ON VISCOSITY SOLUTIONS OF PATH DEPENDENT PDES

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In this paper we propose a notion of viscosity solutions for path dependent semi-linear parabolic PDEs. This can also be viewed as viscosity solutions of non-Markovian backward SDEs, and thus extends the well-known nonlinear Feynman–Kac formula to non-Markovian case. We shall prove the existence, uniqueness, stability and comparison principle for the viscosity solutions. The key ingredient of our approach is a functional Itô calculus recently introduced by Dupire [Functional Itô calculus (2009) Preprint].

1. Introduction. It is well known that a Markovian type backward SDE (BSDE, for short) is associated with a semi-linear parabolic PDE via the so called nonlinear Feynman–Kac formula; see Pardoux and Peng [19]. Such relation was extended to forward–backward SDEs (FBSDE, for short) and quasilinear PDEs; see, for example, Ma, Protter and Yong [17], Pardoux and Tang [21] and Ma, Zhang and Zheng [18], and second order BSDEs (2BSDEs, for short) and fully nonlinear PDEs; see, for example, Cheridito et al. [3] and Soner, Touzi and Zhang [29]. The notable notion *G*-expectation, proposed by Peng [24], was also motivated from connection with fully nonlinear PDEs.

In non-Markovian case, the BSDEs (and FBSDEs, 2BSDEs) become path dependent. Due to its connection with PDE in Markovian case, it has long been discussed that general BSDEs can also be viewed as a PDE. In particular, in his ICM 2010 lecture, Peng [25] proposed the question whether or not a non-Markovian BSDE can be viewed as a path-dependent PDE (PPDE, for short).

The recent work Dupire [6], which was further extended by Cont and Fournie [4], provides a convenient framework for this problem. Dupire introduces the notion of horizontal derivative (that we will refer to as time derivative) and vertical derivative (that we will refer to as space derivative) for nonanticipative stochastic

processes. One remarkable result is the functional Itô formula under his definition. As a direct consequence, if $u(t, \omega)$ is a martingale under the Wiener measure with enough regularity (under their sense), then its drift part from the Itô formula vanishes, and thus it is a classical solution to the following path-dependent heat equation:

$$(1.1) \qquad \partial_t u(t, \omega) + \tfrac{1}{2}\, \partial^2_{\omega\omega} u(t, \omega) = 0.$$

It is then very natural to view BSDEs as semi-linear PPDEs, and 2BSDEs and G-martingales as fully nonlinear PPDEs. However, we shall emphasize that PPDEs can rarely have classical solutions, even for heat equations. We refer to Peng and Wang [27] for some sufficient conditions under which a semi-linear PPDE admits a classical solution.

The present work was largely stimulated by Peng's recent paper [26], which appeared while our investigation of the problem was in an early stage. Peng proposes a notion of viscosity solutions for PPDEs on càdlàg paths using compactness arguments. However, the horizontal derivative (or time derivative) in [26] is defined differently from that in Dupire [6] which leads to a different context than ours. Moreover, Peng [26] derives a uniqueness result for PPDEs on càdlàg paths. Given the nonuniqueness of extension of a function to the càdlàg paths, this does not imply any uniqueness statement in the space of continuous paths. For this reason, our approach uses an alternative definition than that of Peng [26].

The main objective of this paper is to propose a notion of viscosity solutions of PPDEs on the space of continuous paths. To focus on the main idea, we focus on the semi-linear case and leave the fully nonlinear case for future study. We shall prove existence, uniqueness, stability, and comparison principle for viscosity solutions.

The theory of viscosity solutions for standard PDEs has been well developed. We refer to the classical references Crandall, Ishii and Lions [5] and Fleming and Soner [11]. As is well understood, in path-dependent case the main challenge comes from the fact that the space variable is infinite dimensional and thus lacks compactness. Our context does not fall into the framework of Lions [13–15] where the notion of viscosity solutions is extended to Hilbert spaces by using a limiting argument based on the existence of a countable basis. Consequently, the standard techniques for the comparison principle, which rely heavily on the compactness arguments, fail in our context. We shall remark though, for first order PPDEs, by using its special structure Lukoyanov [16] studied viscosity solutions by adapting elegantly the compactness arguments.

To overcome this difficulty, we provide a new approach by decomposing the proof of the comparison principle into two steps. We first prove a partial comparison principle, that is, a classical sub-solution (resp., viscosity sub-solution) is always less than or equal to a viscosity super-solution (resp., classical super-solution). The main idea is to use the classical one to construct a test function for the viscosity one and then obtain a contradiction.

Our second step is a variation of the Perron's method. Let $\underline{u}$ and $\bar{u}$ denote the supremum of classical sub-solutions and the infimum of classical super-solutions, respectively, with the same terminal condition. In standard Perron's approach (see, e.g., Ishii [12] and an interesting recent development by Bayraktar and Sirbu [2]), one shows that

$$(1.2) \qquad\qquad\qquad \underline{u} = \bar{u}$$

by assuming the comparison principle for viscosity solutions, which further implies the existence of viscosity solutions. We shall instead prove (1.2) directly, which, together with our partial comparison principle, implies the comparison principle for viscosity solutions immediately. Our arguments for (1.2) mainly rely on the remarkable result Bank and Baum [1], which was extended to nonlinear case in [28].

We also observe that our results make strong use of the representation of the solution of the semilinear PPDE by means of the corresponding backward SDEs [20]. This is a serious limitation of our approach that we hope to overcome in some future work. However, our approach is suitable for a large class of PPDEs as Hamilton–Jacobi–Bellman equations, which are related to stochastic control problems, and their extension to Hamilton–Jacobi–Bellman–Isaacs equations corresponding to differential games.

The rest of the paper is organized as follows. In Section 2 we introduce the framework of [6] and [4] and adapt it to our problem. We define classical and viscosity solutions of PPDE in Section 3. In Section 4 we introduce the main results, and in Section 5 we prove some basic properties of the solutions, including existence, stability and the partial comparison principle of viscosity solutions. Finally in Section 6 we prove (1.2) and the comparison principle for viscosity solutions.

2. A pathwise stochastic analysis. In this section we introduce the spaces on which we will define the solutions of path dependent PDEs. The key notions of derivatives were proposed by Dupire [6] who introduced the functional Itô calculus, and further developed by Cont and Fournie [4]. We shall also introduce their localization version for our purpose.

2.1. *Derivatives on càdlàg paths.* Let $\hat{\Omega} := \mathbb{D}([0, T], \mathbb{R}^d)$, the set of càdlàg paths, $\hat{\omega}$ denote the elements of $\hat{\Omega}$, $\hat{B}$ the canonical process, $\hat{\mathbb{F}}$ the filtration generated by $\hat{B}$ and $\hat{\Lambda} := [0, T] \times \hat{\Omega}$. We define seminorms on $\hat{\Omega}$ and a pseudometric on $\hat{\Lambda}$ as follows: for any $(t, \hat{\omega}), (t', \hat{\omega}') \in \hat{\Lambda}$,

$$\|\hat{\omega}\|_t := \sup_{0 \le s \le t} |\hat{\omega}_s|,$$

$$(2.1)$$

$$d_\infty((t, \hat{\omega}), (t', \hat{\omega}')) := |t - t'| + \sup_{0 \le s \le T} |\hat{\omega}_{t \wedge s} - \hat{\omega}'_{t' \wedge s}|.$$

Then $(\hat{\Omega}, \| \cdot \|_T)$ is a Banach space and $(\hat{\Lambda}, d_\infty)$ is a complete pseudometric space.

Let $\hat{u} : \hat{\Lambda} \to \mathbb{R}$ be an $\hat{\mathbb{F}}$-progressively measurable random field. Note that the progressive measurability implies that $\hat{u}(t, \hat{\omega}) = \hat{u}(t, \hat{\omega}_{\cdot \wedge t})$ for all $(t, \hat{\omega}) \in \hat{\Lambda}$. Following Dupire [6], we define spatial derivatives of $\hat{u}$, if they exist, in the standard sense: for the basis e_i of $\mathbb{R}^d$, $i = 1, \ldots, d$,

(2.2)
$$\partial_{\omega_i} \hat{u}(t, \hat{\omega}) := \lim_{h \to 0} \frac{1}{h} \big[\hat{u}(t, \hat{\omega} + h \mathbf{1}_{[t,T]} e_i) - \hat{u}(t, \hat{\omega}) \big],$$

$$\partial_{\omega_i \omega_j} \hat{u} := \partial_{\omega_i} (\hat{u}_{\omega_j}), \qquad i, j = 1, \ldots, d,$$

and the right time-derivative of $\hat{u}$, if it exists, as

(2.3) $$\partial_t \hat{u}(t, \hat{\omega}) := \lim_{h \to 0, h > 0} \frac{1}{h} \big[\hat{u}(t + h, \hat{\omega}_{\cdot \wedge t}) - \hat{u}(t, \hat{\omega}) \big], \qquad t < T.$$

For the final time T, we define

(2.4) $$\partial_t \hat{u}(T, \omega) := \lim_{t < T, t \uparrow T} \partial_t \hat{u}(t, \omega).$$

We take the convention that $\hat{\omega}$ are column vectors, but $\partial_\omega \hat{u}$ denotes row vectors, and $\partial^2_{\omega\omega} \hat{u}$ denote $d \times d$-matrices.

DEFINITION 2.1. Let $\hat{u} : \hat{\Lambda} \to \mathbb{R}$ be $\hat{\mathbb{F}}$-progressively measurable.

(i) We say $\hat{u} \in C^0(\hat{\Lambda})$ if $\hat{u}$ is continuous in $(t, \hat{\omega})$ under d_∞.
(ii) We say $\hat{u} \in C_b^0(\hat{\Lambda}) \subset C^0(\hat{\Lambda})$ if $\hat{u}$ is bounded.
(iii) We say $\hat{u} \in C_b^{1,2}(\hat{\Lambda}) \subset C^0(\hat{\Lambda})$ if $\partial_t \hat{u}$, $\partial_\omega \hat{u}$, and $\partial^2_{\omega\omega} \hat{u}$ exist and are in $C_b^0(\hat{\Lambda})$.

REMARK 2.2. To simplify the presentation, in this paper we will consider only bounded viscosity solutions. By slightly more involved estimates, we can extend our results to the cases with polynomial growth. However, the boundedness of the derivatives $\partial_t \hat{u}$, $\partial_\omega \hat{u}$, and $\partial^2_{\omega\omega} \hat{u}$ is crucial for the functional Itô's formula (2.6) below.

2.2. *Derivatives on continuous paths.* We now let $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = \mathbf{0}\}$, the set of continuous paths with initial value $\mathbf{0}$, B the canonical process, $\mathbb{F}$ the filtration generated by B, $\mathbb{P}_0$ the Wiener measure, and $\Lambda := [0, T] \times \Omega$. Here and in the sequel, for notational simplicity, we use $\mathbf{0}$ to denote vectors or matrices with appropriate dimensions whose components are all equal to 0.

Clearly $\Omega \subset \hat{\Omega}$, $\Lambda \subset \hat{\Lambda}$, and each $\omega \in \Omega$ can also be viewed as an element of $\hat{\Omega}$. Then $\| \cdot \|_t$ and d_∞ in (2.1) are well defined on Ω and Λ, $(\Omega, \| \cdot \|_T)$ is a Banach space, and (Λ, d_∞) is a complete pseudometric space. Given $u : \Lambda \to \mathbb{R}$ and $\hat{u} : \hat{\Lambda} \to \mathbb{R}$, we say $\hat{u}$ is consistent with u on Λ if

(2.5) $$\hat{u}(t, \omega) = u(t, \omega) \qquad \text{for all } (t, \omega) \in \Lambda.$$

DEFINITION 2.3.   Let $u : \Lambda \to \mathbb{R}$ be $\mathbb{F}$-progressively measurable.

  (i) We say $u \in C^0(\Lambda)$ if u is continuous in (t, ω) under d_∞.
 (ii) We say $u \in C_b^0(\Lambda) \subset C^0(\Lambda)$ if u is bounded.
(iii) We say $u \in C_b^{1,2}(\Lambda)$ if there exists $\hat{u} \in C_b^{1,2}(\hat{\Lambda})$ such that (2.5) holds.

By [6] and [4], we have the following important results.

THEOREM 2.4.   Let $u \in C_b^{1,2}(\Lambda)$ and $\hat{u} \in C_b^{1,2}(\hat{\Lambda})$ such that (2.5) holds.

  (i) The following definition

$$\partial_t u := \partial_t \hat{u}, \qquad \partial_\omega u := \partial_\omega \hat{u}, \qquad \partial_{\omega\omega}^2 u := \partial_{\omega\omega}^2 \hat{u} \qquad on\ \Lambda$$

is independent of the choice of $\hat{u}$. Namely, if there is another $\hat{u}' \in C_b^{1,2}(\hat{\Lambda})$ such that (2.5) holds, then the derivatives of $\hat{u}'$ coincide with those of $\hat{u}$ on Λ.

 (ii) If $\mathbb{P}$ is a semimartingale measure, then u is a semimartingale under $\mathbb{P}$ and

$$(2.6) \qquad du_t = \partial_t u_t\, dt + \tfrac{1}{2} \mathrm{tr}(\partial_{\omega\omega}^2 u_t\, d\langle B \rangle_t) + \partial_\omega u_t\, dB_t, \qquad \mathbb{P}\text{-}a.s.$$

We note that, for any given $\mathbb{P}$, the quadratic variation $\langle B \rangle$ is well defined. In fact, although not used in this paper, one can construct $\langle B \rangle$ in a pathwise manner, see, for example, [29]. Here and in the sequel, when we emphasize that u is a process, we use the notation $u_t(\omega) := u(t, \omega)$ and often omit ω by simply writing it as u_t. Moreover, when a probability is involved, quite often we use B which by definition satisfies $B_t(\omega) = \omega_t$.

2.3. *Localization of the spaces.*   For our purpose, we need to introduce the localization version of the above notions. Let

$$(2.7) \qquad \begin{aligned} \mathcal{T} := \big\{ &\mathbb{F}\text{-stopping time } \tau : \text{for all } t \in [0, T), \\ &\{\omega : \tau(\omega) > t\} \text{ is an open subset of } (\Omega, \|\cdot\|_T) \big\}. \end{aligned}$$

The following is a typical example of such τ.

EXAMPLE 2.5.   Let $u \in C^0(\Lambda)$. Then, for any constant c,

$$\tau := \inf\{t : u(t, \omega) \geq c\} \wedge T \text{ is in } \mathcal{T}.$$

PROOF.   For any $t < T$, $\{\tau > t\} = \{\sup_{0 \leq s \leq t} u_s < c\}$. Fix $\omega \in \{\tau > t\}$, and set $\varepsilon := \tfrac{1}{2}[c - \sup_{0 \leq s \leq t} u(s, \omega)] > 0$. For any $s \in [0, t]$, since u is continuous at (s, ω), there exists a constant $h_s > 0$ such that $|u(r, \tilde{\omega}) - u(s, \omega)| \leq \varepsilon$ whenever $d_\infty((r, \tilde{\omega}), (s, \omega)) < h_s$. Note that the open intervals $(s - \tfrac{1}{2}h_s, s + \tfrac{1}{2}h_s)$, $s \in [0, t]$, cover the compact set $[0, t]$. Then there exist $0 = s_0 < s_1 < \cdots < s_n = t$ such that $[0, t] \subset \bigcup_{0 \leq i \leq n}(s_i - \tfrac{1}{2}h_{s_i}, s_i + \tfrac{1}{2}h_{s_i})$. Now set $h := \tfrac{1}{2} \min_{0 \leq i \leq n} h_{s_i} > 0$. For

any $\tilde{\omega} \in \Omega$ such that $\|\tilde{\omega} - \omega\|_T < h$, for any $s \in [0, t]$, there exists i such that $|s - s_i| \leq \frac{1}{2}h_{s_i}$. Then

$$d_\infty\big((s, \tilde{\omega}), (s_i, \omega)\big) \leq |s - s_i| + \|\tilde{\omega} - \omega\|_T \leq \tfrac{1}{2}h_{s_i} + h \leq h_{s_i} \qquad \text{for all } s \in [0, t].$$

Thus

$$u(s, \tilde{\omega}) \leq u(s_i, \omega) + \varepsilon \leq \sup_{0 \leq s \leq t} u(s, \omega) + \varepsilon < c \qquad \text{for all } s \in [0, t].$$

This implies that $\tau(\tilde{\omega}) > t$, and therefore $\tau \in \mathcal{T}$. $\quad\square$

Denote

$$(2.8) \quad \Lambda(\tau) := \big\{(t, \omega) \in \Lambda : t < \tau(\omega)\big\} \quad \text{and} \quad \bar{\Lambda}(\tau) := \big\{(t, \omega) \in \Lambda : t \leq \tau(\omega)\big\}.$$

Then clearly $\Lambda(\tau)$ is an open subset of (Λ, d_∞).

DEFINITION 2.6.   Let $\tau \in \mathcal{T}$ and $u : \bar{\Lambda}(\tau) \to \mathbb{R}$. We say $u \in C_b^{1,2}(\bar{\Lambda}(\tau))$ if there exists $\tilde{u} \in C_b^{1,2}(\Lambda)$ such that

$$(2.9) \qquad\qquad\qquad u = \tilde{u} \qquad \text{on } \bar{\Lambda}(\tau).$$

The following result is the localization version of Theorem 2.4.

PROPOSITION 2.7.   *Let* $\tau \in \mathcal{T}$, $u \in C_b^{1,2}(\bar{\Lambda}(\tau))$, $\tilde{u} \in C_b^{1,2}(\Lambda)$ *such that* (2.9) *holds.*

   (i) *One may define*

$$(2.10) \quad \partial_t u := \partial_t \tilde{u}, \qquad \partial_\omega u := \partial_\omega \tilde{u}, \qquad \partial_{\omega\omega}^2 u := \partial_{\omega\omega}^2 \tilde{u} \qquad \text{on } \Lambda(\tau),$$

and the definition is independent of the choice of $\tilde{u}$.

   (ii) *Let* $\mathbb{P}$ *be a semimartingale measure. Then* u *is a* $\mathbb{P}$-*semimartingale on* $[0, \tau]$ *and* (2.6) *holds on* $[0, \tau]$.

PROOF.   First, for the derivatives defined in (2.10), (2.6) follows directly from Theorem 2.4. Next, assume $\tilde{u}' \in C_b^{1,2}(\Lambda)$ also satisfies (2.9). Denote $\bar{u} := \tilde{u} - \tilde{u}'$. Then $\bar{u} = 0$ on $\bar{\Lambda}(\tau)$. Now fix $(t, \omega) \in \Lambda(\tau)$. Since $\Lambda(\tau)$ is open, there exists $h := h(t, \omega) > 0$ such that $(s, \tilde{\omega}) \in \Lambda(\tau)$ whenever $d_\infty((s, \tilde{\omega}), (t, \omega)) < h$. Now following the definition of the time derivative we obtain immediately that $\partial_t \bar{u}(t, \omega) = 0$. Moreover, let $\mathbb{P} = \mathbb{P}_0$, and applying (2.6) to $\bar{u}$, we have

$$0 = \tfrac{1}{2}\operatorname{tr}\big(\partial_{\omega\omega}^2 \bar{u}_t\big)\,dt + \partial_\omega \bar{u}_t\,dB_t, \qquad 0 \leq t < \tau,\ \mathbb{P}_0\text{-a.s.}$$

Thus, since $\partial_\omega \bar{u}$ and $\partial_{\omega\omega}^2 \bar{u}$ are bounded,

$$\partial_\omega \bar{u} = \mathbf{0}, \qquad \partial_{\omega\omega}^2 \bar{u} = \mathbf{0}, \qquad dt \times d\mathbb{P}_0\text{-a.s. on } \Lambda(\tau).$$

Since $\Lambda(\tau)$ is open, and $\partial_\omega \bar{u}$ and $\partial^2_{\omega\omega} \bar{u}$ are continuous in (t, ω) under d_∞, it is clear that

$$\partial_\omega \bar{u} = \mathbf{0}, \qquad \partial^2_{\omega\omega} \bar{u} = \mathbf{0} \qquad \text{on } \Lambda(\tau).$$

This implies that the definition in (2.10) is independent of the choice of $\tilde{u}$. $\square$

2.4. *Space shift.* We first fix $t \in [0, T]$ and introduce the shifted spaces on càdlàg paths:

- Let $\hat{\Omega}^t := \mathbb{D}([t, T], \mathbb{R}^d)$ be the shifted canonical space; $\hat{B}^t$ the shifted canonical process on $\hat{\Omega}^t$; $\hat{\mathbb{F}}^t$ the shifted filtration generated by B^t; and $\hat{\Lambda}^t := [t, T] \times \hat{\Omega}^t$.
- Define $\| \cdot \|_s^t$ and d_∞^t in the spirit of (2.1).
- For $\hat{\mathbb{F}}^t$-progressively measurable $\hat{u} : \hat{\Lambda}^t \to \mathbb{R}$, define the derivatives in the spirit of (2.2) and (2.3), and define the spaces $C^0(\hat{\Lambda}^t)$, $C_b^0(\hat{\Lambda}^t)$ and $C_b^{1,2}(\hat{\Lambda}^t)$ in the spirit of Definition 2.1.

Similarly, we may define the shifted spaces on continuous paths:

- Let $\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d) : \omega_t = \mathbf{0}\}$ be the shifted canonical space, B^t the shifted canonical process on Ω^t, $\mathbb{F}^t$ the shifted filtration generated by B^t, $\mathbb{P}_0^t$ the Wiener measure on Ω^t and $\Lambda^t := [t, T] \times \Omega^t$.
- Define $C^0(\Lambda^t)$, $C_b^0(\Lambda^t)$ and $C_b^{1,2}(\Lambda^t)$ in an obvious way.
- Let $\mathcal{T}^t$ denote the space of $\mathbb{F}^t$-stopping times τ such that, for any $s \in [t, T)$, the set $\{\omega \in \Omega^t : \tau(\omega) > s\}$ is an open subset of Ω^t under $\| \cdot \|_T^t$.
- For each $\tau \in \mathcal{T}^t$, define $\Lambda^t(\tau)$, $\bar{\Lambda}^t(\tau)$, and $C_b^{1,2}(\bar{\Lambda}^t(\tau))$ in an obvious way.

We next introduce the shift and concatenation operators. Let $0 \le s \le t \le T$.

- For $\hat{\omega} \in \hat{\Omega}^s$, $\hat{\omega}' \in \hat{\Omega}^t$ and $\omega \in \Omega^s$, $\omega' \in \Omega^t$, define the concatenation paths $\hat{\omega} \otimes_t \hat{\omega}' \in \hat{\Omega}^s$ and $\omega \otimes_t \omega' \in \Omega^s$ by

$$
\begin{aligned}
(\hat{\omega} \otimes_t \hat{\omega}')(r) &:= \hat{\omega}_r \mathbf{1}_{[s,t)}(r) + (\hat{\omega}_{t-} + \hat{\omega}_r')\mathbf{1}_{[t,T]}(r); \\
(\omega \otimes_t \omega')(r) &:= \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \omega_r')\mathbf{1}_{[t,T]}(r);
\end{aligned}
\qquad \text{for all } r \in [s, T].
$$

- Let $\hat{\omega} \in \hat{\Omega}^s$. For $\hat{\mathcal{F}}_T^s$-measurable random variable $\hat{\xi}$ and $\hat{\mathbb{F}}^s$-progressively measurable process $\hat{X}$ on $\hat{\Omega}^s$, define the shifted $\hat{\mathcal{F}}_T^t$-measurable random variable $\hat{\xi}^{t,\hat{\omega}}$ and $\hat{\mathbb{F}}^t$-progressively measurable process $\hat{X}^{t,\hat{\omega}}$ on $\hat{\Omega}^t$ by

$$\hat{\xi}^{t,\hat{\omega}}(\hat{\omega}') := \hat{\xi}(\hat{\omega} \otimes_t \hat{\omega}'), \qquad \hat{X}^{t,\hat{\omega}}(\hat{\omega}') := \hat{X}(\hat{\omega} \otimes_t \hat{\omega}') \qquad \text{for all } \hat{\omega}' \in \hat{\Omega}^t.$$

- Let $\omega \in \Omega^s$. For $\mathcal{F}_T^s$-measurable random variable ξ and $\mathbb{F}^s$-progressively measurable process X on Ω^s, define the shifted $\mathcal{F}_T^t$-measurable random variable $\xi^{t,\omega}$ and $\mathbb{F}^t$-progressively measurable process $X^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \qquad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega') \qquad \text{for all } \omega' \in \Omega^t.$$

It is clear that all the results in previous subsections can be extended to the shifted spaces, after obvious modifications. Moreover, for any $\tau \in \mathcal{T}$, $(t, \omega) \in \Lambda(\tau)$ and $u \in C_b^{1,2}(\bar{\Lambda}(\tau))$, it is clear that $\tau^{t,\omega} \in \mathcal{T}^t$ and $u^{t,\omega} \in C_b^{1,2}(\bar{\Lambda}^t(\tau^{t,\omega}))$.

For some technical proofs later, we shall also use the following space. Denote

$$(2.11) \qquad \mathcal{T}_+^t := \{\tau \in \mathcal{T}^t : \tau > t\} \qquad \text{for } t < T \text{ and } \mathcal{T}_+^T := \{T\}.$$

DEFINITION 2.8. Let $t \in [0, T]$, $u : \Lambda^t \to \mathbb{R}$ and $\mathbb{P}$ be a semimartingale measure on Ω^t. We say $u \in \bar{C}_{\mathbb{P}}^{1,2}(\Lambda^t)$ if there exist an increasing sequence of $\mathbb{F}^t$-stopping times $t = \tau_0 \le \tau_1 \le \cdots \le T$ such that:

(i) For each $i \ge 0$ and $\omega \in \Omega^t$,

$$\tau_{i+1}^{\tau_i(\omega),\omega} \in \mathcal{T}_+^{\tau_i(\omega)} \quad \text{and} \quad u^{\tau_i(\omega),\omega} \in C_b^{1,2}(\bar{\Lambda}^{\tau_i(\omega)}(\tau_{i+1}^{\tau_i(\omega),\omega}));$$

(ii) For each $i \ge 0$ and $\omega \in \Omega$, $u.(\omega)$ is continuous on $[0, \tau_i(\omega)]$;
(iii) For $\mathbb{P}$-a.s. $\omega \in \Omega^t$, the set $\{i : \tau_i(\omega) < T\}$ is finite.

We shall emphasize that, for $u \in \bar{C}_{\mathbb{P}}^{1,2}(\Lambda^t)$, the derivatives of u are bounded on each interval $[\tau_i(\omega), \tau_{i+1}^{\tau_i(\omega),\omega}]$; however, in general they may be unbounded on the whole interval $[t, T]$. Also, the previous definition and, more specifically the dependence on $\mathbb{P}$ introduced in item (iii), is motivated by the results established in Section 6 below.

The following result is a direct consequence of Proposition 2.7.

PROPOSITION 2.9. *Let $\mathbb{P}$ be a semimartingale measure on Ω^t and $u \in \bar{C}_{\mathbb{P}}^{1,2}(\Lambda^t)$. Then u is a local $\mathbb{P}$-semimartingale on $[t, T]$ and*

$$du_s = \partial_t u_s \, ds + \frac{1}{2} \operatorname{tr}(\partial_{\omega\omega}^2 u_s \, d\langle B^t \rangle_s) + \partial_\omega u_s \, dB_s^t, \qquad t \le s \le T, \mathbb{P}\text{-}a.s.$$

3. PPDEs and definitions. In this paper we study the following semi-linear parabolic Path-dependent PDE (PPDE, for short):

$$(\mathcal{L}u)(t, \omega) = 0, \qquad 0 \le t < T, \omega \in \Omega;$$

$$(3.1) \qquad \text{where } (\mathcal{L}u)(t, \omega) := -\partial_t u(t, \omega) - \frac{1}{2} \operatorname{tr}(\partial_{\omega\omega}^2 u(t, \omega))$$

$$- f(t, \omega, u(t, \omega), \partial_\omega u(t, \omega)).$$

We remark that there is a potential to extend our results to a much more general setting. However, in order to focus on the main ideas, in this paper we content ourselves with the simple PPDE (3.1) under somewhat strong technical conditions, and leave more general cases, for example, fully nonlinear PPDEs, for future studies.

REMARK 3.1.   In the Markovian case, namely $f = f(t, \omega_t, y, z)$ and $u(t, \omega) = v(t, \omega_t)$, the PPDE (3.1) reduces to the following PDE:

$$(\mathcal{L}v)(t, x) = 0, \qquad 0 \le t < T, x \in \mathbb{R}^d,$$

(3.2)   where $(\mathcal{L}v)(t, x) := -\partial_t v(t, x) - \frac{1}{2} \operatorname{tr}[D^2_{xx} v(t, x)]$

$$- f(t, x, v(t, x), D_x v(t, x)).$$

Here D_x and D^2_{xx} denote the standard first and second order derivatives with respect to x. However, slightly different from the PDE literature but consistent with (2.3), ∂_t denotes the right time-derivative.

As usual, we start with classical solutions.

DEFINITION 3.2.   Let $u \in C^{1,2}_b(\Lambda)$. We say u is a classical solution (resp., sub-solution, super-solution) of PPDE (3.1) if

(3.3) $(\mathcal{L}u)(t, \omega) = (\text{resp.}, \le, \ge)0 \qquad$ for all $(t, \omega) \in [0, T) \times \Omega$.

It is clear that, in the Markovian setting as in Remark 3.1,

u is a classical solution (resp., sub-solution, super-solution) of PPDE (3.1) if and only if v is a classical solution (resp., sub-solution, super-solution) of PDE (3.2).

Existence and uniqueness of classical solutions are related to the analogue results for the corresponding backward SDE. In order to avoid diverting the attention from our main purpose in this paper, we report these properties later in Section 5.1, and we move to our notion of viscosity solutions.

For any $L \ge 0$ and $t < T$, let $\mathcal{U}^L_t$ denote the space of $\mathbb{F}^t$-progressively measurable $\mathbb{R}^d$-valued processes β such that each component of β is bounded by L. By viewing β as row vectors, we define

$$M^{t,\beta}_s := \exp\left(\int_t^s \beta_r \, dB^t_r - \frac{1}{2} \int_t^s |\beta_r|^2 \, dr\right),$$

(3.4)
$$\mathbb{P}^t_0\text{-a.s.}, d\mathbb{P}^{t,\beta} := M^{t,\beta}_T \, d\mathbb{P}^t_0,$$

and we introduce for all $t \in [0, T]$ two nonlinear expectations: for any $\xi \in L^2(\mathcal{F}^t_T, \mathbb{P}^t_0)$,

(3.5)
$$\mathcal{E}^L_t[\xi] := \inf\{\mathbb{E}^{\mathbb{P}^{t,\beta}}[\xi] : \beta \in \mathcal{U}^L_t\};$$

$$\bar{\mathcal{E}}^L_t[\xi] := \sup\{\mathbb{E}^{\mathbb{P}^{t,\beta}}[\xi] : \beta \in \mathcal{U}^L_t\}.$$

Moreover, for any $u \in C_b^0(\Lambda)$, define

$$\mathcal{A}^L u(t, \omega) := \Big\{ \varphi \in C_b^{1,2}(\Lambda^t) : \text{there exists } \tau \in \mathcal{T}_+^t \text{ such that}$$

$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) = \min_{\tilde{\tau} \in \mathcal{T}^t} \mathcal{E}_t^L \big[(\varphi - u^{t,\omega})_{\tilde{\tau} \wedge \tau} \big] \Big\};$$

(3.6)

$$\bar{\mathcal{A}}^L u(t, \omega) := \Big\{ \varphi \in C_b^{1,2}(\Lambda^t) : \text{there exists } \tau \in \mathcal{T}_+^t \text{ such that}$$

$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) = \max_{\tilde{\tau} \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L \big[(\varphi - u^{t,\omega})_{\tilde{\tau} \wedge \tau} \big] \Big\}.$$

DEFINITION 3.3.   Let $u \in C_b^0(\Lambda)$.

(i) For any $L \geq 0$, we say u is a viscosity L-subsolution (resp., L-supersolution) of PPDE (3.1) if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \mathcal{A}^L u(t, \omega)$ [resp., $\varphi \in \bar{\mathcal{A}}^L u(t, \omega)$], it holds that

$$(\mathcal{L}^{t,\omega}\varphi)(t, \mathbf{0}) \leq (\text{resp.}, \geq)0,$$

where, for each $(s, \tilde{\omega}) \in [t, T] \times \Omega^t$,

$$(\mathcal{L}^{t,\omega}\varphi)(s, \tilde{\omega}) := -\partial_t \varphi(s, \tilde{\omega}) - \tfrac{1}{2} \operatorname{tr}\big[\partial_{\omega\omega}^2 \varphi(s, \tilde{\omega})\big] - f^{t,\omega}\big(s, \tilde{\omega}, \varphi(s, \tilde{\omega}), \partial_\omega \varphi(s, \tilde{\omega})\big).$$

(ii) We say u is a viscosity subsolution (resp., supersolution) of PPDE (3.1) if u is viscosity L-subsolution (resp., L-supersolution) of PPDE (3.1) for some $L \geq 0$.

(iii) We say u is a viscosity solution of PPDE (3.1) if it is both a viscosity subsolution and a viscosity supersolution.

In the rest of this section we provide several remarks concerning our definition of viscosity solutions. In most places we will comment on the viscosity subsolution only, but obviously similar properties hold for the viscosity supersolution as well.

REMARK 3.4.   As standard in the literature on viscosity solutions of PDEs:

(i) The viscosity property is a local property in the following sense. For any $(t, \omega) \in [0, T) \times \Omega$ and any $\varepsilon > 0$, define

$$\tau_\varepsilon := \inf\{s > t : |B_s^t| \geq \varepsilon\} \wedge (t + \varepsilon).$$

To check the viscosity property of u at (t, ω), it suffices to know the value of $u^{t,\omega}$ on $[t, \tau_\varepsilon]$ for an arbitrarily small $\varepsilon > 0$.

(ii) Typically $\mathcal{A}^L u(t, \omega)$ and $\bar{\mathcal{A}}^L u(t, \omega)$ are disjoint, so u is a viscosity solution does not mean $(\mathcal{L}^{t,\omega}\varphi)(t, \mathbf{0}) = 0$ for φ in some appropriate set. One has to check viscosity subsolution property and viscosity supersolution property separately.

(iii) In general $\mathcal{A}^L u(t, \omega)$ could be empty. In this case automatically u satisfies the viscosity subsolution property at (t, ω).

REMARK 3.5. (i) For $0 \leq L_1 < L_2$, obviously $\mathcal{U}_t^{L_1} \subset \mathcal{U}_t^{L_2}$, $\mathcal{E}_t^{L_2} \leq \mathcal{E}_t^{L_1}$ and $\mathcal{A}^{L_2} u(t, \omega) \subset \mathcal{A}^{L_1} u(t, \omega)$. Then one can easily check that a viscosity L_1-subsolution must be a viscosity L_2-subsolution. Consequently,

> u is a viscosity subsolution if and only if there exists a $L \geq 0$ such that, for all $\tilde{L} \geq L$, u is a viscosity $\tilde{L}$-subsolution.

(ii) However, we require the same L for all (t, ω). We should point out that our definition of viscosity subsolution is not equivalent to the following alternative definition, under which we are not able to prove the comparison principle:

> for any (t, ω) and any $\varphi \in \bigcap_{L \geq 0} \mathcal{A}^L u(t, \omega)$, it holds that $(\mathcal{L}^{t,\omega} \varphi)(t, \mathbf{0}) \leq 0$.

REMARK 3.6. We may replace $\mathcal{A}^L$ with the following $(\mathcal{A}')^L$ which requires strict inequality,

$$(3.7) \quad \mathcal{A}'^L u(t, \omega) := \{\varphi \in C_b^{1,2}(\Lambda^t) : \text{there exists } \tau \in \mathcal{T}_+^t \text{ such that}$$
$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) < \mathcal{E}_t^L[(\varphi - u^{t,\omega})_{\tilde{\tau} \wedge \tau}] \text{ for all } \tilde{\tau} \in \mathcal{T}_+^t\}.$$

Then u is a viscosity L-subsolution of PPDE (3.1) if and only if

$$(\mathcal{L}^{t,\omega} \varphi)(t, \mathbf{0}) \leq 0 \qquad \text{for all } (t, \omega) \in [0, T) \times \Omega \text{ and } \varphi \in \mathcal{A}'^L u(t, \omega).$$

A similar statement holds for the viscosity supersolution.

Indeed, since $\mathcal{A}'^L u(t, \omega) \subset \mathcal{A}^L u(t, \omega)$, then only the if part is clear. To prove the if part, let $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in \mathcal{A}^L u(t, \omega)$. For any $\varepsilon > 0$, denote $\varphi^\varepsilon(s, \tilde{\omega}) := \varphi(s, \tilde{\omega}) + \varepsilon(s - t)$. Then clearly $\varphi^\varepsilon \in \mathcal{A}'^L u(t, \omega)$, and thus

$$(\mathcal{L}^{t,\omega} \varphi^\varepsilon)(t, \mathbf{0}) = -\partial_t \varphi(t, \mathbf{0}) - \varepsilon - \tfrac{1}{2} \operatorname{tr}(\partial_{\omega\omega}^2 \varphi(t, \mathbf{0}))$$
$$- f^{t,\omega}(t, \omega, \varphi(t, \mathbf{0}), \partial_\omega \varphi(t, \mathbf{0}))$$
$$\leq 0.$$

Send $\varepsilon \to 0$, we obtain $(\mathcal{L}^{t,\omega} \varphi)(t, \mathbf{0}) \leq 0$, and thus u is a viscosity L-subsolution.

REMARK 3.7. Consider the Markovian setting in Remark 3.1. One can easily check that u is a viscosity subsolution of PPDE (3.1) in the sense of Definition 3.3 implies that v is a viscosity subsolution of PDE (3.2) in the standard sense.

REMARK 3.8. We have some flexibility to choose $\underline{\mathcal{A}}^L u(t, \omega)$ and $\bar{\mathcal{A}}^L u(t, \omega)$ in Definition 3.3. In principle, the smaller these sets are, the more easily we can prove viscosity properties and thus the existence of viscosity solutions, but the comparison principle and the uniqueness of viscosity solutions become more difficult.

(i) The following $\mathcal{A}''^L u(t, \omega)$ is larger than $\mathcal{A}^L u(t, \omega)$, but all the results in this paper still hold true if we use $\mathcal{A}''^L u(t, \omega)$ [and the corresponding $\bar{\mathcal{A}}''^L u(t, \omega)$],

(3.8)
$$\mathcal{A}''^L u(t, \omega) := \{\varphi \in C_b^{1,2}(\Lambda^t): \text{for any } \tau \in \mathcal{T}_+^t,$$
$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) \leq \mathcal{E}_t^L\big[(\varphi - u^{t,\omega})_{\tilde{\tau} \wedge \tau}\big]$$
$$\text{for some } \tilde{\tau} \in \mathcal{T}_+^t\}.$$

(ii) However, if we use the following smaller alternatives of $\mathcal{A}^L u(t, \omega)$, which do not involve the nonlinear expectation, we are not able to prove the comparison principle and the uniqueness of viscosity solutions,

$$\mathcal{A}^\circ u(t, \omega) := \{\varphi \in C_b^{1,2}(\Lambda^t): \text{there exists } \tau \in \mathcal{T}_+^t \text{ such that}$$
$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) \leq (\varphi - u^{t,\omega})_{\tilde{\tau} \wedge \tau} \text{ for any } \tilde{\tau} \in \mathcal{T}_+^t\};$$

or

$$\mathcal{A}^{\circ\circ} u(t, \omega) := \{\varphi \in C_b^{1,2}(\Lambda^t): \text{for all } (s, \tilde{\omega}) \in (t, T] \times \Omega^t,$$
$$0 = \varphi(t, \mathbf{0}) - u(t, \omega) \leq (\varphi - u^{t,\omega})(s, \tilde{\omega})\}.$$

See also Remark 3.5(ii).

REMARK 3.9. (i) Let u be a viscosity subsolution of PPDE (3.1). Then for any $\lambda \in \mathbb{R}$, $\tilde{u}_t := e^{\lambda t} u_t$ is a viscosity subsolution of the following PPDE:

(3.9) $$\tilde{\mathcal{L}}\tilde{u} := -\partial_t \tilde{u} - \tfrac{1}{2}\operatorname{tr}(\partial_{\omega\omega}^2 \tilde{u}) - \tilde{f}(t, \omega, \tilde{u}, \partial_\omega \tilde{u}) \leq 0,$$

where

$$\tilde{f}(t, \omega, y, z) := -\lambda y + e^{\lambda t} f\big(t, \omega, e^{-\lambda t} y, e^{-\lambda t} z\big).$$

Indeed, assume u is a viscosity L-subsolution of PPDE (3.1). Let $(t, \omega) \in [0, T) \times \Omega$ and $\tilde{\varphi} \in \mathcal{A}^L \tilde{u}(t, \omega)$. For any $\varepsilon > 0$, denote

$$\varphi_s^\varepsilon := e^{-\lambda s} \tilde{\varphi}_s + \varepsilon(s - t).$$

Then, noting that $\tilde{\varphi}_t = e^{\lambda t} u(t, \omega)$,

$$\varphi_s^\varepsilon - u_s^{t,\omega} - e^{-\lambda t}\big(\tilde{\varphi}_s - \tilde{u}_s^{t,\omega}\big)$$
$$= \big(e^{-\lambda s} - e^{-\lambda t}\big)\tilde{\varphi}_s + \big(e^{\lambda(s-t)} - 1\big)u_s + \varepsilon(s - t)$$
$$= \big(e^{-\lambda s} - e^{-\lambda t}\big)(\tilde{\varphi}_s - \tilde{\varphi}_t) + \big(e^{\lambda(s-t)} - 1\big)(u_s - u_t)$$
$$\quad + \big(e^{-\lambda(s-t)} + e^{\lambda(s-t)} - 2\big)u_t + \varepsilon(s - t)$$
$$\geq \varepsilon(s - t) - C(s - t)\big(|\tilde{\varphi}_s - \tilde{\varphi}_t| + |u_s - u_t| + (s - t)\big).$$

Let $\tilde{\tau} \in \mathcal{T}_+^t$ be a stopping time corresponding to $\tilde{\varphi} \in \mathcal{A}^L \tilde{u}(t, \omega)$, and set

$$\tau_\varepsilon := \tilde{\tau} \wedge \inf\Big\{s > t : |\tilde{\varphi}_s - \tilde{\varphi}_t| + |u_s - u_t| + (s - t) \geq \frac{\varepsilon}{C}\Big\} \wedge T.$$

Then $\tau_\varepsilon \in \mathcal{T}_+^t$, by Example 2.5, and for any $\tau \in \mathcal{T}^t$ such that $\tau \leq \tau_\varepsilon$, it follows from the previous inequality that

$$\varphi_\tau^\varepsilon - u_\tau^{t,\omega} \geq e^{-\lambda t}[\tilde{\varphi}_\tau - \tilde{u}_\tau^{t,\omega}].$$

By the increase and the homogeneity of the operator $\mathcal{E}_t^L$, together with the fact that $\tilde{\varphi} \in \mathcal{A}^L \tilde{u}(t, \omega)$, this implies that

$$\mathcal{E}_t^L[\varphi_\tau^\varepsilon - u_\tau^{t,\omega}] \geq \mathcal{E}_t^L[e^{-\lambda t}(\tilde{\varphi}_\tau - \tilde{u}_\tau^{t,\omega})] = e^{-\lambda t}\mathcal{E}_t^L[\tilde{\varphi}_\tau - \tilde{u}_\tau^{t,\omega}] \geq 0 = \varphi_t^\varepsilon - u_t.$$

This implies that $\varphi^\varepsilon \in \mathcal{A}^L u(t, \omega)$, then $\mathcal{L}^{t,\omega}\varphi^\varepsilon(t, 0) \leq 0$. Send $\varepsilon \to 0$, and similar to Remark 3.6 we get $\mathcal{L}^{t,\omega}\varphi^0(t, 0) \leq 0$, where $\varphi_s^0 := e^{-\lambda s}\tilde{\varphi}_s$. Now by straightforward calculation we obtain

$$-\partial_t\tilde{\varphi}(t, 0) - \tfrac{1}{2}\operatorname{tr}[\partial_{\omega\omega}^2\tilde{\varphi}(t, 0)] - \tilde{f}(t, \omega, \tilde{\varphi}(t, 0), \partial_\omega\tilde{\varphi}(t, 0)) \leq 0.$$

That is, $\tilde{u}$ is a viscosity subsolution of PPDE (3.9).

(ii) If we consider more general variable change: $\bar{u}(t, \omega) := \psi(t, u(t, \omega))$, where $\psi \in C^{1,2}([0, T] \times \mathbb{R})$ such that $\partial_y\psi > 0$. Denote by $\bar{\psi} := \psi^{-1}$ the inverse function of ψ with respect to the space variable y. Then one can easily check that u is a classical subsolution of PPDE (3.1) if and only if $\bar{u}$ is a classical subsolution of the following PPDE:

$$\bar{\mathcal{L}}\bar{u} := -\partial_t\bar{u} - \frac{1}{2}\operatorname{tr}(\partial_{\omega\omega}^2\bar{u}) - \bar{f}(t, \omega, \bar{u}, \partial_\omega\bar{u}) \leq 0$$

$$(3.10) \qquad \text{where } \bar{f}(t, \omega, y, z) := \frac{1}{\partial_y\bar{\psi}(t, y)}\left[\partial_t\bar{\psi}(t, y) + \frac{1}{2}\partial_{yy}^2\bar{\psi}(t, y)|z|^2\right.$$

$$\left. + f(t, \omega, \bar{\psi}(t, y), \partial_y\bar{\psi}(t, y)z)\right].$$

However, if u is only a viscosity subsolution of PPDE (3.1), we are not able to prove that $\bar{u}$ is a viscosity subsolution of (3.10). The main difficulty is that the nonlinear expectation $\mathcal{E}_t^L$ and the nonlinear function ψ do not commute. Consequently, given $\bar{\varphi} \in \mathcal{A}^L\bar{u}(t, \omega)$, we are not able to construct as in (i) the corresponding $\varphi \in \mathcal{A}^L u(t, \omega)$.

We conclude this section by connecting the nonlinear expectation operators to backward SDEs, and providing some tools from optimal stopping theory which will be used later.

REMARK 3.10 (Connecting $\mathcal{E}^L$ and $\bar{\mathcal{E}}^L$ to backward SDEs).   For readers who are familiar with BSDE literature, by the comparison principle of BSDEs (see, e.g., El Karoui, Peng and Quenez [10]), one can easily show that $\mathcal{E}_t^L[\xi] = \mathcal{Y}_t$ and

$\bar{\mathcal{E}}_t^L[\xi] = \bar{\mathcal{Y}}_t$, where $(\mathcal{Y}, \mathcal{Z})$ and $(\bar{\mathcal{Y}}, \bar{\mathcal{Z}})$ are the solution to the following BSDEs, respectively:

$$(3.11) \quad \begin{aligned} \mathcal{Y}_s &= \xi - \int_s^T L|\mathcal{Z}_r|\,dr - \int_s^T \mathcal{Z}_r\,dB_r^t, \\ \bar{\mathcal{Y}}_s &= \xi + \int_s^T L|\bar{\mathcal{Z}}_r|\,dr - \int_s^\tau \bar{\mathcal{Z}}_r\,dB_r^t, \qquad t \le s \le T, \ \mathbb{P}_0^t\text{-a.s.} \end{aligned}$$

Moreover, this is a special case of the so called g-expectation; see Peng [22].

REMARK 3.11 (Optimal stopping under nonlinear expectation and reflected backward SDEs). The definition of the set $\mathcal{A}^L$ is closely related to the following optimal stopping problem under nonlinear expectation

$$Y_t := \inf_{\tilde{\tau} \in \mathcal{T}^t} \mathcal{E}_t^L[X_{\tilde{\tau} \wedge \tau}]$$

for some stopping time $\tau \in \mathcal{T}_+^t$ and some adapted bounded pathwise continuous process X. For the ease of presentation here, we provide only heuristic arguments, and we refer to Section 7 of [7] for a rigorous argument and to [8] for the optimal stopping problem under more general nonlinear expectations.

For later use, we provide some key results which can be proved by following the standard corresponding arguments in the standard optimal stopping theory, and we observe that the process Y is pathwise continuous; see (iv) below.

Following the classical arguments in optimal stopping theory, we have:

(i) $\mathcal{E}_t^L[Y_{\tilde{\tau} \wedge \tau}] \ge Y_t$ for all $\tilde{\tau} \in \mathcal{T}^t$, that is, Y is an $\mathcal{E}^L$-submartingale.

(ii) If $\tau^* \in \mathcal{T}^t$ is an optimal stopping rule, then

$$Y_t = \mathcal{E}_t^L[X_{\tau^* \wedge \tau}] = \inf_{\tilde{\tau} \in \mathcal{T}^{\tau^*}} \mathcal{E}_t^L[X_{\tilde{\tau} \wedge \tau}] = \inf_{\tilde{\tau} \in \mathcal{T}^{\tau^*}} \mathcal{E}_t^L[Y_{\tilde{\tau} \wedge \tau}] = \mathcal{E}_t^L[Y_{\tau^* \wedge \tau}],$$

where the last inequality is a consequence of (i), and the third inequality follows from the fact that $X \le Y$ on one hand, and $\inf \mathcal{E}_t^L[\cdot] \ge \mathcal{E}_t^L[\inf \cdot]$ on the other hand. This implies that $Y_{\tau^*} = X_{\tau^*}$ and, by (i), that $Y_{\cdot \wedge \tau^*}$ is an $\mathcal{E}^L$-martingale.

(iii) We then define $\tau_t^1 := \inf\{s > t : Y_t = X_t\}$. Since $Y_T = X_T$, we have $\tau_t^1 \le T$, a.s. Moreover, following the classical arguments in optimal stopping theory, we see that $\{Y_{s \wedge \tau_t^1}\}_{s \ge t}$ is an $\mathcal{E}^L$-martingale. With this in hand, we conclude that τ_t^1 is an optimal stopping time, that is, $Y_t = \mathcal{E}_t^L[X_{\tau_t^1}]$.

(iv) For those readers who are familiar with backward stochastic differential equations, we mention that $Y = \mathcal{Y}^\circ$, where $(\mathcal{Y}^\circ, \mathcal{Z}^\circ, \underline{K}^\circ)$ is the solution to the following reflected BSDEs:

$$(3.12) \qquad \mathcal{Y}_s^\circ = X_\tau - \int_s^\tau L|\mathcal{Z}_r^\circ|\,dr - \int_s^\tau \mathcal{Z}_r^\circ\,dB_r^t - \int_s^\tau d\underline{K}_s^\circ,$$

$$(3.13) \qquad \mathcal{Y}_s^\circ \le X_s \quad \text{and} \quad (\mathcal{Y}_s^\circ - X_s)\,d\underline{K}_s^\circ = 0, \qquad s \in [t, T], \ \mathbb{P}_0^t\text{-a.s.};$$

see, for example, [9]. In particular, it is a well-known result that the process Y is pathwise continuous.

(v) Similar results hold for $\sup_{\tilde{\tau} \in \mathcal{T}^t} \bar{\mathcal{E}}_t[X_{\tilde{\tau} \wedge \tau}]$.

4. The main results.   We start with a stability result.

THEOREM 4.1.   *Let $(f^\varepsilon, \varepsilon > 0)$ be a family of coefficients converging uniformly toward a coefficient $f \in C^0(\Lambda)$ as $\varepsilon \to 0$. For some $L > 0$, let u^ε be a viscosity L-subsolution* (*resp., L-supersolution*) *of PPDE* (3.1) *with coefficients f^ε, for all $\varepsilon > 0$. Assume further that u^ε converges to some u, uniformly in Λ. Then u is a viscosity L-subsolution* (*resp., supersolution*) *of PPDE* (3.1) *with coefficient f.*

The proof of this result is reported in Section 5.3. For our next results, we shall always use the following standing assumptions, where g is a terminal condition associated to the PPDE (3.1).

ASSUMPTION 4.2.   (i) f is bounded, $\mathbb{F}$-progressively measurable, continuous in t, uniformly continuous in ω, and uniformly Lipschitz continuous in (y, z) with a Lipschitz constant $L_0 > 0$.
    (ii) g is bounded and uniformly continuous in ω.

To establish an existence result of viscosity solutions under the above assumption, we note that the PPDE (3.1) with terminal condition $u(T, \omega) = g(\omega)$ is closely related to (and actually motivated from) the following BSDE:

$$
\begin{aligned}
(4.1) \qquad Y_t^0 = g(B_.) &+ \int_t^T f\big(s, B_., Y_s^0, Z_s^0\big) \, ds \\
&- \int_t^T Z_s^0 \, dB_s, \qquad 0 \le t \le T, \mathbb{P}_0\text{-a.s.}
\end{aligned}
$$

We refer to the seminal paper by Pardoux and Peng [20] for the well-posedness of such BSDEs. On the other hand, for any $(t, \omega) \in \Lambda$, by [20] the following BSDE on $[t, T]$ has a unique solution,

$$
\begin{aligned}
(4.2) \qquad Y_s^{0,t,\omega} = g^{t,\omega}(B_.^t) &+ \int_s^T f^{t,\omega}\big(r, B_.^t, Y_r^{0,t,\omega}, Z_r^{0,t,\omega}\big) \, dr \\
&- \int_s^T Z_r^{0,t,\omega} \, dB_r^t, \qquad \mathbb{P}_0^t\text{-a.s.}
\end{aligned}
$$

By the Blumenthal 0–1 law, $Y_t^{0,t,\omega}$ is a constant and we thus define

$$
(4.3) \qquad\qquad u^0(t, \omega) := Y_t^{0,t,\omega}.
$$

THEOREM 4.3.   *Under Assumption 4.2, u^0 is a viscosity solution of PPDE* (3.1) *with terminal condition g.*

The proof is reported in Section 5.2. Similar to the classical theory of viscosity solutions in the Markovian case, we now establish a comparison result which, in particular, implies the uniqueness of viscosity solutions. For this purpose, we need an additional condition:

ASSUMPTION 4.4.   There exist $\hat{f}: \hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfying:

(i) $\hat{f}(t, \omega, y, z) = f(t, \omega, y, z)$ for all $(t, \omega, y, z) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d$.
(ii) $\hat{f}$ is bounded, $\hat{f}(\cdot, y, z) \in C^0(\hat{\Lambda})$ for any fixed (y, z) and $\hat{f}$ is uniformly Lipschitz continuous in (y, z).

REMARK 4.5.   In the Markovian case as in Remark 3.1, we have a natural extension: $\hat{f} = f(t, \hat{\omega}_t, y, z)$ for all $\hat{\omega} \in \hat{\Omega}$. In this case Assumption 4.2 implies Assumption 4.4.

THEOREM 4.6.   *Let Assumptions 4.2 and 4.4 hold. Let u^1 be a viscosity subsolution and u^2 a viscosity supersolution of PPDE (3.1). If $u^1(T, \cdot) \le g \le u^2(T, \cdot)$, then $u^1 \le u^2$ on Λ.*

Consequently, given the terminal condition g, u^0 is the unique viscosity solution of PPDE (3.1).

The proof is reported in Section 6 building on a partial comparison result derived in Section 5.4.

REMARK 4.7.   For technical reasons, we require a uniformly continuous function g between u_T^1 and u_T^2; see Section 6. However, when one of u^1 and u^2 is in $C_b^{1,2}(\Lambda)$, then we need neither the presence of such g nor the existence of $\hat{f}$; see Lemma 5.7 below.

5. Some proofs of the main results.   In this section we provide some proofs of the main results, and provide some more results. We leave the most technical part of the proof for the comparison principle to next section.

5.1. *Properties of classical solutions.*   We first recall from Peng [23] that an $\mathbb{F}$-progressively measurable process Y is called an f-martingale (resp., f-submartingale, f-supermartingale) if, for any $\mathbb{F}$-stopping times $\tau_1 \ge \tau_2$, we have

$$Y_{\tau_1} = (\text{resp.}, \le, \ge)\mathcal{Y}_{\tau_1}(\tau_2, Y_{\tau_2}), \qquad \mathbb{P}_0\text{-a.s.,}$$

where $(\mathcal{Y}, \mathcal{Z}) := (\mathcal{Y}(\tau_2, Y_{\tau_2}), \mathcal{Z}(\tau_2, Y_{\tau_2}))$ is the solution to the following BSDE on $[0, \tau_2]$:

$$\mathcal{Y}_t = Y_{\tau_2} + \int_t^{\tau_2} f(s, B_\cdot, \mathcal{Y}_s, \mathcal{Z}_s)\, ds - \int_t^{\tau_2} Z_s\, dB_s, \qquad 0 \le t \le \tau_2, \mathbb{P}_0\text{-a.s.}$$

Clearly, Y is an f-martingale with terminal condition $g(B_\cdot)$ if and only if it satisfies the BSDE (4.1).

Applying Itô's formula to Proposition 2.9, we obviously have the following:

PROPOSITION 5.1. *Let Assumption* 4.2 *hold and* $u \in C_b^{1,2}(\Lambda)$. *Then* u *is a classical solution* (*resp.*, *subsolution*, *supersolution*) *of PPDE* (3.1) *if and only if the process* u *is an* f-*martingale* (*resp.*, f-*submartingale*, f-*supermartingale*).

In particular, if u *is a classical solution of PPDE* (3.1) *with terminal condition* g, *then*

(5.1) $$Y := u, \qquad Z := \partial_\omega u$$

provides the unique solution of BSDE (4.1).

PROOF.   We shall only prove the subsolution case. Let (Y, Z) be defined by (5.1).

(i) Assume u is a classical subsolution. By Itô's formula,

$$du_t = \left(\partial_t u_t + \tfrac{1}{2}\operatorname{tr}[\partial^2_{\omega\omega} u_t]\right) dt + \partial_\omega u_t \, dB_t$$

(5.2)

$$= -\left(f(t, B_., u_t, \partial_\omega u_t) + (\mathcal{L}u)_t\right) dt + \partial_\omega u_t \, dB_t, \qquad \mathbb{P}_0\text{-a.s.}$$

Then for any $\tau_1 \leq \tau_2$, (Y, Z) satisfies BSDE

$$Y_t = u_{\tau_2} + \int_t^{\tau_2} \left(f(s, B_., Y_s, Z_s) + (\mathcal{L}u)(s, B_.)\right) ds$$

$$- \int_t^{\tau_2} Z_s \, dB_s, \qquad 0 \leq t \leq \tau_2, \mathbb{P}_0\text{-a.s.}$$

Since $\mathcal{L}u \leq 0$, by the comparison principle of BSDEs (see [10]) we obtain $u_{\tau_1} = Y_{\tau_1} \leq \mathcal{Y}_{\tau_1}(\tau_2, u_{\tau_2})$. That is, u is an f-submartingale.

(ii) Assume u is an f-submartingale. For any $0 \leq t < t+h \leq T$, denote $\delta Y_s := \mathcal{Y}_s(t+h, u_{t+h}) - Y_s$, $\delta Z_s := \mathcal{Z}_s(t+h, u_{t+h}) - Z_s$. By (5.2) we have

$$\delta Y_s = \int_s^{t+h} \left[\alpha_r \delta Y_r + \langle \beta, \delta Z \rangle_r - (\mathcal{L}u)_r\right] dr$$

$$- \int_s^{t+h} \delta Z_s \, dB_s, \qquad t \leq s \leq t+h, \mathbb{P}_0\text{-a.s.}$$

where $|\alpha|, |\beta| \leq L_0$. Define

(5.3) $$\Gamma_s := \exp\left(\int_t^s \beta_r \, dB_r + \int_t^s \left(\alpha_r - \frac{1}{2}|\beta_r|^2\right) dr\right),$$

and we have

$$\delta Y_t = -\mathbb{E}_t^{\mathbb{P}_0}\left[\int_t^{t+h} \Gamma_s (\mathcal{L}u)_s \, ds\right].$$

Since $Y = u$ is an f-submartingale, we get

$$0 \leq \frac{1}{h}\delta Y_t = -\mathbb{E}_t^{\mathbb{P}_0}\left[\frac{1}{h}\int_t^{t+h} \Gamma_s (\mathcal{L}u)_s \, ds\right].$$

Send $h \to 0$, and we obtain

$$\mathcal{L}u(t, B_.) \leq 0, \qquad \mathbb{P}_0\text{-a.s.}$$

Note that $\mathcal{L}u$ is continuous in ω and obviously any support of $\mathbb{P}_0$ is dense, and we have

$$\mathcal{L}u(t, \omega) \leq 0 \qquad \text{for all } \omega \in \Omega.$$

That is, u is a classical subsolution of PPDE (3.1).

(iii) When u is a classical solution similar to (i), we know Y is a f-martingale, and thus (5.1) provides a solution to the BSDE. Finally, the uniqueness follows from the uniqueness of BSDEs. □

REMARK 5.2.   This proposition extends the well-known nonlinear Feynman–Kac formula of Pardoux and Peng [19] to non-Markovian case.

We next prove a simple comparison principle for classical solutions.

LEMMA 5.3.   *Let Assumption* 4.2 *hold true. Let* u^1 *be a classical subsolution and* u^2 *a classical supersolution of PPDE* (3.1). *If* $u^1(T, \cdot) \leq u^2(T, \cdot)$, *then* $u^1 \leq u^2$ *on* Λ.

PROOF.   Denote $Y^i := u^i$, $Z^i := \partial_\omega u^i$, $i = 1, 2$. By (5.2) we have

$$dY_t^i = -[f(t, B_., Y^i, Z^i) + (\mathcal{L}u^i)_t] dt + Z_t^i dB_t, \qquad 0 \leq t \leq T, \mathbb{P}_0\text{-a.s.}$$

Since $Y_T^1 \leq Y_T^2$ and $\mathcal{L}u^1 \leq 0 \leq \mathcal{L}u^2$, by the comparison principle for BSDEs we obtain $Y^1 \leq Y^2$. That is, $u^1 \leq u^2$, $\mathbb{P}_0$-a.s. Since u^1 and u^2 are continuous, and the support of $\mathbb{P}_0$ is dense in Ω, we obtain $u^1 \leq u^2$ on Λ.   □

5.2. *Existence of viscosity solutions.*   We first establish the regularity of u^0 as defined in (4.3).

PROPOSITION 5.4.   *Under Assumption* 4.2, u^0 *is uniformly continuous in* Λ *under* d_∞.

PROOF.   Since f and g are bounded, clearly u^0 is bounded. To show the uniform continuity, let $(t_i, \omega^i) \in \Lambda$, $i = 1, 2$, and assume without loss of generality that $0 \leq t_1 \leq t_2 \leq T$. By taking conditional expectations $\mathbb{E}_{t_2}^{\mathbb{P}_0^{t_1}}$, one can easily see that Y^{0,t_1,ω^1} can be viewed as the solution to the following BSDE on $[t_2, T]$: for $\mathbb{P}_0^{t_1}$-a.s. B^{t_1},

$$Y_s^{0,t_1,\omega^1} = g^{t_2,\omega^1 \otimes_{t_1} B^{t_1}}(B_.^{t_2}) + \int_s^T f^{t_2,\omega^1 \otimes_{t_1} B^{t_1}}(r, B_.^{t_2}, Y_r^{0,t_1,\omega^1}, Z_r^{0,t_1,\omega^1}) dr$$

$$- \int_s^T Z_r^{0,t_1,\omega^1} dB_r^{t_2}, \qquad t_2 \leq s \leq T, \mathbb{P}_0^{t_2}\text{-a.s.}$$

Denote

$$\delta\omega := \omega^1 - \omega^2, \qquad \delta Y := Y^{0,t_1,\omega^1} - Y^{0,t_2,\omega^2}, \qquad \delta Z := Z^{0,t_1,\omega^1} - Z^{0,t_2,\omega^2}.$$

Then

$$\delta Y_s = \delta Y_T + \int_s^T \left(\gamma_r + \alpha_r \delta Y_r + \langle \beta, \delta Z \rangle_r\right) dr$$

$$- \int_s^T \delta Z_r \, dB_r^{t_2}, \qquad t_2 \le s \le T, \mathbb{P}_0^{t_2}\text{-a.s.,}$$

where

$$|\alpha| \le L_0, \qquad \beta \in \mathcal{U}_t^{L_0}$$

and

$$\gamma_r := \left(f^{t_2,\omega^1 \otimes_{t_1} B^{t_1}} - f^{t_2,\omega^2}\right)\left(r, B_{\cdot}^{t_2}, Y^{0,t_1,\omega^1}, Z_r^{0,t_1,\omega^1}\right).$$

Define Γ as in (5.3) with initial time t_2, then

$$\delta Y_{t_2} = \Gamma_T \delta Y_T + \int_{t_2}^T \Gamma_r \gamma_r \, dr - \int_{t_2}^T \Gamma_r [\delta Z_r + \delta Y_r \beta_r] \, dB_r^{t_2}, \qquad \mathbb{P}_0^{t_2}\text{-a.s.}$$

Let ρ denote the modulus of continuity function of f and g with respect to ω. Note that

$$|\delta Y_T| = \left|g\left(\omega^1 \otimes_{t_1} B^{t_1} \otimes_{t_2} B^{t_2}\right) - g\left(\omega^2 \otimes_{t_2} B^{t_2}\right)\right|$$

$$\le \rho\left(\left\|\omega^1 \otimes_{t_1} B^{t_1} - \omega^2\right\|_{t_2}\right) \le \rho\left(d_\infty\left((t_1, \omega^1), (t_2, \omega^2)\right) + \left\|B^{t_1}\right\|_{t_2}^{t_1}\right).$$

Similarly,

$$|\gamma_s| \le \rho\left(d_\infty\left((t_1, \omega^1), (t_2, \omega^2)\right) + \left\|B^{t_1}\right\|_{t_2}^{t_1}\right).$$

Then

$$|\delta Y_{t_2}| = \left|\mathbb{E}^{\mathbb{P}_0^{t_2}}\left[\Gamma_T \delta Y_T + \int_{t_2}^T \Gamma_s \gamma_s \, ds\right]\right| \le C\rho\left(d_\infty\left((t_1, \omega^1), (t_2, \omega^2)\right) + \left\|B^{t_1}\right\|_{t_2}^{t_1}\right).$$

Thus, noting that f is bounded,

$$|u_{t_1}^0(\omega^1) - u_{t_2}^0(\omega^2)|$$

$$= |Y_{t_1}^{0,t_1,\omega^1} - Y_{t_2}^{0,t_2,\omega^2}|$$

$$(5.4) \quad = \left|\mathbb{E}^{\mathbb{P}_0^{t_1}}\left[Y_{t_2}^{0,t_1,\omega^1} + \int_{t_1}^{t_2} f^{t_1,\omega^1}\left(s, B_{\cdot}^{t_1}, Y_s^{0,t_1,\omega^1}, Z_s^{0,t_1,\omega^1}\right) ds - Y_{t_2}^{0,t_2,\omega^2}\right]\right|$$

$$\le C[t_2 - t_1] + \mathbb{E}^{\mathbb{P}_0^{t_1}}\left[|\delta Y_{t_2}|\right]$$

$$\le C[t_2 - t_1] + C\mathbb{E}^{\mathbb{P}_0^{t_1}}\left[\rho\left(d_\infty\left((t_1, \omega^1), (t_2, \omega^2)\right) + \left\|B^{t_1}\right\|_{t_2}^{t_1}\right)\right].$$

For any $\varepsilon > 0$, there exists $h > 0$ such that $\rho(h) \leq \frac{\varepsilon}{2C}$ for the above C. Since f, g are bounded, we may assume ρ is also bounded and denote by $\|\rho\|_\infty$ its bound. Now for $d_\infty((t_1, \omega^1), (t_2, \omega^2)) \leq \frac{h}{2}$, we obtain

$$|u_{t_1}^0(\omega^1) - u_{t_2}^0(\omega^2)|$$

$$\leq C d_\infty((t_1, \omega^1), (t_2, \omega^2)) + C\rho(h) + C\|\rho\|_\infty \mathbb{P}_0^{t_1}\left[\|B^{t_1}\|_{t_2}^{t_1} > \frac{h}{2}\right]$$

$$\leq \frac{\varepsilon}{2} + C d_\infty((t_1, \omega^1), (t_2, \omega^2)) + 4C\|\rho\|_\infty h^{-2}\mathbb{E}_0^{\mathbb{P}_0^{t_1}}\left[(\|B^{t_1}\|_{t_2}^{t_1})^2\right]$$

$$= \frac{\varepsilon}{2} + C d_\infty((t_1, \omega^1), (t_2, \omega^2)) + 4C\|\rho\|_\infty h^{-2}(t_2 - t_1)$$

$$\leq \frac{\varepsilon}{2} + C(1 + 4\|\rho\|_\infty h^{-2})d_\infty((t_1, \omega^1), (t_2, \omega^2)).$$

By choosing $d_\infty((t_1, \omega^1), (t_2, \omega^2))$ small enough, we see that $|u_{t_1}^0(\omega^1) - u_{t_2}^0(\omega^2)| \leq \varepsilon$. This completes the proof. $\square$

However, in general one cannot expect u^0 to be a classical solution to PPDE (3.1). We refer to Peng and Wang [27] for some sufficient conditions, in a slightly different setting.

PROOF OF THEOREM 4.3. We just show that u^0 is a viscosity subsolution. We prove by contradiction. Assume u^0 is not a viscosity subsolution. Then, for all $L > 0$, u^0 is not an L-viscosity subsolution. For the purpose of this proof, it is sufficient to consider an arbitrary $L \geq L_0$, the Lipschitz constant of f introduced in Assumption 4.2(i). Then, there exist

$$(t, \omega) \in [0, T) \times \Omega \quad \text{and} \quad \varphi \in \mathcal{A}^L u^0(t, \omega) \qquad \text{such that } c := (\mathcal{L}^{t,\omega}\varphi)(t, 0) > 0.$$

Denote, for $s \in [t, T]$,

$$\tilde{Y}_s := \varphi(s, B^t), \qquad \tilde{Z}_s := \partial_\omega\varphi(s, B^t),$$

$$\delta Y_s := \tilde{Y}_s - Y_s^{0,t,\omega}, \qquad \delta Z_s := \tilde{Z}_s - Z_s^{0,t,\omega}.$$

Applying Itô's formula, we have

$$d(\delta Y_s) = -\left[(\mathcal{L}^{t,\omega}\varphi)(s, B^t) + f^{t,\omega}(s, B^t_\cdot, \tilde{Y}_s, \tilde{Z}_s) - f^{t,\omega}(s, B^t_\cdot, Y_s^{0,t,\omega}, Z_s^{0,t,\omega})\right]ds$$

$$+ \delta Z_s \, dB_s^t$$

$$= -\left[(\mathcal{L}^{t,\omega}\varphi)(s, B^t_\cdot) + \alpha_s \delta Y_s + \langle\beta, \delta Z\rangle_s\right]ds + \delta Z_s \, dB_s^t, \qquad \mathbb{P}_0^t\text{-a.s.},$$

where $|\alpha| \leq L_0$ and $\beta \in \mathcal{U}_t^{L_0} \subset \mathcal{U}_t^L$. Observing that $\delta Y_t = 0$, we define

$$\tau_0 := T \wedge \inf\left\{s > t : (\mathcal{L}^{t,\omega}\varphi)(s, B^t_\cdot) - L_0|\delta Y_s| \leq \frac{c}{2}\right\}.$$

Then, by Proposition 5.4 and Example 2.5, $\tau_0 \in \mathcal{T}_+^t$ and

(5.5) $$\left(\mathcal{L}^{t,\omega}\varphi\right)(s, B_\cdot^t) + \alpha_s \delta Y_s \geq \frac{c}{2} \qquad \text{for all } s \in [t, \tau_0].$$

Now for any $\tau \in \mathcal{T}^t$ such that $\tau \leq \tau_0$, we have

$$0 = \delta Y_t = \delta Y_\tau + \int_t^\tau \left[\left(\mathcal{L}^{t,\omega}\varphi\right)(s, B^t) + \alpha_s \delta Y_s + \langle \beta, \delta Z \rangle_s\right] ds - \int_t^\tau \delta Z_s \, dB_s^t$$

$$\geq \varphi(\tau, B^t) - u^{0,t,\omega}(\tau, B^t) + \frac{c}{2}(\tau - t) - \int_t^\tau \delta Z_s \left(dB_s^t - \beta_s \, ds\right).$$

Then $\mathcal{E}_t^L[(\varphi - u^{0,t,\omega})(\tau, B^t)] \leq \mathbb{E}_t^{\mathbb{P}^\beta}[(\varphi - u^{0,t,\omega})(\tau, B^t)] \leq 0$. This contradicts with $\varphi \in \mathcal{A}^L u^0(t, \omega)$. $\square$

Following similar arguments, one can easily prove the following:

PROPOSITION 5.5.   *Under Assumption 4.2, a bounded classical subsolution* (*resp., supersolution*) *of the PPDE* (3.1) *must be a viscosity subsolution* (*resp., supersolution*).

5.3. *Stability of viscosity solutions.*

PROOF OF THEOREM 4.1.   We shall prove only the viscosity subsolution property by contradiction. By Remark 3.6, without loss of generality we assume there exists $\varphi \in \underline{\mathcal{A}}'^L u(0, \mathbf{0})$ such that $c := \mathcal{L}\varphi(0, \mathbf{0}) > 0$, where $\underline{\mathcal{A}}'^L u(0, \mathbf{0})$ is defined in (3.7).

Denote

(5.6)
$$X^0 := \varphi - u, \qquad X^\varepsilon := \varphi - u^\varepsilon \quad \text{and}$$
$$\tau_0 := \inf\left\{t > 0 : \mathcal{L}\varphi(t, B) \leq \frac{c}{2}\right\} \wedge T.$$

Since $f \in C^0(\Lambda)$, it follows from Example 2.5 that $\tau_0 \in \mathcal{T}_+^0$. By (3.7), there exists $\tau_1 \in \mathcal{T}_+^0$ such that $\tau_1 \leq \tau_0$ and

$$\mathcal{E}_0^L\left(\tau_1, X_{\tau_1}^0\right) > 0 = X_0^0.$$

Since u^ε converges toward u uniformly, we have

(5.7) $$\mathcal{E}_0^L\left(\tau_1, X_{\tau_1}^\varepsilon\right) > X_0^\varepsilon \qquad \text{for sufficiently small } \varepsilon > 0.$$

Consider the optimal stopping problem, under nonlinear expectation, together with the corresponding optimal stopping rule,

(5.8) $$Y_t := Y_t^\varepsilon := \inf_{\tau \in \mathcal{T}^t} \mathcal{E}_t^L\left[X_{\tau \wedge \tau_1}^\varepsilon\right] \quad \text{and} \quad \tau_0^* := \inf\{t \geq 0 : Y_t = X_t^\varepsilon\};$$

see Remark 3.11. We claim that

$$\mathbb{P}_0\big[\tau_0^* < \tau_1\big] > 0, \tag{5.9}$$

because otherwise $X_0^\varepsilon \geq Y_0 = \mathcal{E}_0^L[X_{\tau_1}^\varepsilon]$, contradicting (5.7).

Since X^ε and Y are continuous, $\mathbb{P}_0$-a.s. there exists $E \subset \{\tau_0^* < \tau_1\}$ such that $\mathbb{P}_0(E) = \mathbb{P}_0(\tau_0^* < \tau_1) > 0$, and for any $\omega \in E$, denoting $t := \tau_0^*(\omega)$ we have $X_t^\varepsilon(\omega) = Y_t(\omega)$. Notice that $\tau_1^{t,\omega} \in \mathcal{T}_+^t$. By standard arguments using the regular conditional probability distributions (see, e.g., [30] or [28]), it follows from the definition of τ_0^* together with the $\mathcal{E}^L$-submartingale property of Y that

$$X_t^\varepsilon(\omega) = Y_t(\omega) = Y_t^{t,\omega}(\omega) \leq \mathcal{E}_t^L\big[Y_\tau^{t,\omega}\big] \leq \mathcal{E}_t^L\big[X_\tau^{\varepsilon,t,\omega}\big] \qquad \text{for all } \tau \in \mathcal{T}^t, \tau \leq \tau_1^{t,\omega}.$$

This implies that

$$0 \leq \mathcal{E}_t^L\big[X_\tau^{\varepsilon,t,\omega} - X_t^\varepsilon(\omega)\big]$$
$$= \mathcal{E}_t^L\big[\varphi_\tau^{t,\omega} - \varphi(t,\omega) + u^\varepsilon(t,\omega) - u_\tau^{\varepsilon,t,\omega}\big] \qquad \text{for all } \tau \in \mathcal{T}, \tau \leq \tau_1^{t,\omega}.$$

Define

$$\varphi_s^\varepsilon := \varphi_s^{t,\omega} - \varphi(t,\omega) + u^\varepsilon(t,\omega).$$

Then we have $\varphi^\varepsilon \in \underline{\mathcal{A}}^L u^\varepsilon(t,\omega)$. Since u^ε is a viscosity L-subsolution of PPDE (3.1) with coefficients f^ε, we have

$$0 \geq -\partial_t \varphi^\varepsilon(t,0) - \frac{1}{2}\text{tr}[\partial_{\omega\omega}^2 \varphi^\varepsilon](t,0) - f^\varepsilon\big(t,\omega,\varphi^\varepsilon(t,0),\partial_\omega \varphi^\varepsilon(t,0)\big)$$

$$= -\partial_t \varphi(t,\omega) - \frac{1}{2}\text{tr}[\partial_{\omega\omega}^2 \varphi](t,\omega) - f^\varepsilon\big(t,\omega,u^\varepsilon(t,\omega),\partial_\omega \varphi(t,\omega)\big)$$

$$= (\mathcal{L}\varphi)(t,\omega) + f\big(t,\omega,u(t,\omega),\partial_\omega \varphi(t,\omega)\big) - f^\varepsilon\big(t,\omega,u^\varepsilon(t,\omega),\partial_\omega \varphi(t,\omega)\big)$$

$$\geq \frac{c}{2} + f\big(t,\omega,u(t,\omega),\partial_\omega \varphi(t,\omega)\big) - f^\varepsilon\big(t,\omega,u^\varepsilon(t,\omega),\partial_\omega \varphi(t,\omega)\big),$$

thanks to (5.6). Send $\varepsilon \to 0$, we obtain $0 \geq \frac{c}{2}$, a contradiction. $\square$

REMARK 5.6. (i) We need the same L in the proof of Theorem 4.1. If u^ε is only a viscosity subsolution of PPDE (3.1) with coefficient f^ε, but with possibly different L_ε, we are not able to show that u is a viscosity subsolution of PPDE (3.1) with coefficients (3.1).

(ii) However, if u^ε is a viscosity solution of PPDE (3.1) with coefficient f^ε, by Theorems 4.3 and 4.6, it follows immediately from the stability of BSDEs that u is the unique viscosity solution of PPDE (3.1) with coefficient f.

5.4. *Partial comparison principle.* The following partial comparison principle, which improves Lemma 5.3, is crucial for this paper. The main argument is very much similar to that of Theorem 4.1.

LEMMA 5.7.   *Let Assumption* 4.2 *hold true. Let* u^1 *be a viscosity subsolution and* u^2 *a viscosity supersolution of PPDE* (3.1). *If* $u^1(T, \cdot) \leq u^2(T, \cdot)$ *and one of* u^1 *and* u^2 *is in* $C_b^{1,2}(\Lambda)$, *then* $u^1 \leq u^2$ *on* Λ.

PROOF.   First, by Remark 3.9(i), by otherwise changing the variable we may assume without loss of generality that

(5.10) f is strictly decreasing in y.

We assume $u^2 \in C_b^{1,2}(\Lambda)$ and u^1 is a viscosity L-subsolution for some $L \geq 0$. We shall prove by contradiction. Without loss of generality, we assume

$$-c := u_0^2 - u_0^1 < 0.$$

For future purposes, we shall obtain the contradiction under the following slightly weaker assumptions:

(5.11)
$$u^2 \in \bar{C}_{\mathbb{P}_0}^{1,2}(\Lambda) \qquad \text{bounded} \quad \text{and}$$
$$(\mathcal{L}u^2) \geq 0, \qquad u^2(T, \cdot) \geq u^1(T, \cdot), \qquad \mathbb{P}_0\text{-a.s.}$$

Denote

$$X := u^2 - u^1 \quad \text{and} \quad \tau_0 := \inf\{t > 0 : X_t \geq 0\} \wedge T.$$

Note that $X_0 = -c < 0$, $X_T \geq 0$, and X is continuous, $\mathbb{P}_0$-a.s. Then

(5.12) $\tau_0 > 0, \qquad X_t < 0, \qquad t \in [0, \tau_0), \quad \text{and} \quad X_{\tau_0} = 0, \qquad \mathbb{P}_0\text{-a.s.}$

Similar to Remark 3.11, define the process Y by the optimal stopping problem under nonlinear expectation,

$$Y_t := \inf_{\tau \in \mathcal{T}^t} \mathcal{E}_t^L[X_{\tau \wedge \tau_0}], \qquad t \in [0, \tau_0],$$

together with the corresponding optimal stopping rule,

$$\tau_0^* := \inf\{t \geq 0 : Y_t = X_t\}.$$

Then $\tau_0^* \leq \tau_0$, and we claim that

(5.13) $\mathbb{P}_0[\tau_0^* < \tau_0] > 0,$

because otherwise $X_0 \geq Y_0 = \mathcal{E}_0^L[X_{\tau_0}]$, contradicting (5.12).

As in the proof of Theorem 4.1, there exists $E \subset \{\tau_0^* < \tau_0\}$ such that $\mathbb{P}_0(E) = \mathbb{P}_0[\tau_0^* < \tau_0] > 0$, and for any $\omega \in E$, by denoting $t := \tau_0^*(\omega)$ we have $\tau_0^{t,\omega} \in \mathcal{T}_+^t$ and

$$X_t(\omega) = Y_t(\omega) = \inf_{\tau \in \mathcal{T}^t} \mathcal{E}_t^L[X_{\tau \wedge \tau_0^{t,\omega}}], \qquad \mathbb{P}_0^t\text{-a.s.}$$

Let $\{\tau_i, i \geq 0\}$ be the sequence of stopping times in Definition 2.8 corresponding to u^2. Then $\mathbb{P}_0[\{\tau_0^* < \tau_i\} \cap E] > 0$ for i large enough, and thus there exists $\omega \in E$ such that $t := \tau_0^*(\omega) < \tau_i(\omega)$. Without loss of generality, we assume $\tau_{i-1}(\omega) \leq t$. It is clear that $(\tau_0 \wedge \tau_i)^{t,\omega} \in \mathcal{T}_+^t$ and $(u^2)^{t,\omega} \in C_b^{1,2}(\bar{\Lambda}^t((\tau_0 \wedge \tau_i)^{t,\omega}))$. In particular, there exists $\tilde{u}^2 \in C_b^{1,2}(\Lambda^t)$ such that $(u^2)^{t,\omega} = \tilde{u}^2$ on $(\tau_0 \wedge \tau_i)^{t,\omega}$.

Now for any $\tau \in \mathcal{T}_+^t$ such that $\tau \leq (\tau_0 \wedge \tau_i)^{t,\omega}$, it follows from Remark 3.11 that

$$X_t(\omega) = Y_t(\omega) = Y_t^{t,\omega} \leq \mathcal{E}_t^L[Y_\tau^{t,\omega}] \leq \mathcal{E}_t^L[X_\tau^{t,\omega}].$$

Thus

$$0 \leq \mathcal{E}_t^L[(\tilde{u}^2)_\tau^{t,\omega} - (u^1)_\tau^{t,\omega} - X_t(\omega)].$$

Denote $\varphi_s := (\tilde{u}^2)_s^{t,\omega} - X_t(\omega)$, $s \in [t, T]$. Then $\varphi \in \mathcal{A}^L u^1(t, \omega)$. Since u^1 is a viscosity L-subsolution, and u^2 is a classical supersolution, we have

$$0 \geq (\mathcal{L}\varphi)(t, 0) = -\partial_t \tilde{u}^2(t, 0) - \tfrac{1}{2} \text{tr}[\partial_{\omega\omega}^2 \tilde{u}^2(t, 0)] - f(t, \omega, u^1(t, \omega), \partial_\omega \tilde{u}^2(t, 0))$$

$$= -\partial_t u^2(t, \omega) - \tfrac{1}{2} \text{tr}[\partial_{\omega\omega}^2 u^2(t, \omega)] - f(t, \omega, u^1(t, \omega), \partial_\omega u^2(t, \omega))$$

$$= (\mathcal{L}u^2)(t, \omega) + f(t, \omega, u^2(t, \omega), \partial_\omega u^2(t, \omega)) - f(t, \omega, u^1(t, \omega), \partial_\omega u^2(t, \omega))$$

$$\geq f(t, \omega, u^2(t, \omega), \partial_\omega u^2(t, \omega)) - f(t, \omega, u^1(t, \omega), \partial_\omega u^2(t, \omega)).$$

By (5.12), $u^2(t, \omega) < u^1(t, \omega)$. Then the above inequality contradicts with (5.10). $\square$

6. A variation of Perron's approach.

To prove Theorem 4.6, we define

$$\bar{u}(t, \omega) := \inf\{\varphi(t, \mathbf{0}) : \varphi \in \bar{\mathcal{D}}(t, \omega)\},$$

(6.1)

$$\underline{u}(t, \omega) := \sup\{\varphi(t, \mathbf{0}) : \varphi \in \underline{\mathcal{D}}(t, \omega)\},$$

where, in light of (5.11),

$$\bar{\mathcal{D}}(t, \omega) := \{\varphi \in \bar{C}_{\mathbb{P}_0^t}^{1,2}(\Lambda^t) \text{ bounded: } (\mathcal{L}\varphi)_s^{t,\omega} \geq 0,$$

$$s \in [t, T] \text{ and } \varphi_T \geq g^{t,\omega}, \mathbb{P}_0^t\text{-a.s.}\};$$

(6.2)

$$\underline{\mathcal{D}}(t, \omega) := \{\varphi \in \bar{C}_{\mathbb{P}_0^t}^{1,2}(\Lambda^t) \text{ bounded: } (\mathcal{L}\varphi)_s^{t,\omega} \leq 0,$$

$$s \in [t, T] \text{ and } \varphi_T \leq g^{t,\omega}, \mathbb{P}_0^t\text{-a.s.}\}.$$

By Lemma 5.7, in particular by its proof under the weaker condition (5.11), it is clear that

(6.3) $$\underline{u} \leq u^0 \leq \bar{u}.$$

The following result is important for our proof of Theorem 4.6.

THEOREM 6.1. *Let Assumptions* 4.2 *and* 4.4 *hold true. Then*

$$(6.4) \qquad\qquad \underline{u} = \bar{u}.$$

PROOF OF THEOREM 4.6.   By Lemma 5.7, in particular by its proof under the weaker condition (5.11), we have $u^1 \leq \bar{u}$ and $\underline{u} \leq u^2$. Then Theorem 6.1 implies that $u^1 \leq u^2$.

This clearly leads to the uniqueness of viscosity solution, and therefore, by Theorem 4.3 u^0 is the unique viscosity solution of PPDE (3.1) with terminal condition g. □

REMARK 6.2.   In standard Perron's method, one shows that $\underline{u}$ (resp., $\bar{u}$) is a viscosity super-solution (resp., viscosity sub-solution) of the PDE. Assuming that the comparison principle for viscosity solutions holds true, then (6.4) holds.

In our situation, we shall instead prove (6.4) directly first, which in turn is used to prove the comparison principle for viscosity solutions. Roughly speaking, the comparison principle for viscosity solutions is more or less equivalent to the partial comparison principle Lemma 5.7 and the equality (6.4). To our best knowledge, such an approach is novel in the literature.

We decompose the proof of Theorem 6.1 into several lemmas. First, let $t < T$ and $\theta \in (C_b^0(\Lambda^t))^d$ satisfy

$$(6.5) \qquad \text{there exists } \hat{\theta} \in (C_b^0(\hat{\Lambda}^t))^d \text{ such that } \theta = \hat{\theta} \text{ in } \Lambda \text{ and } \hat{\theta} \text{ is uniformly continuous in } \hat{\omega} \text{ under the uniform norm } \|\cdot\|_T^t.$$

Define

$$(6.6) \qquad Z_s = z + \int_t^s \theta_r \, dr, \qquad v_s := \int_t^s Z_s \, dB_s^t, \qquad t \leq s \leq T, \mathbb{P}_0^t\text{-a.s.}$$

By Itô's formula, we have

$$v_s = Z_s B_s^t - \int_t^s \theta_r B_r^t \, dr.$$

Denote

$$(6.7) \qquad \begin{aligned} \hat{Z}_s(\hat{\omega}) &:= z + \int_t^s \hat{\theta}_r(\hat{\omega}) \, dr, \\ \hat{v}(s, \hat{\omega}) &:= \hat{Z}_s(\hat{\omega})\hat{\omega}_s - \int_t^s \hat{\theta}_r(\hat{\omega})\hat{\omega}_r \, dr, \qquad \hat{\omega} \in \hat{\Omega}^t. \end{aligned}$$

Now for any $\omega \in \Omega$ and $x \in \mathbb{R}$, let $\hat{u}^{t,\omega}$ denote the unique solution to the following ODE (with random coefficients) on $[t, T]$:

$$(6.8) \qquad \begin{aligned} \hat{u}^{t,\omega}(s, \hat{\omega}) &:= x - \int_t^s \hat{f}^{t,\omega}\big(r, \hat{\omega}, \hat{u}^{t,\omega}(r, \hat{\omega}), \hat{Z}_r(\hat{\omega})\big) \, dr \\ &\quad + \hat{v}(s, \hat{\omega}), \qquad t \leq s \leq T, \hat{\omega} \in \hat{\Omega}^t, \end{aligned}$$

and define

(6.9) $$u^{t,\omega}(s, \tilde{\omega}) := \hat{u}^{t,\omega}(s, \tilde{\omega}) \qquad \text{for } (s, \tilde{\omega}) \in \Lambda^t.$$

We then have the following:

LEMMA 6.3. *Let Assumptions* 4.2 *and* (6.5) *hold true. Then for each* $(t, \omega) \in \Lambda$, *the above* $u^{t,\omega} \in C_b^{1,2}(\Lambda^t)$ *and* $\mathcal{L}^{t,\omega} u^{t,\omega} = 0$.

PROOF. We first show that $\hat{u}^{t,\omega} \in C_b^{1,2}(\hat{\Lambda}^t)$, which implies that $u^{t,\omega} \in C_b^{1,2}(\Lambda^t)$. For $t \le s_1 < s_2 \le T$ and $\hat{\omega}^1, \hat{\omega}^2 \in \hat{\Omega}^t$, we have

$$|\hat{Z}_{s_1}(\hat{\omega}^1) - \hat{Z}_{s_2}(\hat{\omega}^2)| \le \int_{s_1}^{s_2} |\hat{\theta}_r(\hat{\omega}^1)| \, dr + \int_t^{s_1} |\hat{\theta}_r(\hat{\omega}^1) - \hat{\theta}_r(\hat{\omega}^2)| \, dr$$

$$\le C[s_2 - s_1] + \int_t^{s_1} |\hat{\theta}_r(\hat{\omega}^1) - \hat{\theta}_r(\hat{\omega}^2)| \, dr.$$

Note that $d_\infty^t((r, \hat{\omega}^1), (r, \hat{\omega}^2)) \le d_\infty^t((s_1, \hat{\omega}^1), (s_2, \hat{\omega}^2))$ for $t \le r \le s_1$. Then one can easily see that $\hat{Z} \in C_b^0(\hat{\Lambda}^t)$. Similarly one can show that $\hat{v}, \hat{u}^{t,\omega} \in C^0(\hat{\Lambda}^t)$.

Next, one can easily check that, for all $\hat{\omega} \in \hat{\Omega}^t$,

$$\partial_t \hat{Z}_s(\hat{\omega}) = \hat{\theta}_s(\hat{\omega}), \qquad \partial_\omega \hat{Z}_s(\hat{\omega}) = \mathbf{0};$$

$$\partial_t \hat{v}(s, \hat{\omega}) = \hat{\theta}_s(\hat{\omega})\hat{\omega}_s - \hat{\theta}_s(\hat{\omega})\hat{\omega}_s = 0,$$

$$\partial_\omega \hat{v}(s, \hat{\omega}) = \hat{Z}_s(\hat{\omega}), \qquad \partial_{\omega\omega}^2 \hat{v}(s, \hat{\omega}) = \mathbf{0};$$

$$\partial_t \hat{u}^{t,\omega}(s, \hat{\omega}) = -\hat{f}^{t,\omega}(s, \hat{\omega}, \hat{u}^{t,\omega}(s, \hat{\omega}), \hat{Z}_s(\hat{\omega})),$$

$$\partial_\omega \hat{u}^{t,\omega}(s, \hat{\omega}) = \hat{Z}_s(\hat{\omega}), \qquad \partial_{\omega\omega}^2 \hat{u}^{t,\omega}(s, \hat{\omega}) = \mathbf{0}.$$

Since $\hat{\theta}$ and $\hat{f}$ are bounded, it is straightforward to see that $\hat{u}^{t,\omega} \in C_b^{1,2}(\hat{\Lambda}^t)$.

Finally, from the above derivatives we see immediately that $\mathcal{L}^{t,\omega} u^{t,\omega} = 0$. □

Our next two lemmas rely heavily on the remarkable result Bank and Baum [1], which is extended to BSDE case in [28].

LEMMA 6.4. *Let Assumption* 4.2 *hold true. Let* $\tau \in \mathcal{T}$, Z *be* $\mathbb{F}$-*progressively measurable such that* $\mathbb{E}^{\mathbb{P}_0}[\int_\tau^T |Z_s|^2 \, ds] < \infty$, *and* $X_\tau, \tilde{X}_\tau \in L^2(\mathcal{F}_\tau, \mathbb{P}_0)$. *For any* $\varepsilon > 0$, *there exists* $\mathbb{F}$-*progressively measurable process* Z^ε *such that*:

(i) *For the Lipschitz constant* L_0 *in Assumption* 4.2(ii), *it holds that*

(6.10) $$\mathbb{P}_0\Big[\sup_{\tau \le t \le T} e^{-L_0 t} |X_t^\varepsilon - X_t| \ge \varepsilon + e^{-L_0 \tau} |\tilde{X}_\tau - X_\tau|\Big] \le \varepsilon,$$

where X, X^ε are the solutions to the following ODEs with random coefficients,

$$X_t = X_\tau - \int_\tau^t f(s, B, X_s, Z_s) \, ds + \int_\tau^t Z_s \, dB_s,$$

(6.11) $$X_t^\varepsilon = \tilde{X}_\tau - \int_\tau^t f(s, B, X_s^\varepsilon, Z_s^\varepsilon) \, ds$$

$$+ \int_\tau^t Z_s^\varepsilon \, dB_s, \qquad \tau \le t \le T, \mathbb{P}_0\text{-}a.s.;$$

(ii) $\theta_t^\varepsilon := \frac{d}{dt} Z_t^\varepsilon$ *exists for* $t \in [\tau, T)$, *where* θ_τ^ε *is understood as the right derivative, and for each* $\omega \in \Omega$, $(\theta^\varepsilon)^{\tau(\omega),\omega}$ *satisfies* (6.5) *with* $t := \tau(\omega)$.

PROOF.   First, let $h := h_\varepsilon > 0$ be a small number which will be specified later. By standard arguments there exists a time partition $0 = t_0 < \cdots < t_n = T$ and a smooth function $\psi : [0, T] \times \mathbb{R}^{n \times d} \to \mathbb{R}^d$ such that ψ and its derivatives are bounded and

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_\tau^T |\tilde{Z}_t - Z_t|^2 \, dt \right] < h$$

(6.12)
          where $\tilde{Z}_t(\omega) := \psi(t, \omega_{t_1 \wedge t}, \ldots, \omega_{t_n \wedge t})$ for all $(t, \omega) \in \Lambda$.

Next, for some $\tilde{h} := \tilde{h}_\varepsilon > 0$ which will be specified later, denote

(6.13) $$Z_t^\varepsilon := \frac{1}{\tilde{h}} \int_{t-\tilde{h}}^t \tilde{Z}_{\tau \vee s} \, ds \qquad \text{for } t \in [\tau, T].$$

By choosing $\tilde{h} > 0$ small enough (which may depend on h_ε), we have

(6.14) $$\mathbb{E}^{\mathbb{P}_0}\left[\int_\tau^T |Z_t^\varepsilon - Z_t|^2 \, dt \right] < 2h.$$

Now denote

$$\delta Z^\varepsilon := Z^\varepsilon - Z, \qquad \delta X^\varepsilon := X^\varepsilon - X.$$

Then

$$\delta X_t^\varepsilon = \delta X_\tau^\varepsilon - \int_\tau^t \left[\alpha_s \delta X_s^\varepsilon + \langle \beta, \delta Z^\varepsilon \rangle_s \right] ds + \int_\tau^t \delta Z_s^\varepsilon \, dB_s,$$

where $|\alpha| \le L_0$ and $\beta \in \mathcal{U}_t^{L_0}$. Denote $\Gamma_t^\varepsilon := \exp(\int_\tau^t \alpha_s \, ds)$. We get

$$\Gamma_t^\varepsilon \delta X_t^\varepsilon = \delta X_\tau^\varepsilon - \int_\tau^t \Gamma_s^\varepsilon \langle \beta, \delta Z^\varepsilon \rangle_s \, ds + \int_\tau^t \Gamma_s^\varepsilon \delta Z_s^\varepsilon \, dB_s.$$

Then

$$0 \leq \sup_{\tau \leq t \leq T} e^{-L_0 t} |\delta X_t^\varepsilon| - e^{-L_0 \tau} |\delta X_\tau^\varepsilon| \leq e^{-L_0 \tau} \Big[\sup_{\tau \leq t \leq T} \Gamma_t^\varepsilon |\delta X_t^\varepsilon| - |\delta X_\tau^\varepsilon| \Big]$$

$$\leq \sup_{\tau \leq t \leq T} |\Gamma_t^\varepsilon \delta X_t^\varepsilon - \delta X_\tau^\varepsilon| = \sup_{\tau \leq t \leq T} \left| -\int_\tau^t \Gamma_s^\varepsilon \langle \beta, \delta Z^\varepsilon \rangle_s \, ds + \int_\tau^t \Gamma_s^\varepsilon \delta Z_s^\varepsilon \, dB_s \right|$$

$$\leq C \int_\tau^T |\delta Z_s^\varepsilon| \, ds + \sup_{\tau \leq t \leq T} \left| \int_\tau^t \Gamma_s^\varepsilon \delta Z_s^\varepsilon \, dB_s \right|.$$

Thus

$$\mathbb{P}_0 \Big[\sup_{\tau \leq t \leq T} e^{-L_0 t} |X_t^\varepsilon - X_t| \geq \varepsilon + e^{-L_0 \tau} |\tilde{X}_\tau - X_\tau| \Big]$$

$$= \mathbb{P}_0 \Big[\sup_{\tau \leq t \leq T} e^{-L_0 t} |X_t^\varepsilon - X_t| - e^{-L_0 \tau} |\tilde{X}_\tau - X_\tau| \geq \varepsilon \Big]$$

$$\leq \mathbb{P}_0 \Big[C \int_\tau^T |\delta Z_s^\varepsilon| \, ds + \sup_{\tau \leq t \leq T} \left| \int_\tau^t \Gamma_s^\varepsilon \delta Z_s^\varepsilon \, dB_s \right| \geq \varepsilon \Big]$$

$$\leq \frac{C}{\varepsilon^2} \mathbb{E}^{\mathbb{P}_0} \Big[\Big(\int_\tau^T |\delta Z_s^\varepsilon| \, ds \Big)^2 + \sup_{\tau \leq t \leq T} \left| \int_\tau^t \Gamma_s^\varepsilon \delta Z_s^\varepsilon \, dB_s \right|^2 \Big]$$

$$\leq \frac{C}{\varepsilon^2} \mathbb{E}^{\mathbb{P}_0} \Big[\int_\tau^T |\delta Z_s^\varepsilon|^2 \, ds \Big] \leq \frac{Ch}{\varepsilon^2},$$

thanks to (6.14). Now set $h := \frac{\varepsilon^3}{C}$, and we prove (6.10).

Finally, by (6.13) and (6.12) we have

$$\theta_s^\varepsilon = \frac{1}{\tilde{h}} [\tilde{Z}_s - \tilde{Z}_{(s-\tilde{h}) \vee \tau}], \qquad s \in [\tau, T].$$

Fix $\omega \in \Omega$ and set $t := \tau(\omega)$. For each $\hat{\omega} \in \hat{\Omega}^t$, set $\bar{\omega} := \omega \otimes_t \hat{\omega} \in \hat{\Omega}$, and define

$$\hat{Z}_s^{t,\omega}(\hat{\omega}) := \psi(s, \bar{\omega}_{t_1 \wedge s}, \dots, \bar{\omega}_{t_n \wedge s}),$$

$$(\theta^\varepsilon)_s^{t,\omega}(\hat{\omega}) := \frac{1}{\tilde{h}} [\hat{Z}_s^{t,\omega}(\hat{\omega}) - \hat{Z}_{(s-\tilde{h}) \vee t}^{t,\omega}(\hat{\omega})], \qquad s \in [\tau, T].$$

Then we can easily check that $(\theta^\varepsilon)^{t,\omega}$ satisfies (6.5). $\quad\square$

LEMMA 6.5. *Assume Assumption 4.2 holds. Let $x \in \mathbb{R}$ and Z be $\mathbb{F}$-progressively measurable such that $\mathbb{E}^{\mathbb{P}_0}[\int_0^T |Z_s|^2 \, ds] < \infty$. For any $\varepsilon > 0$, there exists $\mathbb{F}$-progressively measurable process Z^ε and an increasing sequence of $\mathbb{F}$-stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq T$ such that:*

(i) *It holds that*

$$\text{(6.15)} \qquad \sup_{0 \le t \le T} |X_t^\varepsilon - X_t| \le \varepsilon, \qquad \mathbb{P}_0\text{-}a.s.,$$

where X, X^ε *are the solutions to the following ODEs with random coefficients*

$$X_t = x - \int_0^t f(s, B, X_s, Z_s)\, ds + \int_0^t Z_s\, dB_s,$$

$$\text{(6.16)} \qquad X_t^\varepsilon = x - \int_0^t f(s, B, X_s^\varepsilon, Z_s^\varepsilon)\, ds$$

$$+ \int_0^t Z_s^\varepsilon\, dB_s, \qquad 0 \le t \le T, \mathbb{P}_0\text{-}a.s.$$

(ii) *For each* i, $\theta_t^\varepsilon := \frac{d}{dt} Z_t^\varepsilon$ *exists for* $t \in [\tau_i, \tau_{i+1})$, *where* θ_τ^ε *is understood as the right derivative. Moreover, there exists* $\tilde\theta^{i,\varepsilon}$ *on* $[\tau_i, T]$ *such that* $\tilde\theta_t^{i,\varepsilon} = \theta_t^\varepsilon$ *for* $t \in [\tau_i, \tau_{i+1})$, *and for each* $\omega \in \Omega$, $(\theta^{i,\varepsilon})^{\tau_i(\omega),\omega}$ *satisfies* (6.5) *with* $t := \tau_i(\omega)$.

(iii) *For* $\mathbb{P}_0$-*a.s.* $\omega \in \Omega$, *for each* i, $\tau_i < \tau_{i+1}$ *whenever* $\tau_i < T$, *and the set* $\{i : \tau_i(\omega) < T\}$ *is finite.*

PROOF.   Let $\varepsilon > 0$ be fixed, and set $\varepsilon_i := 2^{-i-2} e^{-L_0 T} \varepsilon$, $i \ge 0$. We construct τ_i and $(Z^{i,\varepsilon}, X^{i,\varepsilon})$ by induction as follows.

First, for $i = 0$, set $\tau_0 := 0$. Apply Lemma 6.4 with initial time τ_0, initial value x and error level ε_0, we can construct $Z^{0,\varepsilon}$ and $X^{0,\varepsilon}$ on $[\tau_0, T]$ satisfying the properties in Lemma 6.4. In particular,

$$\mathbb{P}_0\Big[\sup_{\tau_0 \le t \le T} e^{-L_0 t} |X_t^{0,\varepsilon} - X_t| \ge \varepsilon_0 \Big] \le \varepsilon_0.$$

Denote

$$\text{(6.17)} \qquad \tau_1 := \inf\{t \ge \tau_0 : e^{-L_0 t} |X_t^{0,\varepsilon} - X_t| \ge \varepsilon_0\} \wedge T.$$

Since X and $X^{0,\varepsilon}$ are continuous, we have $\tau_1 > \tau_0$, $\mathbb{P}_0$-a.s. We now define

$$Z_t^\varepsilon := Z_t^{0,\varepsilon}, \qquad X_t^\varepsilon := X_t^{0,\varepsilon}, \qquad t \in [\tau_0, \tau_1).$$

Assume we have defined τ_i, Z^ε, X^ε on $[\tau_0, \tau_i)$ and $X^{i-1,\varepsilon}$ on $[\tau_{i-1}, T]$. Apply Lemma 6.4 with initial time τ_i, initial value $X_{\tau_i}^{i-1,\varepsilon}$ and error level ε_i, we can construct $Z^{i,\varepsilon}$ and $X^{i,\varepsilon}$ on $[\tau_i, T]$ satisfying the properties in Lemma 6.4. In particular,

$$\mathbb{P}_0\Big[\sup_{\tau_i \le t \le T} e^{-L_0 t} |X_t^{i,\varepsilon} - X_t| \ge \varepsilon_i + e^{-L_0 \tau_i} |X_{\tau_i}^{i-1,\varepsilon} - X_{\tau_i}| \Big] \le \varepsilon_i.$$

Denote

$$\tau_{i+1} := \inf\{t \ge \tau_i : e^{-L_0 t} |X_t^{i,\varepsilon} - X_t| \ge \varepsilon_i + e^{-L_0 \tau_i} |X_{\tau_i}^{i-1,\varepsilon} - X_{\tau_i}|\} \wedge T.$$

Since X and $X^{i,\varepsilon}$ are continuous, we have $\tau_{i+1} > \tau_i$ whenever $\tau_i < T$. We then define

$$Z_t^\varepsilon := Z_t^{i,\varepsilon}, \qquad X_t^\varepsilon := X_t^{i,\varepsilon}, \qquad t \in [\tau_i, \tau_{i+1}).$$

From our construction we have $\mathbb{P}_0(\tau_{i+1} < T) \leq \varepsilon_i$. Then

$$\sum_{i=0}^\infty \mathbb{P}_0(\tau_{i+1} < T) \leq \sum_{i=0}^\infty \varepsilon_i < \infty.$$

It follows from the Borel–Cantelli lemma that the set $\{i : \tau_i(\omega) < T\}$ is finite, for $\mathbb{P}_0$-a.s. $\omega \in \Omega$, which proves (iii).

We thus have defined Z^ε, X^ε on $[0, T]$, and the statements in (ii) follow directly from Lemma 6.4. So it remains to prove (i). For each i, by the definition of τ_i we see that

$$e^{-L_0\tau_{i+1}}\left|X_{\tau_{i+1}}^\varepsilon - X_{\tau_{i+1}}\right| \leq \varepsilon_i + e^{-L_0\tau_i}\left|X_{\tau_i}^\varepsilon - X_{\tau_i}\right|, \qquad \mathbb{P}_0\text{-a.s.}$$

Since $X_{\tau_0}^\varepsilon = X_{\tau_0} = x$, by induction we get

$$\sup_i e^{-L_0\tau_i}\left|X_{\tau_i}^\varepsilon - X_{\tau_i}\right| \leq \sum_{i=0}^\infty \varepsilon_i \leq \sum_{i=0}^\infty 2^{-i-2}e^{-L_0T}\varepsilon = \frac{1}{2}e^{-L_0T}\varepsilon, \qquad \mathbb{P}_0\text{-a.s.}$$

Then for each i,

$$\sup_{\tau_i \leq t \leq \tau_{i+1}} \left|X_t^\varepsilon - X_t\right| \leq e^{L_0T}\left[\varepsilon_i + \left|X_{\tau_i}^\varepsilon - X_{\tau_i}\right|\right]$$

$$\leq e^{L_0T}\left[2^{-i-2}e^{-L_0T}\varepsilon + \frac{1}{2}e^{-L_0T}\varepsilon\right] \leq \varepsilon, \qquad \mathbb{P}_0\text{-a.s.,}$$

which implies (6.15). $\square$

PROOF OF THEOREM 6.1.   Without loss of generality, we shall only prove $\bar{u}_0 = u_0^0$. Recall that (Y^0, Z^0) is the solution to BSDE (4.1). Set $Z := Z^0$ and $x := Y_0^0$ in Lemma 6.5, we see that $X = Y^0 = u^0$ and thus X satisfies the regularity in Proposition 5.4.

From the construction in Lemma 6.5 and then by Lemma 6.4 we see that $\tilde{\theta}_t^{0,\varepsilon} := \frac{d}{dt}Z_t^{0,\varepsilon}$ exists for all $t \in [0, T)$ and satisfies (6.5). Then by Lemma 6.3 we see that $X^{0,\varepsilon} \in C_b^{1,2}(\Lambda)$ and $\mathcal{L}X^{0,\varepsilon} = 0$. This implies that, for the τ_1 defined in (6.17), $\tau_1(\omega) > \tau_0$ for all $\omega \in \Omega$ and, by Example 2.5, $\tau_1 \in \mathcal{T}_+$.

For $i = 1, 2, \ldots$, repeat the above arguments and by induction we can show that, for each i and each $\omega \in \Omega$, $\tau_{i+1}^{\tau_i(\omega),\omega} \in \mathcal{T}_+^{\tau_i(\omega)}$. Moreover, by Lemma 6.5, $\{i : \tau_i < T\}$ is finite, $\mathbb{P}_0$-a.s.

We now let u^ε be the solution to the following ODE:

$$u_t^\varepsilon = X_0^\varepsilon + e^{L_0 T}\varepsilon - \int_0^t f(s, B, u_s^\varepsilon, Z_s^\varepsilon)\, ds + \int_0^t Z_s^\varepsilon\, dB_s.$$

For $i = 0, 1, \ldots,$ by the construction of Z^ε in Lemma 6.5 and following the arguments in Lemma 6.3, one can easily show that

$$(6.18) \qquad u^\varepsilon \in \bar{C}_{\mathbb{P}_0}^{1,2}([0, T]) \quad \text{and} \quad \mathcal{L}u^\varepsilon = 0.$$

Moreover, note that

$$u_t^\varepsilon - X_t^\varepsilon = e^{L_0 T}\varepsilon - \int_0^t \alpha_s[u_s^\varepsilon - X_s^\varepsilon]\, ds,$$

where $|\alpha| \le L_0$. By standard arguments one has

$$\sup_{0 \le t \le T} |u_t^\varepsilon - X_t^\varepsilon| \le e^{2L_0 T}\varepsilon \quad \text{and} \quad u_T^\varepsilon - X_T^\varepsilon \ge e^{-LT}[u_0^\varepsilon - X_0^\varepsilon] = \varepsilon.$$

Therefore, by (6.15) and noting that u^0 is bounded, u^ε is bounded and

$$u_T^\varepsilon(\omega) \ge X_T^\varepsilon(\omega) + \varepsilon \ge X_T(\omega) = Y_T^0(\omega) = g(\omega) \qquad \text{for } \mathbb{P}_0\text{-a.s. } \omega.$$

This, together with (6.18), implies that $u^\varepsilon \in \bar{\mathcal{D}}(0, 0)$. Then, by the definition of $\bar{u}$,

$$\bar{u}_0 \le u_0^\varepsilon = X_0^\varepsilon + e^{L_0 T}\varepsilon \le u_0^0 + \varepsilon + e^{L_0 T}\varepsilon.$$

Since ε is arbitrary, we obtain $\bar{u}_0 \le u_0^0$. $\square$

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