}_{G_\varepsilon}^1} \le C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta^m\|_{\mathbb{M}_{G_{\varepsilon_k}}} \le 2^k C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta^m\|_{\mathbb{M}_G^*} \to 0, \quad \text{as } n, m \to \infty.$$

Since $(\mathcal{M}_{G_\varepsilon}^1, \|\cdot\|_{\mathcal{M}_{G_\varepsilon}^1})$ is complete, there exists unique (in $\mathcal{P}_\varepsilon$-q.s. sense) $\eta^{(\varepsilon)} \in \mathcal{M}_{G_\varepsilon}^1$ such that $\lim_{n\to\infty}\|\eta^n - \eta^{(\varepsilon)}\|_{\mathcal{M}_{G_\varepsilon}^1} = 0$. By the uniqueness, clearly $\eta^{(\varepsilon)} = \eta^{(\tilde{\varepsilon})}$, $\mathcal{P}_\varepsilon$-q.s. for any $0 < \tilde{\varepsilon} < \varepsilon \le c_0$. Thus, there exists $\eta \in \mathcal{M}_{G_0}^1$ such that $\eta^{(\varepsilon)} = \eta$, $\mathcal{P}_\varepsilon$-q.s. for all $0 < \varepsilon \le c_0$.

We now show that

$$\lim_{n\to\infty}\|\eta^n - \eta\|_{\mathbb{M}_G^*} = 0. \tag{3.21}$$

Indeed, for any $\delta > 0$, there exists N_δ such that

$$\|\eta^n - \eta^m\|_{\mathbb{M}_G^*} \le \delta, \quad \text{for all } n, m \ge N_\delta.$$

Note that, by the right inequality of (3.15),

$$\|\eta^m - \eta\|_{\mathbb{M}_{G_\varepsilon}} \leq C_0 \|\eta^m - \eta^{(\varepsilon)}\|_{\mathcal{M}^1_{G_\varepsilon}} \to 0, \quad \text{as } m \to \infty.$$

Then for any $n \geq N_\delta$ and $K \geq 1$,

$$\sum_{k=1}^{K} 2^{-k} \|\eta^n - \eta\|_{\mathbb{M}_{G_{\varepsilon_k}}} = \lim_{m\to\infty} \sum_{k=0}^{K} 2^{-k} \|\eta^n - \eta^m\|_{\mathbb{M}_{G_{\varepsilon_k}}} \leq \varliminf_{m\to\infty} \|\eta^n - \eta^m\|_{\mathbb{M}^*_G} \leq \delta.$$

Send $K \to \infty$ we obtain $\|\eta^n - \eta\|_{\mathbb{M}^*_G} \leq \delta$ for all $n \geq N_\delta$. This proves (3.21), and hence the theorem. □

4. The G-martingale representation theorem

We first note that, assuming (Y^i, Z^i, η^i), $i = 1, 2$, satisfy (2.19), then

$$\mathrm{d}\big(Y^1_t - Y^2_t\big) = \big(Z^1_t - Z^2_t\big)\cdot \mathrm{d}B_t - \big[G\big(\eta^1_t\big) - G\big(\eta^2_t\big)\big]\,\mathrm{d}t + \frac{1}{2}\big[\eta^1_t - \eta^2_t\big] : \mathrm{d}\langle B\rangle_t. \tag{4.1}$$

By Lemma 3.3, we have

$$\|\eta^1 - \eta^2\|_{\mathbb{M}_{G_\varepsilon}} = 2 \lim_{n\to\infty} \mathbb{E}^{G_\varepsilon}\left[\int_0^T \delta_n(t)\,\mathrm{d}(Y^1_t - Y^2_t)\right], \quad \text{for all } 0 < \varepsilon \leq c_0. \tag{4.2}$$

In light of (3.18), for any $p > 1$ we define:

$$\mathrm{d}_{G,p}(\xi_1, \xi_2) := \|Y^1 - Y^2\|_{\mathbb{D}^p_G} + \sum_{k=1}^{\infty} 2^{-k} \lim_{n\to\infty} \mathbb{E}^{G_{\varepsilon_k}}\left[\int_0^T \delta_n(t)\,\mathrm{d}\big(Y^1_t - Y^2_t\big)\right], \tag{4.3}$$

where $\xi_i \in \mathcal{L}_{ip}$ and $Y^i_t := \mathbb{E}^G_t[\xi_i]$, $i = 1, 2$.

Then clearly $d_{G,p}$ is a metric on $\mathcal{L}_{ip}$, and we let $\mathbb{L}^{*p}_G \subset \mathcal{L}^p_G$ denote the closure of $\mathcal{L}_{ip}$ under $d_{G,p}$. We remark that

$$\|\xi_1 - \xi_2\|_{\mathcal{L}^p_G} \leq \|Y^1 - Y^2\|_{\mathbb{D}^p_G} \leq \|\xi_1 - \xi_2\|_{\mathbb{L}^p_G}.$$

Remark 4.1. We allow the metric $d_{G,p}(\xi_1, \xi_2)$ to depend on Y^i, but not on (Z^i, η^i) explicitly. The component Y has a representation, namely as the conditional G-expectation of ξ, but in general we do not have a desirable representation for Z or η. Thus, it is relatively easier to verify conditions imposed on Y than those on Z or η. See also [21] for a similar idea.

Given (Z, η) and y, let $Y^{y,Z,\eta}$ denote the G-martingale defined by (2.19) with initial value $Y_0 = y$. We first have

Lemma 4.2. For any $p > 1$, $y \in \mathbb{R}$ and $(Z, \eta) \in \mathcal{H}^p_G \times \mathcal{M}^p_G$, we have $Y^{y,Z,\eta}_T \in \mathbb{L}^{*p}_G$. Moreover, for any such (y_i, Z^i, η^i), $i = 1, 2$, it holds that

$$\mathrm{d}_{G,p}(Y^{y_1,Z^1,\eta^1}_T, Y^{y_2,Z^2,\eta^2}_T) \leq C_p\Big[|y_1 - y_2| + \|Z^1 - Z^2\|_{\mathcal{H}^p_G} + |\eta^1 - \eta^2\|_{\mathcal{M}^p_G}\Big]. \tag{4.4}$$

Proof. We first prove the *a priori* estimate (4.4). Denote $Y^i := Y^{y_i,Z^i,\eta^i}$, $i = 1,2$; $\Delta Y := Y^1 - Y^2$ and similarly for other notations. By (4.1), it is obvious that

$$\|\Delta Y\|_{\mathbb{D}_G^p} \le C_p\Big[|\Delta y| + \|\Delta Z\|_{\mathcal{H}_G^p} + \|\Delta\eta\|_{\mathcal{M}_G^p}\Big]. \tag{4.5}$$

Moreover, by (4.2) and the right inequality of (3.15), we have

$$\lim_{n\to\infty}\mathbb{E}^{G_{\varepsilon_k}}\left[\int_0^T \delta_n(t)\,\mathrm{d}(\Delta Y_t)\right] = \frac{1}{2}\|\Delta\eta\|_{\mathbb{M}_{G_{\varepsilon_k}}} \le C\|\Delta\eta\|_{\mathcal{M}^1_{G_{\varepsilon_k}}} \le C\|\Delta\eta\|_{\mathcal{M}_G^1} \le C\|\Delta\eta\|_{\mathcal{M}_G^p}.$$

Then,

$$\sum_{k=1}^{\infty}2^{-k}\lim_{n\to\infty}\mathbb{E}^{G_{\varepsilon_k}}\left[\int_0^T \delta_n(t)\,\mathrm{d}(\Delta Y_t)\right] \le C\sum_{k=1}^{\infty}2^{-k}\|\Delta\eta\|_{\mathcal{M}_G^p} = C\|\Delta\eta\|_{\mathcal{M}_G^p}.$$

This, together with (4.5), implies (4.4).

We now show that $Y_T := Y_T^{y,Z,\eta} \in \mathbb{L}_G^{*p}$ in two steps.

Step 1. Assume $\eta = 0$. By (4.4) and the definition of $\mathcal{H}_G^p$, we may assume without loss of generality that $Z = \sum_{i=0}^{n-1} Z_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \in \mathcal{H}_G^0$. Then

$$Y_T = Y_0 + \sum_{i=0}^{n-1} Z_{t_i}B^{t_i}_{t_{i+1}} \in \mathcal{L}_{ip} \subset \mathbb{L}_G^{*p}.$$

Step 2. For the general case, by (4.4) and the definition of $\mathcal{M}_G^p$, we may assume without loss of generality that $\eta = \sum_{i=0}^{n-1}\eta_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \in \mathcal{M}_G^0$. Then

$$Y_T = Y_0 + \int_0^T Z_t\cdot\mathrm{d}B_t - \sum_{i=0}^{n-1}\left[G(\eta_{t_i})[t_{i+1}-t_i] - \frac{1}{2}\eta_{t_i} : \left[\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right]\right].$$

For each i, applying Itô's formula we have

$$\mathrm{d}\big(B_t^{t_i}(B_t^{t_i})^T\big) = 2B_t^{t_i}\,\mathrm{d}(B_t^{t_i})^T + \mathrm{d}\langle B^{t_i}\rangle_t = 2B_t^{t_i}\,\mathrm{d}B_t^T + \mathrm{d}\langle B\rangle_t, \quad t\in[t_i,t_{i+1}].$$

Then

$$\eta_{t_i} : \left[\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right] = \eta_{t_i} : \left[B^{t_i}_{t_{i+1}}\left(B^{t_i}_{t_{i+1}}\right)^T\right] - 2\int_{t_i}^{t_{i+1}}\left(\eta_{t_i}B_t^{t_i}\right)\cdot\mathrm{d}B_t.$$

Thus, denoting $\tilde{Z}_t := Z_t - \sum_{i=0}^{n-1}\eta_{t_i}B_t^{t_i}\mathbf{1}_{[t_i,t_{i+1})}(t)$,

$$Y_T = Y_0 + \int_0^T \tilde{Z}_t\cdot\mathrm{d}B_t - \sum_{i=0}^{n-1}\left[G(\eta_{t_i})[t_{i+1}-t_i] - \frac{1}{2}\eta_{t_i} : [B^{t_i}_{t_{i+1}}(B^{t_i}_{t_{i+1}})^T]\right]. \tag{4.6}$$

One can easily check that $\tilde{Z}\in\mathcal{H}_G^p$. Then by Step 1, $\int_0^T\tilde{Z}_t\cdot\mathrm{d}B_t \in \mathbb{L}_G^{*p}$. Moreover, it is obvious that

$$\sum_{i=0}^{n-1}\left[G(\eta_{t_i})[t_{i+1}-t_i] - \frac{1}{2}\eta_{t_i} : \left[B^{t_i}_{t_{i+1}}(B^{t_i}_{t_{i+1}})^T\right]\right] \in \mathcal{L}_{ip}.$$

Then it follows from (4.6) that $Y_T \in \mathbb{L}_G^{*p}$. □

Our main result of the paper is the following representation theorem, which is in the opposite direction of Lemma 4.2.

THEOREM 4.3. Let $p > 1$.

(i) For any $\xi \in \mathbb{L}_G^{*p}$ and denoting $Y_t := \mathbb{E}_t^G[\xi]$, there exist unique $Z \in \mathcal{H}_G^p$ and $\eta \in \mathbb{M}_G^*$ such that (2.19) holds $\mathcal{P}_0$-q.s. Moreover, there exists a constant $C_p > 0$ such that

$$\|Y\|_{\mathbb{D}_G^p} + \|Z\|_{\mathcal{H}_G^p} + \|\eta\|_{\mathbb{M}_G^*} \le C_p d_{G,p}(\xi, 0). \tag{4.7}$$

(ii) For any $\xi_1, \xi_2 \in \mathbb{L}_G^{*p}$, let (Y^i, Z^i, η^i) denote the corresponding terms. Then

$$\begin{aligned} &\|Y^1 - Y^2\|_{\mathbb{D}_G^p} + \|\eta^1 - \eta^2\|_{\mathbb{M}_G^*} \le C_p d_{G,p}(\xi_1, \xi_2), \\ &\|Z^1 - Z^2\|_{\mathcal{H}_G^p} \le C_p (d_{G,p}(\xi_1, \xi_2))^{1/2}. \end{aligned} \tag{4.8}$$

Remark 4.4. We can only prove the representation (2.19) in $\mathcal{P}_0$-q.s. sense. This is mainly because we are not able to prove the equivalence of $\|\cdot\|_{\mathbb{M}_G}$ and $\|\cdot\|_{\mathcal{M}_G^1}$ in Theorem 3.7. See also Remark 3.8 (iii). If the above η happens to fall in the space $\mathcal{M}_G^1$ (which is a strict subset of $\mathbb{M}_G^*$), then both sides of (2.19) will lie in $\mathcal{L}_G^1$, and thus, the representation (2.19) will hold $\mathcal{P}$-q.s. However, we are not able to provide natural sufficient conditions (in terms of ξ and/or Y) for this. It is still an open problem to establish the representation (2.19) in $\mathcal{P}$-q.s. sense and we shall leave it for future research.

Proof. We proceed in two steps.

Step 1. We first prove *a priori* estimates (4.7) and (4.8) by assuming (Y, Z, η) and (Y^i, Z^i, η^i), $i = 1, 2$, are in $\mathbb{D}_G^p \times \mathcal{H}_G^p \times \mathbb{M}_G^*$ and satisfy (2.19) $\mathcal{P}_0$-q.s. Indeed, by (4.2) and (4.3) it is clear that

$$\|Y\|_{\mathbb{D}_G^p} + \|\eta\|_{\mathbb{M}_G^*} \le C_p d_{G,p}(\xi, 0), \quad \|Y^1 - Y^2\|_{\mathbb{D}_G^p} + \|\eta^1 - \eta^2\|_{\mathbb{M}_G^*} \le C d_{G,p}(\xi_1, \xi_2).$$

Moreover, combining the arguments in [22] and [9], or following the arguments in [25], one can easily prove

$$\|Z\|_{\mathcal{H}_G^p} \le C_p \|Y\|_{\mathbb{D}_G^p}, \quad \|Z^1 - Z^2\|_{\mathcal{H}_G^p} \le C_p \left(\|Y^1 - Y^2\|_{\mathbb{D}_G^p}\right)^{1/2}.$$

Then (4.7) and (4.8) hold.

Step 2. We next prove the existence of (Z, η). For any $\xi \in \mathbb{L}_G^{*p}$, by definition there exist $\xi_n \in \mathcal{L}_{ip}$ such that $\lim_{n\to\infty} \rho_G^p(\xi_n, \xi) = 0$. Let (Y^n, Z^n, η^n) correspond to ξ_n. As $n, m \to \infty$, by (4.8) we have

$$\begin{aligned} &\|Y^n - Y^m\|_{\mathbb{D}_G^p} + \|\eta^n - \eta^m\|_{\mathbb{M}_G^*} + \|Z^n - Z^m\|_{\mathcal{H}_G^p} \\ &\le C_p \left[d_{G,p}(\xi_n, \xi_m) + \left(d_{G,p}(\xi_n, \xi_m)\right)^{1/2} \right] \to 0. \end{aligned}$$

Then there exist $(Y, Z, \eta) \in \mathbb{D}_G^p \times \mathcal{H}_G^p \times \mathbb{M}_G^*$ such that

$$\|Y^n - Y\|_{\mathbb{D}_G^p} + \|\eta^n - \eta\|_{\mathbb{M}_G^*} + \|Z^n - Z\|_{\mathcal{H}_G^p} \to 0, \quad \text{as } n \to \infty.$$

Moreover, for any $0 < \varepsilon \le c_0$, choose k large enough so that $\varepsilon_k < \varepsilon$. Then

$$\|\eta^n - \eta\|_{\mathcal{M}^1_{G_\varepsilon}} \le C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta\|_{\mathbb{M}^1_{G_{\varepsilon_k}}} \le 2^k C_{\varepsilon,\varepsilon_k}\|\eta^n - \eta\|_{\mathbb{M}^*_G} \to 0, \quad \text{as } n \to \infty.$$

Thus,

$$\int_0^t G(\eta^n_s)\,\mathrm{d}s \to \int_0^t G(\eta_s)\,\mathrm{d}s, \quad \mathcal{P}_\varepsilon\text{-q.s.}$$

Since (Y^n, Z^n, η^n) satisfy (2.19) $\mathcal{P}_\varepsilon$-q.s., then it is clear that (Y, Z, η) also satisfy (2.19) $\mathcal{P}_\varepsilon$-q.s. By the arbitrariness of ε, we see that (Y, Z, η) also satisfy (2.19) $\mathcal{P}_0$-q.s.

Finally, the uniqueness of $(Z, \eta) \in \mathcal{H}^p_G \times \mathbb{M}^*_G$ follows from (4.8). □

We conclude this paper by providing a nontrivial example of ξ which has the representation, but is not in $\mathcal{L}_{ip}$.

Example 4.5. Let $d = 1$ and $B^*_t := \sup_{0\le s\le t} B_s$. Then $B^*_T \in \mathbb{L}^{*p}_G \backslash \mathcal{L}_{ip}$ for any $p > 1$.

Proof. It is clear that $B^*_T \notin \mathcal{L}_{ip}$. We prove $B^*_T \in \mathbb{L}^{*p}_G$ in several steps.

Step 1. Assume $\xi : \Omega \to \mathbb{R}$ is uniformly Lipschitz continuous and convex in ω. We show that $\mathbb{E}^G[\xi] = \mathbb{E}^{\bar{\mathbb{P}}}[\xi]$, where $\bar{\mathbb{P}} := \mathbb{P}^{\bar{\sigma}}$.

Indeed, for any n, denote $t^n_i := iT/n$, $i = 0, \ldots, n$, $x_0 := 0$, and define

$$\begin{aligned} g_n(x_1, \ldots, x_n) :&= \xi\left(\sum_{i=1}^n \frac{1}{t^n_i - t^n_{i-1}}\left[x_{i-1}\left(t^n_i - t\right) + x_i\left(t - t^n_{i-1}\right)\right]\mathbf{1}_{(t^n_{i-1}, t^n_i]}(t)\right); \\ \xi_n :&= g_n\left(B_{t^n_1}, \ldots, B_{t^n_n}\right). \end{aligned}$$

Since ξ is convex, clearly g_n is convex. Then $\mathbb{E}^G[\xi_n] = \mathbb{E}^{\bar{\mathbb{P}}}[\xi_n]$. Since ξ is uniformly Lipschitz continuous, then $|\xi_n - \xi| \le C\max_{1\le i\le n}\sup_{t^n_{i-1}\le t\le t^n_i}|B_t - B_{t^n_i}|$. This implies that $\mathbb{E}^G[|\xi_n - \xi|] \to 0$ and $\mathbb{E}^{\bar{\mathbb{P}}}[|\xi_n - \xi|] \to 0$ as $n \to \infty$, and therefore, $\mathbb{E}^G[\xi] = \mathbb{E}^{\bar{\mathbb{P}}}[\xi]$.

Step 2. For simplicity, we assume that $\bar{\sigma} = 1$, and thus, $\bar{\mathbb{P}} = \mathbb{P}_0$. Note that $\xi := B^*_T$ is uniformly Lipschitz continuous and convex in ω. Then by adapting Step 1 to conditional G-expectations, we have

$$Y_t := \mathbb{E}^G_t[\xi] = \mathbb{E}^{\mathbb{P}_0}_t[B^*_T] = u(t, B_t, B^*_t),$$

where, for $x \le y$,

$$u(t, x, y) := \mathbb{E}^{\mathbb{P}_0}\left[y \vee \left[x + \sup_{t\le s\le T} B^t_s\right]\right] = \mathbb{E}^{\mathbb{P}_0}\left[y \vee [x + B^*_{T-t}]\right].$$

Note that, under $\mathbb{P}_0$, B^*_{T-t} has the same distribution as $|B_{T-t}|$. Then?

$$\begin{aligned} u(t, x, y) &= \sqrt{\frac{2}{\pi}}\int_0^\infty y \vee (x + \sqrt{T-t}z)\mathrm{e}^{-(z^2/2)}\,\mathrm{d}z \\ &= \sqrt{\frac{2}{\pi}}\int_0^{(y-x)/(\sqrt{T-t})} y\,\mathrm{e}^{-(z^2/2)}\,\mathrm{d}z + \sqrt{\frac{2}{\pi}}\int_{(y-x)/(\sqrt{T-t})}^\infty (x + \sqrt{T-t}z)\,\mathrm{e}^{-(z^2/2)}\,\mathrm{d}z. \end{aligned}$$

For $t \in [0, T)$ and $x < y$, we have

$$\partial_t u(t,x,y) = -\frac{1}{\sqrt{2\pi(T-t)}} e^{-((y-x)^2/2(T-t))}; \quad \partial_y u(t,x,y) = \sqrt{\frac{2}{\pi}} \int_0^{(y-x)/(\sqrt{T-t})} e^{-(z^2/2)}\,dz;$$
$$\partial_x u(t,x,y) = \sqrt{\frac{2}{\pi}} \int_{(y-x)/(\sqrt{T-t})}^{\infty} e^{-(z^2/2)}\,dz; \quad \partial_{xx} u(t,x,y) = \sqrt{\frac{2}{\pi(T-t)}} e^{-((y-x)^2/2(T-t))} > 0. \tag{4.9}$$

Then

$$\partial_t u + \frac{1}{2} G(\partial_{xx} u) = \partial_t u + \frac{1}{2} \partial_{xx} u = 0, \quad \text{and} \quad \partial_y u(t,y,y) = 0.$$

Note that dB_t^* has support on $\{t : B_t^* = B_t\}$. Then by Itô's formula, we have?

$$\begin{aligned} dY_t &= du(t, B_t, B_t^*) \\ &= \partial_t u(t, B_t, B_t^*)\,dt + \partial_x u(t, B_t, B_t^*)\,dB_t + \partial_y u(t, B_t, B_t^*)\,dB_t^* + \frac{1}{2}\partial_{xx} u(t, B_t, B_t^*)\,d\langle B \rangle_t \\ &= \partial_x u(t, B_t, B_t^*)\,dB_t - G(\partial_{xx} u(t, B_t, B_t^*))\,dt + \frac{1}{2}\partial_{xx} u(t, B_t, B_t^*)\,d\langle B \rangle_t. \end{aligned}$$

Thus, we obtain the representation with

$$Z_t = \partial_x u(t, B_t, B_t^*), \quad \eta_t = \partial_{xx} u(t, B_t, B_t^*). \tag{4.10}$$

Step 3. By Lemma 4.2, it remains to show that $(Z, \eta) \in \mathcal{H}_G^p \times \mathcal{M}_G^p$. For any n, denote

$$Z_t^n := Z_t \mathbf{1}_{[0, T-(1/n)]}, \quad \eta_t^n := \eta_t \mathbf{1}_{[0, T-(1/n)]}.$$

Note that, in the interval $[0, T - (1/n)]$, $\partial_x u$ and $\partial_{xx} u$ are bounded and uniformly Lipschitz continuous in (t, x, y), then clearly $(Z^n, \eta^n) \in \mathcal{H}_G^p \times \mathcal{M}_G^p$. Moreover, by (4.9) we have $|\partial_x u(t,x,y)| \le 1$ and $|\partial_{xx} u(t,x,y)| \le (C/\sqrt{T-t})$. Then, as $n \to \infty$,

$$\begin{aligned} &\mathbb{E}^G\left[\left(\int_0^T |Z_t - Z_t^n|^2\,d\langle B \rangle_t\right)^{p/2}\right] = \mathbb{E}^G\left[\left(\int_{T-(1/n)}^T |Z_t|^2\,d\langle B \rangle_t\right)^{p/2}\right] \\ &\le \mathbb{E}^G\left[\left(\langle B \rangle_T - \langle B \rangle_{T-(1/n)}\right)^{p/2}\right] = \frac{C_p}{n^{p/2}} \to 0; \\ &\mathbb{E}^G\left[\left(\int_0^T |\eta_t - \eta_t^n|\,dt\right)^p\right] = \mathbb{E}^G\left[\left(\int_{T-(1/n)}^T |\eta_t|\,dt\right)^p\right] \\ &\le C\mathbb{E}^G\left[\left(\int_{T-(1/n)}^T \frac{dt}{\sqrt{T-t}}\right)^p\right] = \frac{C_p}{n^{p/2}} \to 0; \end{aligned}$$

This proves that $(Z, \eta) \in \mathcal{H}_G^p \times \mathcal{M}_G^p$ and completes the proof. □

Remark 4.6. From the proof above, we see there exist $(Z, \eta) \in \mathcal{H}_G^p \times \mathcal{M}_G^p$ such that (2.19) holds for B_T^*. By Remark 4.4 we conclude that the representation (2.19) holds $\mathcal{P}$-q.s.

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