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DUAL FORMULATION OF SECOND ORDER TARGET PROBLEMS

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This paper provides a new formulation of second order stochastic target problems introduced in [*SIAM J. Control Optim.* **48** (2009) 2344–2365] by modifying the reference probability so as to allow for different scales. This new ingredient enables us to prove a dual formulation of the target problem as the supremum of the solutions of standard backward stochastic differential equations. In particular, in the Markov case, the dual problem is known to be connected to a fully nonlinear, parabolic partial differential equation and this connection can be viewed as a stochastic representation for all nonlinear, scalar, second order, parabolic equations with a convex Hessian dependence.

1. Introduction. The connection between the backward stochastic differential equations (BSDE hereafter) and the nonlinear, parabolic partial differential equations (PDE hereafter) is well documented. Indeed, the standard BSDEs, as introduced by Pardoux and Peng [14], are known to provide a stochastic representation for the solutions of semi-linear PDEs in the Markov case. In this representation, the diffusion coefficient of the underlying process is the linear coefficient of the Hessian variable in the PDE. Therefore, the connection to fully nonlinear equations requires an extension that should allow for stochastic processes with different diffusion coefficients. Indeed, [6] develops such a generalization to the second order and also proves a Markovian uniqueness result in an appropriate class. However, no existence theory is available for this generalization, with the one exception in the Markov context. In this case any smooth solution of the related PDE, if it exists, is easily seen to be a solution of the second order BSDE. A closely related class of control problems, called the *second order stochastic target problem*, was introduced in [20] as well.

In this paper we provide a new formulation for the second order stochastic target problems. A better understanding of the target problem is essential for a coherent

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theory of second order BSDEs. Indeed, we develop this theory in our accompanying work [21], including existence and uniqueness results with minimal assumptions.

We continue with the description of the target problem. Let B be a Brownian motion under the probability measure $\mathbb{P}_0$ and $\{\mathcal{F}_t, t \geq 0\}$ be the corresponding filtration. For a continuous semimartingale Z, we denote by Γ the density of its covariation with B. We then define the controlled process Y by

$$Y_t := y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)\, ds + \int_0^t Z_s \circ dB_s, \qquad d\langle Z, B\rangle_t = \Gamma_t\, dt, \tag{1.1}$$

where $\circ$ denotes the Fisk–Stratonovich stochastic integration. We assume that the given random nonlinear function H satisfies the standard Lipschitz and measurability conditions. Then, for any reasonable process Z and an initial condition y, a unique solution, which is denoted by $Y^{y,Z}$, exists. We now fix a time horizon, say, $T = 1$, and a class of admissible controls $\mathcal{Z}^0$. Then, given an $\mathcal{F}_1$ measurable random variable ξ, [20] defines the second order stochastic target problem by

$$\mathcal{V}^0 := \inf\{y : Y_1^{y,Z} \geq \xi \ \mathbb{P}_0\text{-a.s. for some } Z \in \mathcal{Z}^0\}. \tag{1.2}$$

In this formulation, the structure of the set of admissible controls is crucial. In fact, if $\mathcal{Z}^0$ is not properly defined, then the dependence of the problem on the variable Γ can be trivialized. We refer to [3] for a detailed discussion of this issue in a particular example of mathematical finance. One of the achievements of the approach given below is to avoid this strong dependence on the control set and simply to work with standard spaces.

As in many optimization problems, convex duality results provide a deeper understanding and powerful technical tools. Indeed, they are an essential step for the well-posedness of the second order backward stochastic differential equations, as proved in our accompanying paper [21]. Motivated by these, we adopt a new point of view for the target problems which also allows for the construction of the dual. This new formulation differs from that of [20] in two instances. First, we reinforce the constraint $Y_1^{y,Z} \geq \xi$ in (1.2) by requiring that it should hold under various mutually singular measures and not only on the support of $\mathbb{P}_0$. Second, the set of admissible controls utilized here is more natural and, as discussed above, it avoids the technical aspects of [20].

Our reformulation is motivated by the work of Denis and Martini [8] on the deep theory of quasi-sure stochastic analysis. An important related probabilistic notion, introduced by Peng [17], is the G-Brownian motion. Here instead of using these two powerful tools, we employ a direct approach by assuming sufficient regularity. One drawback of all these approaches is the implicit regularity assumption. Indeed, in all these approaches, integrability in any power is possible only if the random variable is quasi-surely continuous. This is a Lusin type of result and is not restrictive when there is only countably many measures. However, in general, this is an additional constraint. In one of our accompanying papers [22], we provide an alternative approach through aggregation of random variables. The general aggregation result of [22] allows us to consider a larger class of random variables, but then the class of probability measures must be slightly restricted.

We believe our approach has several advantages:

- It avoids redeveloping an appropriate theory of stochastic integration from scratch, as it is done in [8] and [17].
- More importantly, a representation theorem is available in our framework as proved in [23].
- Finally, by deriving appropriate estimates, it is shown in [21] that one can extend these concepts to a larger space with regularity conditions. Indeed, a similar extension of G-martingales is given in [7], showing that they cover the same space as in the quasi-sure analysis of [8].

We next provide an intuitive description of our formulation. For this heuristic explanation we assume a Markov structure. Namely, we assume that H in (1.1) and ξ in (1.2) are given by

$$H_t(y,z,\gamma) = h(t, X_t, y, z, \gamma), \qquad \xi = g(X_T), \tag{1.3}$$

where $dX_t = dB_t$ and h, g are deterministic scalar functions. Let $\mathcal{V}^0(t,x)$ be defined as in (1.2) with time origin at t and $X_t = x$. As it is usual, we assume that $\gamma \mapsto h(t,x,y,z,\gamma)$ is nondecreasing. Then, by an appropriate choice of admissible controls $\mathcal{Z}$, it is shown in [20] that this problem is a viscosity solution of the corresponding dynamic programming equation,

$$-\frac{\partial u}{\partial t} - h(t,x,u(t,x), Du(t,x), D^2u(t,x)) = 0, \qquad u(1,x) = g(x). \tag{1.4}$$

We further assume that $\gamma \mapsto h(t,x,r,p,\gamma)$ is convex. Then,

$$h(t,x,r,p,\gamma) = \sup_{a\geq 0}\left\{\frac{1}{2}a\gamma - f(t,x,r,p,a)\right\}, \tag{1.5}$$

where f is the (partial) convex conjugate of h with respect to γ. Let D_f be the domain of f as a function of a. By the classical maximum principle of parabolic differential equations, we expect that, for every $a \in D_f$, the solution $u \geq u^a$, where u solves (1.4) and u^a is defined as the solution of the following semi-linear PDE:

$$\begin{aligned} &-\frac{\partial u}{\partial t} - \frac{1}{2}aD^2u(t,x) + f(t,x,u(t,x),Du(t,x),a) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(1,x) = g(x). \end{aligned} \tag{1.6}$$

In turn, by standard results, $u^a(t,x) = Y_t^a$, where, for $s \in [t,T]$,

$$\begin{aligned} X_s^a &= x + \int_t^s a_r^{1/2}\,dB_r, \\ Y_s^a &= g(X_T^a) - \int_s^T f(r, X_r^a, Y_r^a, Z_r^a, a)\,dr - \int_t^T Z_r^a a^{1/2}\,dB_s. \end{aligned} \tag{1.7}$$

We have formally argued that $\mathcal{V}^0(t,x) \geq Y_t^a$ for any $a \in D_f$. Let $\mathcal{A}^f$ be the collection of all processes with values in D_f. By extending (1.7) to processes a, it is then natural to consider the problem

$$V_t := \sup_{a \in \mathcal{A}^f} Y_t^a \tag{1.8}$$

as the dual of the primal stochastic target problem. Indeed, the optimization problem (1.8) corresponds to the dual formulation of the second order target problem in the Markov case. Such a duality relation was suggested in the specific example of [19] and can be proved rigorously by showing that $v(t,x) := V_t$ is a viscosity solution of the fully nonlinear PDE (1.4). This, by uniqueness, implies that $v = \mathcal{V}^0$. Of course, such an argument requires some technical conditions at least to guarantee that comparison of viscosity supersolutions and subsolutions holds true for the PDE (1.4).

The main object of this paper is to provide a purely probabilistic proof of this duality result. Moreover, our duality result does not require to restrict the problem to the Markov framework.

We should mention that we use weak formulation in our approach, that is, instead of controlling the state process X^a in (1.7), our control is the distribution of X^a on its canonical space. See (2.3) below for the precise definition. Such weak formulation is important for modeling model uncertainty, as in [8] and [17]. In the contexts of stochastic control, which naturally uses strong formulation, some ideas have already appeared in the literature; see, for example, El Karoui and Quenez [10] and Peng [16]. In particular, [16] uses the notion of r.c.p.d. which turns out to be crucial in our approach.

This paper is organized as follows. After introducing the probabilistic structures in the next section, we provide the definition of the stochastic target problem in Section 3. Two relaxations, which are also shown to be equivalent to the original problem, are also introduced in that section. The main duality result is stated and proved in the following section. Section 5 is devoted to a weaker formulation. An extension is outlined in the next section and in the Appendix we provide the proofs of two technical results.

2. The setup. Let $\Omega := \{\omega \in C([0,1], \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0 \leq t \leq 1} |\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq 1}$ the filtration generated by B, and $\mathbb{F}^+ := \{\mathcal{F}_t^+, 0 \leq t \leq 1\}$ the right limit of $\mathbb{F}$.

We say a probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Föllmer [11] (see also Karandikar [12] for a more general result), there exists an $\mathbb{F}$-progressively measurable process, denoted as $\int_0^t B_s\,dB_s$, which coincides with Itô's integral, $\mathbb{P}$-a.s. for all local martingale measures $\mathbb{P}$. In particular, this provides a pathwise definition of

$$\langle B \rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s\,dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \varlimsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\langle B \rangle_t - \langle B \rangle_{t-\varepsilon}),$$

where $^{\mathrm{T}}$ denotes the transposition, and the $\overline{\lim}$ is taken componentwise and pointwise in ω. Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}$-quadratic variation of B, $\mathbb{P}$-a.s. for all local martingale measures $\mathbb{P}$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}, \qquad \mathbb{P}\text{-a.s.,} \tag{2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real-valued positive definite matrices. We note that, for different $\mathbb{P}_1, \mathbb{P}_2 \in \overline{\mathcal{P}}_W$, in general $\mathbb{P}_1$ and $\mathbb{P}_2$ are mutually singular. For any $\mathbb{P}\in\overline{\mathcal{P}}_W$, it follows from the Lévy characterization that Itô's stochastic integral under $\mathbb{P}$,

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_s^{-1/2}\,dB_s, \qquad t\in[0,1], \mathbb{P}\text{-a.s.} \tag{2.2}$$

defines a $\mathbb{P}$-Brownian motion. As in [22], we abuse the terminology of Denis and Martini [8] as follows:

DEFINITION 2.1. For any subset $\mathcal{P}\subset\overline{\mathcal{P}}_W$, we say a property holds $\mathcal{P}$-quasi-surely ($\mathcal{P}$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P}\in\mathcal{P}$.

In this paper we concentrate on the subclass $\overline{\mathcal{P}}_S\subset\overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0\circ(X^\alpha)^{-1} \qquad \text{where } X_t^\alpha := \int_0^t \alpha_s^{1/2}\,dB_s, t\in[0,1], \mathbb{P}_0\text{-a.s.} \tag{2.3}$$

for some $\mathbb{F}$-progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^1|\alpha_t|\,dt<\infty$, $\mathbb{P}_0$-a.s. We recall from [22] that

$$\overline{\mathcal{P}}_S = \{\mathbb{P}\in\overline{\mathcal{P}}_W : \overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}\}, \tag{2.4}$$

where $\overline{\mathbb{F}}^{\mathbb{P}}$ (resp., $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$) is the $\mathbb{P}$-augmentation of the filtration generated by B (resp., by $W^{\mathbb{P}}$). Moreover,

(2.5) every $\mathbb{P}\in\overline{\mathcal{P}}_S$ satisfies the Blumenthal zero–one law and the martingale representation property.

Notice that an $\mathbb{F}$-progressively measurable process can be viewed as a mapping from $[0,T]\times\Omega$ to $\mathbb{R}^d$. Moreover, X^α takes values in Ω and, thus, its canonical space is also Ω and the canonical filtration is still $\mathbb{F}$. We have the following simple lemma.

LEMMA 2.2. *Let α be an $\mathbb{F}$-progressively measurable process taking values in $\mathbb{S}_d^{>0}$ with $\int_0^1|\alpha_t|\,dt<\infty$, $\mathbb{P}_0$-a.s. Then there exists an $\mathbb{F}$-progressively measurable mapping $\beta_\alpha:[0,T]\times\Omega\to\mathbb{R}^d$ such that*

$$B=\beta_\alpha(X^\alpha), \qquad \mathbb{P}_0\text{-a.s.} \quad \textit{and}$$
$$W^{\mathbb{P}^\alpha}=\beta_\alpha(B), \qquad \hat{a}(B)=\alpha\circ\beta_\alpha(B), \qquad dt\times\mathbb{P}^\alpha\text{-a.s.}$$

PROOF. First, by [22], Lemma 8.1, we know $\overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0} = \overline{\mathbb{F}^B}^{\mathbb{P}_0}$, and, in particular, B is $\overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0}$ progressively measurable. By [22], Lemma 2.4 and Remark 2.3 below, there exists an $\mathbb{F}^{X^\alpha}$-progressively measurable process $\tilde{B}$ such that $\tilde{B} = B$, $\mathbb{P}_0$-a.s. Then, by viewing Ω as the canonical space of X^α, one may identify the process $\tilde{B}$ as an $\mathbb{F}$-progressively measurable mapping β_α. Changing back to the canonical space of B and noting that X^α takes values in Ω, we have $\tilde{B}(\omega) = \beta_\alpha(X^\alpha(\omega))$ for all $\omega \in \Omega$, and, therefore, $B = \beta_\alpha(X^\alpha)$, $\mathbb{P}_0$-a.s.

Now it follows from the definition of $\mathbb{P}^\alpha$ that

$$(B, \tilde{W}^\alpha)_{\mathbb{P}^\alpha} = (X^\alpha, B)_{\mathbb{P}_0} \qquad \text{where } \tilde{W}^\alpha := \beta_\alpha(B), \tag{2.6}$$

that is, the $\mathbb{P}^\alpha$-distribution of $(B, \tilde{W}^\alpha)$ is equal to the $\mathbb{P}_0$-distribution of (X^α, B). Note that $d\langle B\rangle_t = \hat{a}_t(B)\,dt$, $\mathbb{P}^\alpha$-a.s. and $d\langle X^\alpha\rangle_t = \alpha(B)\,dt = \alpha \circ \beta_\alpha(X^\alpha)\,dt$, $\mathbb{P}_0$-a.s. Then

$$(B, \tilde{W}^\alpha, \hat{a}(B))_{\mathbb{P}^\alpha} = \big(X^\alpha, B, \alpha \circ \beta_\alpha(X^\alpha)\big)_{\mathbb{P}_0}.$$

This implies that $\hat{a}(B) = \alpha \circ \beta_\alpha(B)$, $dt \times \mathbb{P}^\alpha$-a.s. Moreover, since $dB_t = \alpha_t^{-1/2}(B)\,dX_t^\alpha = \alpha_t^{-1/2}(\beta(X^\alpha))\,dX_t^\alpha$, $\mathbb{P}_0$-a.s. it follows from (2.6) that

$$\tilde{W}_t^\alpha = \int_0^t \alpha_s^{-1/2}(\beta(B))\,dB_s = \int_0^t \hat{a}_s^{-1/2}(B)\,dB_s = W_t^{\mathbb{P}^\alpha}, \qquad t \in [0,1], \mathbb{P}^\alpha\text{-a.s.}$$

$\square$

REMARK 2.3. In the standard stochastic analysis literature, the theory is developed under the augmented filtration. Because we are working under mutually singular measures, unless otherwise stated, we shall use the filtration $\mathbb{F}$. We recall from [22] that, for every probability measure $\mathbb{P}$, every $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process X has an $\mathbb{F}$-progressively measurable version $\tilde{X}$, that is, $X = \tilde{X}$, $\mathbb{P}$-a.s. Therefore, given $\mathbb{P}$, all processes involved in this paper will be considered in their $\mathbb{F}$-version. However, notice that such a version may depend on $\mathbb{P}$. See also Remark 3.6 below.

Moreover, following similar arguments, the above result still holds true if we replace $\mathbb{F}$ by an arbitrary filtration. In the proof of Lemma 2.2, we have used the result on the filtration $\mathbb{F}^{X^\alpha}$.

Finally, we clarify that by the statement "$\tilde{X} = X$, $\mathbb{P}$-a.s." we mean that these processes are equal to $dt \times d\mathbb{P}$-a.s. When both of them are càdlàg, clearly $\tilde{X}_t = X_t$, $0 \le t \le 1$, $\mathbb{P}$-a.s.

3. Second order target problem and relaxations. In this section we start with the definitions and assumptions related to the nonlinearity H and its convex dual. Several spaces used in the paper are also introduced in Section 3.1. We then give the definition of the original problem, two relaxed problems and the dual. We provide an easy first string of inequalities in the final subsection.

3.1. *Definitions and assumptions.* Let $H_t(\omega, y, z, \gamma): [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$ be $\mathbb{F}$-progressively measurable, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0. We assume throughout the following:

ASSUMPTION 3.1. For all $\omega \in \Omega$, H is Lipschitz continuous in (y, z), uniformly in (t, ω, γ) and it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm. Moreover, we assume that it is lower-semicontinuous in γ and the conjugate F defined at (3.1) below is measurable.

In the sequel, we denote by $A : B := \mathrm{Tr}[A^{\mathrm{T}} B]$ for $A, B \in \mathbb{R}^{d\times n}$. We introduce the conjugate of H with respect to γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \qquad a \in \mathbb{S}_d^{>0}. \tag{3.1}$$

We notice that F is measurable if H is upper-semicontinuous (and hence continuous) in γ or if D_H is compact; see, for example, [2]. Moreover, since H is uniformly continuous in (ω, y, z), the domain of F as a function of a is independent of (ω, y, z). Thus, we denote it by D_{F_t}. By the uniform Lipschitz continuity of H in (y, z), we know that

(3.2) $F(\cdot, a)$ is uniformly Lipschitz continuous in (y, z) and uniformly continuous in ω, uniformly on (t, a), for every $a \in D_{F_t}$.

Moreover, for our duality result of Section 4, we need to further assume the following:

ASSUMPTION 3.2. There is a constant C such that, for all (t, ω, y, z_1, z_2) and all $a \in D_{F_t}$:

$$|F_t(\omega, y, z_1, a) - F_t(\omega, y, z_2, a)| \le C|a^{1/2}(z_1 - z_2)|.$$

We also define

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0). \tag{3.3}$$

In order to focus on our main idea, in this section we shall restrict the probability measures in a subset $\mathcal{P}_H \subset \overline{\mathcal{P}}_S$ defined below. We will extend our results to more general cases, as well as allowing H to take value ∞, in Section 6 below.

DEFINITION 3.3. Let $\mathcal{P}_H$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \qquad dt \times d\mathbb{P}\text{-a.s. for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}, \tag{3.4}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 (|\hat{F}_t^0|^2 + |H_t^0|^2)\, dt\right] < \infty. \tag{3.5}$$

REMARK 3.4. In our accompanying paper [21] we consider a slightly more general class $\mathcal{P}_H^\kappa$ with a parameter $\kappa \in (1,2]$. The $\mathcal{P}_H$ in this paper coincides with the case $\kappa = 2$ there. All the results in this paper can be easily extended to the general case $\kappa \in (1,2]$. In particular, Theorem 4.5 and Proposition 4.10 in this paper still hold true for general κ, which are used in [21], Theorem 4.6.

It is clear that $\hat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$, and by (3.2) together with Assumption 3.2,

$$
\begin{aligned}
|\hat{F}_t(y_1, z_1) - \hat{F}_t(y_2, z_2)| \le C\bigl(|y_1 - y_2| + |\hat{a}_t^{1/2}(z_1 - z_2)|\bigr),& \\
dt \times d\mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_H.&
\end{aligned} \tag{3.6}
$$

REMARK 3.5. The Lipschitz continuity in z in (3.6) is implied by the following condition on H:

$$
|H_t(y, z_1, \gamma) - H_t(y, z_2, \gamma)| \le C|\hat{a}_t^{1/2}(z_1 - z_2)|, \qquad dt \times d\mathbb{P}\text{-a.s.}
$$

for some constant C which does not depend on (t, ω, y, γ).

We conclude this subsection by introducing the spaces which will be needed for the formulation of the second order target problems. For any domain D in an Euclidean space with appropriate dimension, let $\mathbb{L}^0(D)$ denote the space of all $\mathcal{F}_1$-measurable random variables taking values in D, and $\mathbb{H}^0(D)$ the space of all $\mathbb{F}^+$-progressively measurable processes taking values in D. Notice that here we use the right limit filtration $\mathbb{F}^+$. For any $\mathbb{P} \in \overline{\mathcal{P}}_W$, let $\mathbb{D}^0(\mathbb{P}, D)$ be the subspace of $\mathbb{H}^0(D)$ whose elements have càdlàg paths, $\mathbb{P}$-a.s.; $\mathbb{I}^0(\mathbb{P}, D)$ the subspace of $\mathbb{D}^0(\mathbb{P}, D)$ whose elements K have nondecreasing paths with $K_0 = 0$, $\mathbb{P}$-a.s.; and $\mathbb{S}^0(\mathbb{P}, D)$ the subspace of $\mathbb{D}^0(\mathbb{P}, D)$ whose elements have continuous paths, $\mathbb{P}$-a.s.

Moreover, let

$$
\begin{aligned}
&\mathbb{L}^2(\mathbb{P}, D) := \{\xi \in \mathbb{L}^0(D) : \mathbb{E}^{\mathbb{P}}[|\xi|^2] < \infty\},\\
&\mathbb{H}^2(\mathbb{P}, D) := \left\{H \in \mathbb{H}^0(D) : \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |H_t|^2\,dt\right] < \infty\right\},\\
&\mathbb{D}^2(\mathbb{P}, D) := \left\{Y \in \mathbb{D}^0(\mathbb{P}, D) : \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^2\right] < \infty\right\},\\
&\mathbb{I}^2(\mathbb{P}, D) := \mathbb{D}^2(\mathbb{P}, D) \cap \mathbb{I}^0(\mathbb{P}, D),\\
&\mathbb{S}^2(\mathbb{P}, D) := \mathbb{D}^2(\mathbb{P}, D) \cap \mathbb{S}^0(\mathbb{P}, D),
\end{aligned} \tag{3.7}
$$

and denote

$$
\hat{\mathbb{L}}_H^2(D) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{L}^2(\mathbb{P}, D), \qquad \hat{\mathbb{H}}_H^2(D) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{H}^2(\mathbb{P}, D),
$$

and the corresponding subsets of càdlàg, continuous processes, nondecreasing processes: $\hat{\mathbb{D}}^2_H(D) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{D}^2(\mathbb{P}, D)$, $\hat{\mathbb{S}}^2_H(D) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{S}^2(\mathbb{P}, D)$, $\hat{\mathbb{I}}^2_H(D) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{I}^2(\mathbb{P}, D)$.

Finally, let

$$\hat{\mathbb{G}}^2_H(D_H) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathbb{G}^2(\mathbb{P}, D_H) \quad \text{and} \quad \widehat{\mathcal{SM}}^2_H(\mathbb{R}^d) := \bigcap_{\mathbb{P}\in\mathcal{P}_H} \mathcal{SM}^2_H(\mathbb{P}, \mathbb{R}^d),$$

where

$$\mathbb{G}^2(\mathbb{P}, D_H) := \left\{\Gamma \in \mathbb{H}^0(D_H) : \tfrac{1}{2}\hat{a} : \Gamma - H(0,0,\Gamma) \in \mathbb{H}^2(\mathbb{P}, \mathbb{R})\right\}$$

and $\mathcal{SM}^2_H(\mathbb{P}, \mathbb{R}^d) \subset \mathbb{D}^2(\mathbb{P}, \mathbb{R}^d)$ is the space of all square integrable $(\mathbb{P}, \mathbb{F}^+)$-semimartingales Z with $\Gamma \in \mathbb{G}^2(\mathbb{P}, D_H)$, where Γ is defined by $d\langle Z, B\rangle_t = \Gamma_t : d\langle B\rangle_t$, $\mathbb{P}$-a.s.

REMARK 3.6. We emphasize that in the above spaces we require the processes to be $\mathbb{F}^+$-progressively measurable. This is important because the process V^+ in (4.21) is in general $\mathbb{F}^+$-progressively measurable. See also Proposition 4.11 and the paragraph before it.

However, for fixed $\mathbb{P} \in \overline{\mathcal{P}}_S$, it follows from the Blumenthal zero–one law that $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t^+]$, $\mathbb{P}$-a.s. for any $t \in [0,1]$ and $\mathbb{P}$-integrable ξ. In particular, this shows that any $\mathcal{F}_t^+$-measurable random variable has an $\mathcal{F}_t$-measurable $\mathbb{P}$-modification. Consequently, for any fixed $\mathbb{P}$, we may view the processes in $\mathbb{L}^2(\mathbb{P}, D)$ as $\mathbb{F}$-progressively measurable.

3.2. *The second order target problem.* For $Z \in \widehat{\mathcal{SM}}^2_H(\mathbb{R}^d)$, it follows from Karandikar [12] that Itô's stochastic integrals

$$\int_0^t Z_s\, dB_s \quad \text{and} \quad \int_0^t B_s\, dZ_s \qquad \text{are defined } \mathcal{P}_H\text{-q.s.}$$

In particular, the quadratic covariation between Z and B is well defined $\mathcal{P}_H$-q.s. and has a density process Γ:

$$d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t = \Gamma_t \hat{a}_t\, dt, \qquad \mathcal{P}_H\text{-q.s.} \tag{3.8}$$

For any $y \in \mathbb{R}$ and $Z \in \widehat{\mathcal{SM}}^2_H(\mathbb{R}^d)$, let $Y := Y^{y,Z} \in \hat{\mathbb{S}}^2_H(\mathbb{R})$ denote the controlled process defined by the following ODE (with random coefficients):

$$\begin{aligned} Y_t &= y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)\, ds + \int_0^t Z_s \circ dB_s \\ &= y + \int_0^t \left(\frac{1}{2}\hat{a}_s : \Gamma_s - H_s(Y_s, Z_s, \Gamma_s)\right) ds \\ &\quad + \int_0^t Z_s\, dB_s, \qquad t \in [0,1], \mathcal{P}_H\text{-q.s.,} \end{aligned} \tag{3.9}$$

where $\circ$ denotes the Stratonovich stochastic integral. We note that the well-posedness of (3.9) follows directly from the assumptions that $\Gamma \in \hat{\mathbb{G}}^2_H(D_H)$, Z is square integrable under each $\mathbb{P} \in \mathcal{P}_H$, and H is uniformly Lipschitz continuous in (y, z).

Let $\xi \in \mathbb{L}^0(\mathbb{R})$. Following Soner and Touzi [20], we introduce the second order stochastic target problem:

$$\mathcal{V}(\xi) := \inf\{y : Y_1^{y,Z} \geq \xi, \mathcal{P}_H\text{-q.s. for some } Z \in \widehat{\mathcal{SM}}^2_H(\mathbb{R}^d)\}. \tag{3.10}$$

3.3. *Relaxations*. We relax the target problem (3.10) by removing the constraint that Z is a semimartingale. For any $y \in \mathbb{R}$, $\bar{Z} \in \hat{\mathbb{H}}^2_H(\mathbb{R}^d)$, $\bar{\Gamma} \in \hat{\mathbb{G}}^2_H(D_H)$, and $\mathbb{P} \in \mathcal{P}_H$, let $\bar{Y} := \bar{Y}^{\mathbb{P},y,\bar{Z},\bar{\Gamma}} \in \mathbb{S}^2(\mathbb{P}, \mathbb{R})$ denote the unique solution of

$$\bar{Y}_t = y + \int_0^t \left(\frac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(\bar{Y}_s, \bar{Z}_s, \bar{\Gamma}_s)\right) ds + \int_0^t \bar{Z}_s \, dB_s, \qquad t \in [0, 1], \mathbb{P}\text{-a.s.} \tag{3.11}$$

Here, we observe that the stochastic integral $\int_0^t Z_s \, dB_s$ may not have a $\mathcal{P}_H$-q.s. version, in general, and thus we can only define (3.11) under each $\mathbb{P} \in \mathcal{P}_H$.

Our relaxed target problem is

$$\bar{\mathcal{V}}(\xi) := \inf\{y : \exists(\bar{Z}, \bar{\Gamma}) \in \hat{\mathbb{H}}^2_H(\mathbb{R}^d) \times \hat{\mathbb{G}}^2_H(D_H) \text{ such that } \bar{Y}_1^{\mathbb{P},y,\bar{Z},\bar{\Gamma}} \geq \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_H\}. \tag{3.12}$$

The main duality result of this paper relies on the following further relaxation of the above target problems. For $y \in \mathbb{R}$, $\bar{\bar{Z}} \in \hat{\mathbb{H}}^2_H(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_H$, let $\bar{\bar{Y}} := \bar{\bar{Y}}^{\mathbb{P},y,\bar{\bar{Z}}} \in \mathbb{S}^2(\mathbb{P}, \mathbb{R})$ be the unique solution of

$$\bar{\bar{Y}}_t = y + \int_0^t \hat{F}_s(\bar{\bar{Y}}_s, \bar{\bar{Z}}_s) \, ds + \int_0^t \bar{\bar{Z}}_s \, dB_s, \qquad t \in [0, 1], \mathbb{P}\text{-a.s.}, \tag{3.13}$$

where existence and uniqueness of $\bar{\bar{Y}}$ follows from (3.5) and (3.6). Here, again, the stochastic integral $\int_0^t \bar{\bar{Z}}_s \, dB_s$ may not have a $\mathcal{P}_H$-q.s. version. Our further relaxed second order target problem does not involve the processes Γ and $\bar{\Gamma}$, and is defined by

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\{y : \exists\bar{\bar{Z}} \in \hat{\mathbb{H}}^2_H(\mathbb{R}^d) \text{ s.t. } \bar{\bar{Y}}_1^{\mathbb{P},y,\bar{\bar{Z}}} \geq \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_H\}. \tag{3.14}$$

3.4. *Dual formulation*. By (2.5), each $\mathbb{P} \in \mathcal{P}_H \subset \overline{\mathcal{P}}_S$ satisfies the martingale representation property. Let τ be an $\mathbb{F}$-stopping time and η an $\mathcal{F}_\tau$-measurable and $\mathbb{P}$-square integrable random variable. By (3.5), (3.6) and the standard BSDE theory, the following BSDE has a unique solution $(\mathcal{Y}^{\mathbb{P}}(\tau, \eta), \mathcal{Z}^{\mathbb{P}}(\tau, \eta)) \in \mathbb{S}^2(\mathbb{P}, \mathbb{R}) \times$

$\mathbb{H}^2(\mathbb{P}, \mathbb{R}^d)$:

$$
\begin{aligned}
\mathcal{Y}_t^{\mathbb{P}}(\tau, \eta) = \eta - \int_t^\tau \hat{F}_s(\mathcal{Y}_s^{\mathbb{P}}(\tau, \eta), \mathcal{Z}_s^{\mathbb{P}}(\tau, \eta))\, ds \\
- \int_t^\tau \mathcal{Z}_s^{\mathbb{P}}(\tau, \eta)\, dB_s, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{3.15}
$$

Now for any $\xi \in \hat{\mathbb{L}}_H^2(\mathbb{R})$, our dual formulation is

$$
v(\xi) := \sup_{\mathbb{P} \in \mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(1, \xi). \tag{3.16}
$$

By the Blumenthal zero–one law (2.5), we know $\mathcal{Y}_0^{\mathbb{P}}(1, \xi)$ is a constant, and thus $v(\xi)$ is deterministic.

Our main focus of this paper is to provide conditions which guarantee that the problems $\bar{\mathcal{V}}(\xi)$, $\bar{\bar{\mathcal{V}}}(\xi)$ and $v(\xi)$ agree. In order to connect these problems to $\mathcal{V}(\xi)$, we will need an appropriate reformulation; see Section 5.

3.5. *Some preliminary results*. In this subsection we prove a straightforward string of inequalities.

PROPOSITION 3.7. *Let Assumptions* 3.1 *and* 3.2 *hold true. Then, for any* $\xi \in \hat{\mathbb{L}}_H^2(\mathbb{R})$,

$$
\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) \geq v(\xi). \tag{3.17}
$$

PROOF. (i) The first inequality holds true by definition of $\mathcal{V}$ and $\bar{\mathcal{V}}$.

(ii) To prove that $\bar{\mathcal{V}}(\xi) \geq \bar{\bar{\mathcal{V}}}(\xi)$, let $y \in \mathbb{R}$, $\bar{Z} \in \hat{\mathbb{H}}_H^2(\mathbb{R}^d)$ and $\bar{\Gamma} \in \hat{\mathbb{G}}_H^2(D_H)$ be such that $\bar{Y}_1^{\mathbb{P}, y, \bar{Z}, \bar{\Gamma}} \geq \xi$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$. By the definition of the conjugate function F,

$$
\tfrac{1}{2}\hat{a}_s : \bar{\Gamma}_s - H_s(y, \bar{Z}_s, \bar{\Gamma}_s) \leq \hat{F}_s(y, \bar{Z}_s) \qquad \text{for all } y \in \mathbb{R}.
$$

By the comparison theorem for ODEs, we conclude that $\bar{Y}_1^{\mathbb{P}, y, \bar{Z}, \bar{\Gamma}} \leq \bar{\bar{Y}}_1^{\mathbb{P}, y, \bar{Z}}$, $\mathbb{P}$-a.s. Thus, $\bar{\bar{Y}}_1^{\mathbb{P}, y, \bar{Z}} \geq \xi$, $\mathbb{P}$-a.s. and, therefore, $y \geq \bar{\bar{\mathcal{V}}}(\xi)$.

(iii) Similarly, to see that $\bar{\bar{\mathcal{V}}}(\xi) \geq \bar{\mathcal{V}}(\xi)$, we consider some $y > \bar{\bar{\mathcal{V}}}(\xi)$ so that there exists $\bar{\bar{Z}} \in \hat{\mathbb{H}}_H^2(\mathbb{R}^d)$ such that

$$
\bar{\bar{Y}}_1^{\mathbb{P}, y\bar{\bar{Z}}} \geq \xi, \qquad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_H.
$$

Then, for any $\varepsilon > 0$, it follows from the lower-semicontinuity of H in γ that there exists a progressively measurable process $\bar{\Gamma} \in \mathbb{H}^0(D_H)$ such that

$$
\hat{F}(\bar{\bar{Y}}, \bar{\bar{Z}}) - \varepsilon \leq \tfrac{1}{2}\hat{a} : \bar{\Gamma} - H(\bar{\bar{Y}}, \bar{\bar{Z}}, \bar{\Gamma}) \leq \hat{F}(\bar{\bar{Y}}, \bar{\bar{Z}}).
$$

Then, $\bar{\Gamma} \in \hat{\mathbb{G}}_H^2(D_H)$ and it follows from classical estimates on ODEs that there exists a constant C such that, with $\bar{y} := y + C\varepsilon$, we have

$$\bar{Y}_1^{\mathbb{P},\bar{y},\bar{\bar{Z}},\bar{\Gamma}} \geq \bar{\bar{Y}}_1^{\mathbb{P},y\bar{\bar{Z}}} \geq \xi \qquad \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H.$$

Hence, $\bar{y} \geq \bar{\mathcal{V}}(\xi)$. Since $\varepsilon > 0$ and $y > \bar{\bar{\mathcal{V}}}(\xi)$ are arbitrary, we conclude that $\bar{\bar{\mathcal{V}}}(\xi) \geq \bar{\mathcal{V}}(\xi)$.

(iv) The final inequality $\bar{\bar{\mathcal{V}}}(\xi) \geq v(\xi)$ can be proved similarly to (ii) above by using the comparison theorem for BSDEs. □

REMARK 3.8. Consider the Markovian case $H_t(y, z, \gamma) = h(t, B_t, y, z, \gamma)$ and $\xi = g(B_1)$, for some deterministic functions h, g. Assume in addition that the PDE (1.4) has a solution $u \in C^{1,2}$ with appropriate growth. Then, by the classical verification argument of stochastic control, one can prove that $u(0, 0) = v(\xi)$. Moreover, if H is convex, then it follows from a direct application of Itô's formula that $u(0, 0) = \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$. If in addition $\{Du(t, B_t), t \in [0, 1]\} \in \widehat{\mathcal{SM}}_H^2(\mathbb{R}^d)$, then we also have $u(0, 0) = \mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$. Finally, any optimal $\mathbb{P}^*$ (if exists) for the problem $v(\xi)$ satisfies

$$\tfrac{1}{2}\hat{a}_t : D^2u(t, B_t) - H.(\cdot, u, Du, D^2u)(t, B_t) = F.(\cdot, u, Du, \hat{a}.)(t, B_t), \qquad \mathbb{P}^*\text{-a.s.}$$

In the non-Markovian case, we shall prove in the next section our main duality result $\bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$ and that the optimal $(\bar{Z}, \bar{\Gamma})$, $\bar{\bar{Z}}$, for the problems $\bar{\mathcal{V}}(\xi)$ and $\bar{\bar{\mathcal{V}}}(\xi)$, respectively, exist. However, we are not able to prove $\mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi)$ in general. In order to obtain a result of this type, we shall introduce a slight modification of these problems by restricting $\mathbb{P}$ to smaller sets; see Section 5 below.

4. The main results. This section is devoted to the proof of reverse inequalities.

4.1. *Conditional expectation.* We first establish a dynamic programming principle to prove our duality result $\bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$. The understanding of the regular conditional probability distributions (r.c.p.d.) is crucial for this result. Indeed, let $\mathbb{P}$ be an arbitrary probability measure on Ω and τ be an $\mathbb{F}$-stopping time. By Stroock and Varadhan [24], there exists a r.c.p.d. $\mathbb{P}_\tau^\omega$ for all $\omega \in \Omega$ satisfying the following:

– For each $\omega \in \Omega$, $\mathbb{P}_\tau^\omega$ is a probability measure on $\mathcal{F}_1$;
– For each $E \in \mathcal{F}_1$, the mapping $\omega \to \mathbb{P}_\tau^\omega(E)$ is $\mathcal{F}_\tau$-measurable;
– For $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathbb{P}_\tau^\omega$ is the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, that is, for every bounded $\mathcal{F}_1$-measurable random variable ξ we have

$$\mathbb{E}^{\mathbb{P}}(\xi | \mathcal{F}_\tau)(\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}(\xi), \qquad \mathbb{P}\text{-a.s.};$$

– For each $\omega \in \Omega$,

$$\mathbb{P}^\omega_\tau(\Omega^\omega_\tau) = 1 \qquad \text{where } \Omega^\omega_\tau := \{\omega' \in \Omega : \omega'(s) = \omega(s), 0 \le s \le \tau(\omega)\}. \tag{4.1}$$

The goal of this subsection is to understand $\mathbb{P}^\omega_\tau$ for $\mathbb{P} \in \mathcal{P}_H$. Roughly, we shall prove that $\mathbb{P}^\omega_\tau$ satisfies the properties of Definition 3.3 on a shifted space; see Lemma 4.3 below. To do that, we introduce some notation:

– For $0 \le t \le 1$, denote by $\Omega^t := \{\omega \in C([t, 1], \mathbb{R}^d) : \omega(t) = 0\}$ the shifted canonical space; B^t the shifted canonical process on Ω^t; $\mathbb{P}^t_0$ the shifted Wiener measure; $\mathbb{F}^t$ the shifted filtration generated by B^t.
– For $0 \le s \le t \le 1$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$:

$$\omega^t_r := \omega_r - \omega_t \qquad \text{for all } r \in [t, 1];$$

– For $0 \le s \le t \le 1$ and $\omega \in \Omega^s$, $\tilde\omega \in \Omega^t$, define the concatenation path $\omega \otimes_t \tilde\omega \in \Omega^s$ by

$$(\omega \otimes_t \tilde\omega)(r) := \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \tilde\omega_r)\mathbf{1}_{[t,1]}(r) \qquad \text{for all } r \in [s, 1].$$

– For $0 \le s \le t \le 1$ and an $\mathcal{F}^s_1$-measurable random variable ξ on Ω^s, for each $\omega \in \Omega^s$, define the shifted $\mathcal{F}^t_1$-measurable random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\tilde\omega) := \xi(\omega \otimes_t \tilde\omega) \qquad \text{for all } \tilde\omega \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s$-progressively measurable process X on $[s, 1]$ and $(t, \omega) \in [s, 1] \times \Omega^s$, the shifted process $\{X^{t,\omega}_r, r \in [t, 1]\}$ is $\mathbb{F}^t$-progressively measurable.
– For $\mathbb{F}$-stopping time τ, we shall simplify the notation as follows:

$$\omega \otimes_\tau \tilde\omega := \omega \otimes_{\tau(\omega)} \tilde\omega, \qquad \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \qquad X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

The r.c.p.d. $\mathbb{P}^\omega_\tau$ induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}^{\tau(\omega)}_1$ such that the $\mathbb{P}^{\tau,\omega}$-distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}^\omega_\tau$-distribution of $\{B_t - B_{\tau(\omega)}, t \in [\tau(\omega), 1]\}$. By (4.1), it is clear that for every bounded and $\mathcal{F}_1$-measurable random variable ξ,

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]. \tag{4.2}$$

We shall also call $\mathbb{P}^{\tau,\omega}$ the r.c.p.d. of $\mathbb{P}$.

For $0 \le t \le 1$, following the same arguments as in Section 2 but restricting to the canonical space Ω^t, we may define martingale measures $\mathbb{P}^{t,\alpha}$ for each $\mathbb{F}^t$-progressively measurable $\mathbb{S}^{>0}_d$-valued process α such that $\int_t^1 |\alpha_r|\,dr < \infty$, $\mathbb{P}^t_0$-a.s. Let $\overline{\mathcal{P}}^t_S$ denote the set of all such measures $\mathbb{P}^{t,\alpha}$. Similarly, we may define the density process $\hat a^t$ of the quadratic variation process $\langle B^t \rangle$.

We first have the following result.

LEMMA 4.1. *Let $\mathbb{P} \in \overline{\mathcal{P}}_S$ and τ be an $\mathbb{F}$-stopping time. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathbb{P}^{\tau,\omega} \in \overline{\mathcal{P}}_S^{\tau(\omega)}$ and*

$$\hat{a}_s^{\tau,\omega}(\tilde{\omega}) = \hat{a}_s^{\tau(\omega)}(\tilde{\omega}) \qquad \text{for } ds \times d\mathbb{P}^{\tau,\omega}\text{-a.e. } (s,\tilde{\omega}) \in [\tau(\omega), 1] \times \Omega^{\tau(\omega)}, \tag{4.3}$$

where the left-hand side above is the shifted process of original density process $\hat{a}$ on $\Omega = \Omega_0$ and the right-hand side is the density process on the shifted space $\Omega^{\tau(\omega)}$.

PROOF. The proof of $\mathbb{P}^{\tau,\omega} \in \overline{\mathcal{P}}_S^{\tau(\omega)}$ is relegated to the Appendix. We now prove (4.3).

Since $d\langle B_\cdot - B_\tau\rangle_t = \hat{a}_t\,dt$, $\mathbb{P}$-a.s., then $d\langle B_\cdot - B_\tau\rangle_t = \hat{a}_t\,dt$, $\mathbb{P}^\omega_\tau$-a.s. for $\mathbb{P}$-a.e. $\omega \in \Omega$. Note that, for each $\omega \in \Omega$ and $t \ge \tau(\omega)$,

$$\hat{a}_t(\omega) = \hat{a}_t\big(\omega \otimes_\tau \omega^{\tau(\omega)}\big) = \hat{a}_t^{\tau,\omega}\big(\omega^{\tau(\omega)}\big).$$

This implies that $d\langle B_\cdot^{\tau(\omega)}\rangle_t = \hat{a}_t^{\tau,\omega}\,dt$, $\mathbb{P}^{\tau,\omega}$-a.s. for $\mathbb{P}$-a.e. $\omega \in \Omega$. Now (4.3) follows from the definition of $\hat{a}^{\tau(\omega)}$. $\square$

We next study the r.c.p.d. for $\mathbb{P} \in \mathcal{P}_H$. For each $(t,\omega) \in [0,1] \times \Omega$, let

$$\begin{aligned} H_s^{t,\omega}(\tilde{\omega}, y, z, \gamma) &:= H_s(\omega \otimes_t \tilde{\omega}, y, z, \gamma), \\ \hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) &:= F_s\big(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^t(\tilde{\omega})\big) \end{aligned} \tag{4.4}$$

for all $(s,\tilde{\omega}) \in [t,1] \times \Omega^t$ and $(y,z,\gamma) \in \mathbb{R} \times \mathbb{R}^d \times D_H$. We emphasize that in the definition of $\hat{F}^{t,\omega}$ we use the density process $\hat{a}^t$ in the shifted space. This is important in (4.5) below. However, by Lemma 4.1 we actually have

$$\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) = F_s\big(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^{t,\omega}(\tilde{\omega})\big) = \hat{F}_s(\omega \otimes_t \tilde{\omega}, y, z),$$
$$ds \times d\mathbb{P}^{t,\omega}\text{-a.e. } (s,\tilde{\omega}) \in [t,1] \times \Omega^t, \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Since H and F are uniformly continuous in ω under the $\mathbb{L}^\infty$-norm, by Assumption 3.1 and (3.2), we also have

$$H_s^{t,\omega}(\tilde{\omega}, y, z, \gamma) \text{ and } \hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) \text{ are uniformly continuous in } \omega \text{ under the } \mathbb{L}^\infty\text{-norm.} \tag{4.5}$$

We remark that $F_s(\omega \otimes_t \tilde{\omega}, y, z, \hat{a}_s^{t,\omega}(\tilde{\omega}))$ is in general not continuous in ω because $\hat{a}$ is not continuous in ω, in general; see Lemma 2.2. Similarly, as a consequence of (4.5), we see that for any $\mathbb{P}^t \in \overline{\mathcal{P}}_S^t$,

$$\mathbb{E}^{\mathbb{P}^t}\left[\int_t^1 \big(|H_s^{t,\omega}(0)|^2 + |\hat{F}_s^{t,\omega}(0)|^2\big)\,ds\right] < \infty \quad \text{for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \tag{4.6}$$

We now extend Definition 3.3 to the shifted space.

DEFINITION 4.2. Let $\mathcal{P}_H^t$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S^t$ such that

$$\underline{a}_{\mathbb{P}} \le \hat{a}^t \le \overline{a}_{\mathbb{P}},\ ds \times d\mathbb{P}\text{-a.e. on } [t,1] \times \Omega^t \text{ for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0},$$

$$\mathbb{E}^{\mathbb{P}}\left[\int_t^1 \left(|H_s^{t,\omega}(0)|^2 + |\hat{F}_s^{t,\omega}(0)|^2\right) ds\right] < \infty \tag{4.7}$$

for all or, equivalently, some $\omega \in \Omega$.

Then we have the following.

LEMMA 4.3. *Let Assumption 3.1 hold true. Then, for any $\mathbb{F}$-stopping time τ and $\mathbb{P} \in \mathcal{P}_H$, the r.c.p.d. $\mathbb{P}^{\tau,\omega} \in \mathcal{P}_H^{\tau(\omega)}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

PROOF. Let $\mathbb{P} = \mathbb{P}^\alpha \in \mathcal{P}_H \subset \overline{\mathcal{P}}_S$. By Lemma 4.1 we have $\mathbb{P}^{\tau,\omega} \in \overline{\mathcal{P}}_S^{\tau(\omega)}$, $\mathbb{P}$-a.s. By (3.4) and (3.5), it holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ that

$$\underline{a}_{\mathbb{P}} \le \hat{a}_s^{\tau,\omega}(\tilde{\omega}) \le \overline{a}_{\mathbb{P}}, ds \times d\mathbb{P}^{\tau,\omega}\text{-a.e. } (s,\tilde{\omega}) \in [\tau(\omega),1] \times \Omega^{\tau(\omega)},$$

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\left[\int_{\tau(\omega)}^1 \left(\left|F_s(\omega \otimes_\tau \tilde{\omega}, 0, 0, \hat{a}_s^{\tau,\omega}(\tilde{\omega}))\right|^2 + |H_s(\omega \otimes_\tau \tilde{\omega}, 0, 0, 0)|^2\right) ds\right] < \infty.$$

This, together with (4.3) and (4.4), implies (4.7), and thus completes the proof. $\square$

We remark that in this paper we actually use the r.c.p.d. only on deterministic times. However, the r.c.p.d. on stopping times will be important in our accompanying paper [21].

4.2. *The duality result.* To establish our main duality result, we need the following assumption on the terminal data.

ASSUMPTION 4.4. ξ is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm.

Under Assumptions 3.1 and 4.4, there exists a modulus of continuity function ρ for ξ and H in ω. Then, for any $0 \le t \le s \le 1$, $(y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}^d$, and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$|\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| \le \rho(\|\omega - \omega'\|_t) \quad \text{and}$$

$$|\hat{F}_s^{t,\omega}(\tilde{\omega}, y, z) - \hat{F}_s^{t,\omega'}(\tilde{\omega}, y, z)| \le \rho(\|\omega - \omega'\|_t),$$

where $\|\omega\|_t := \sup_{0 \le s \le t} |\omega_s|$, $0 \le t \le 1$. We next define for all $\omega \in \Omega$,

$$\Lambda(\omega) := \sup_{0 \le t \le 1} \Lambda_t(\omega)$$

$$\text{where } \Lambda_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \left(\mathbb{E}^{\mathbb{P}}\left[|\xi^{t,\omega}|^2 + \int_t^1 |\hat{F}_s^{t,\omega}(0)|^2 ds\right]\right)^{1/2}. \tag{4.8}$$

By (4.5) and following the same arguments as for (4.6), we have

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \tag{4.9}$$

Moreover, when Λ is finite, it is uniformly continuous in ω under the $\mathbb{L}^\infty$-norm and is therefore $\mathcal{F}_1$-measurable.

Our main duality result is as follows:

THEOREM 4.5. *Let Assumptions 3.1, 3.2, 4.4 hold, and assume further that*

$$\mathbb{E}^{\mathbb{P}}[|\Lambda|^2] < \infty \qquad \text{for all } \mathbb{P} \in \mathcal{P}_H. \tag{4.10}$$

Then $\bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$, and existence holds for the problem $\bar{\bar{\mathcal{V}}}(\xi)$. Moreover, if F has a progressively measurable optimizer, existence also holds for the problem $\bar{\mathcal{V}}(\xi)$.

We first provide several examples that satisfy the hypothesis of the theorem and then prove it in Section 4.4.

4.3. *Examples.*

EXAMPLE 1 (Linear generator). Assume that H is linear in γ:

$$H_t(y, z, \gamma) = f_t(y, z) + \tfrac{1}{2}\sigma_t\sigma_t^{\mathrm{T}} : \gamma,$$

where $f_t(y, z)$ and σ_t satisfy appropriate conditions for our assumptions to hold. Notice that the domain of F is reduced to a one-point set:

$$F_t(y, z, a) = f_t(y, z)\mathbf{1}_{\{a=\sigma_t\sigma_t^{\mathrm{T}}\}} + \infty\mathbf{1}_{\{a\neq\sigma_t\sigma_t^{\mathrm{T}}\}}.$$

Then, the present formulation of the second order target problem is clearly equivalent to the classical formulation under the reference measure $\mathbb{P}^{\sigma\sigma^{\mathrm{T}}}$ which ignores any uncertainty on the diffusion coefficient.

EXAMPLE 2 (Uncertain volatility models). Set $H_t(y, z, \gamma) := G(\gamma) := \frac{1}{2}[\bar{\sigma}^2\gamma^+ - \underline{\sigma}^2\gamma^-]$, where $\bar{\sigma} > \underline{\sigma} \geq 0$. This is the context studied by Denis and Martini [8]. By straightforward calculation, we find $\mathrm{dom}(F_t) = [\underline{\sigma}^2, \bar{\sigma}^2]$, and for any $a \in [\underline{\sigma}^2, \bar{\sigma}^2]$, $F(a) = 0$. It is easily seen that all our assumptions are satisfied. Moreover, we have $\bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = \mathbb{E}^G(\xi)$ for appropriate random variable ξ, where $\mathbb{E}^G$ is the G-expectation defined in Peng [17]. More connections between this paper and G-martingales are established in our accompanying paper [23].

EXAMPLE 3 (Hedging under gamma constraints). Let $\underline{\Gamma}, \overline{\Gamma} \geq 0$ be two given constants. The problem of superhedging under Gamma constraint, as introduced in [5, 18] and [19], corresponds to the specification $H_s(y, z, \gamma) = H(\gamma) = \frac{1}{2}\sigma^2\gamma$ for $\gamma \in [-\underline{\Gamma}, \overline{\Gamma}]$, and $+\infty$ otherwise. By straightforward calculation, we see that $F(a) = \frac{1}{2}(\overline{\Gamma}(a - \sigma^2)^+ + \underline{\Gamma}(a - \sigma^2)^-)$. If both bounds are finite, the domain of the dual function F is the nonnegative real line. The dual formulation of this paper coincides with that of [19].

4.4. *Proof of the duality result.* The rest of this section is devoted to the proof of Theorem 4.5. From now on, we shall always assume Assumptions 3.1, 3.2, 4.4 and that (4.10) hold. In particular, we notice that (4.10) and (4.9) imply that

$$\Lambda_t(\omega) < \infty \qquad \text{for all } (t,\omega) \in [0,1] \times \Omega. \tag{4.11}$$

To prove the theorem, we define the following value process V_t pathwise:

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(1,\xi) \qquad \text{for all } (t,\omega) \in [0,1] \times \Omega, \tag{4.12}$$

where, for any $(t_1,\omega) \in [0,1] \times \Omega$, $\mathbb{P} \in \mathcal{P}_H^{t_1}$, $t_2 \in [t_1, 1]$, and any $\eta \in \mathbb{L}^2(\mathbb{P}, \mathcal{F}_{t_2})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2,\eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$, where $(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega})$ is the solution to the following BSDE on the shifted space Ω^{t_1} under $\mathbb{P}$:

$$\begin{aligned} y_s^{\mathbb{P},t_1,\omega} = \eta^{t_1,\omega} - \int_s^{t_2} \hat{F}_r^{t_1,\omega}(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega})\,dr - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega}\,dB_r^{t_1}, \\ ds \in [t_1,t_2], \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.13}$$

In view of the Blumenthal zero–one law (2.5), $\mathcal{Y}_t^{\mathbb{P},t,\omega}(1,\xi)$ is constant for any given (t,ω) and $\mathbb{P} \in \mathcal{P}_H^t$. Moreover, since $\omega_0 = 0$ for all $\omega \in \Omega$, it is clear that, for the $\mathcal{Y}^{\mathbb{P}}$ defined in (3.15),

$$\mathcal{Y}^{\mathbb{P},0,\omega}(t,\eta) = \mathcal{Y}^{\mathbb{P}}(t,\eta) \quad \text{and} \quad V_0(\omega) = v(\xi) \qquad \text{for all } \omega \in \Omega.$$

LEMMA 4.6. *Assume all the conditions in Theorem 4.5 hold. Then for all $(t,\omega) \in [0,1] \times \Omega$, we have $|V_t(\omega)| \le C\Lambda_t(\omega)$. Moreover, for all $(t,\omega,\omega') \in [0,1] \times \Omega^2$, $|V_t(\omega) - V_t(\omega')| \le C\rho(\|\omega - \omega'\|_t)$. Consequently, V_t is $\mathcal{F}_t$-measurable for every $t \in [0,1]$.*

PROOF. (i) For each $(t,\omega) \in [0,1] \times \Omega$ and $\mathbb{P} \in \mathcal{P}_H^t$, on $[t,1]$ we have

$$\begin{aligned} y_s^{\mathbb{P},t,\omega} = \xi^{t,\omega} - \int_s^1 [\hat{F}_r^{t,\omega}(0) + \gamma_s y_r^{\mathbb{P},t,\omega} + z_r^{\mathbb{P},t,\omega}(\hat{a}_r^t)^{1/2}\eta_r^T]\,dr - \int_s^1 z_r^{\mathbb{P},t,\omega}\,dB_r^t, \\ \mathbb{P}\text{-a.s.,} \end{aligned}$$

where γ, η are bounded, thanks to (3.6). Define

$$M_s := \exp\biggl(-\int_t^s \eta_r\,dB_r^t - \int_t^s \biggl[\gamma_r + \frac{1}{2}|(\hat{a}_r^t)^{1/2}\eta_r^T|^2\biggr]dr\biggr). \tag{4.14}$$

Applying Itô's formula, we obtain

$$y_t^{\mathbb{P},t,\omega} = M_t y_t^{\mathbb{P},t,\omega} = M_T \xi^{t,\omega} - \int_t^1 M_s \hat{F}_s^{t,\omega}(0)\,ds - \int_t [\cdots]\,dB_s^t, \qquad \mathbb{P}\text{-a.s.}$$

Thus,

$$
\begin{aligned}
|y_t^{\mathbb{P},t,\omega}|^2 &= \left|\mathbb{E}^{\mathbb{P}}\left[M_T \xi^{t,\omega} - \int_t^1 M_s \hat{F}_s^{t,\omega}(0)\,ds\right]\right|^2 \\
&\le \left|\mathbb{E}^{\mathbb{P}}\left[\sup_{t\le s\le T} M_s|\xi^{t,\omega}| + \int_t^1 |\hat{F}_s^{t,\omega}(0)|\,ds\right]\right|^2 \\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le T}|M_s|^2\Big]\mathbb{E}^{\mathbb{P}}\left[|\xi^{t,\omega}|^2 + \int_t^1 |\hat{F}_s^{t,\omega}(0)|^2\,ds\right].
\end{aligned}
$$

Since γ, η are bounded, by standard arguments we see that

$$
|y_t^{\mathbb{P},t,\omega}|^2 \le C\mathbb{E}^{\mathbb{P}}\left[|\xi^{t,\omega}|^2 + \int_t^1 |\hat{F}_s^{t,\omega}(0)|^2\,ds\right] \le C|\Lambda_t(\omega)|^2.
$$

Since $\mathbb{P}\in\mathcal{P}_H^t$ is arbitrary, we get $|V_t(\omega)| \le C\Lambda_t(\omega)$.

(ii) Similarly, for $(t,\omega,\omega')\in[0,1]\times\Omega^2$ and $\mathbb{P}\in\mathcal{P}_H^t$, denote

$$
\begin{aligned}
&\delta y := y^{\mathbb{P},t,\omega} - y^{\mathbb{P},t,\omega'}, \qquad \delta z := z_t^{\mathbb{P},t,\omega} - z^{\mathbb{P},t,\omega'}, \qquad \delta\xi := \xi^{t,\omega} - \xi^{t,\omega'}, \\
&\delta F := \hat{F}^{t,\omega} - \hat{F}^{t,\omega'}.
\end{aligned}
$$

Then, for $s\in[t,1]$, $|\delta\xi| + |\delta F_s| \le C\rho(\|\omega-\omega'\|_t)$ and

$$
\delta y_s = \delta\xi - \int_s^1 [\delta F_r(y_r^{\mathbb{P},t,\omega}, z_r^{\mathbb{P},t,\omega}) + \tilde{\gamma}_r\delta y_r + \delta z_r(\hat{a}_r^t)^{1/2}\tilde{\eta}_r^T]\,dr - \int_s^1 \delta z_r\,dB_r^t, \qquad \mathbb{P}\text{-a.s.},
$$

where $\tilde{\gamma}$, $\tilde{\eta}$ are bounded, thanks to (3.6) again. Define $\tilde{M}$ as in (4.14) but corresponding to $(\tilde{\gamma},\tilde{\eta})$. Then following the arguments in (i), we obtain $|\delta y_t| \le C\rho(\|\omega-\omega'\|_t)$. Since $\mathbb{P}$ is arbitrary, we prove the lemma. $\square$

The following dynamic programming principle plays a central role in our analysis.

PROPOSITION 4.7. *Assume all the conditions in Theorem 4.5 hold. Then*

$$
V_{t_1}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}_H^{t_1}} \mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega}) \qquad \textit{for all } 0\le t_1<t_2\le 1 \textit{ and } \omega\in\Omega.
$$

PROOF. To simplify the presentation, we assume without loss of generality that $t_1=0$ and $t_2=t$. That is, we shall prove

$$
v(\xi) = \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t). \tag{4.15}
$$

Denote $(y^{\mathbb{P}}, z^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(1,\xi), \mathcal{Z}^{\mathbb{P}}(1,\xi))$.

(i) For any $\mathbb{P} \in \mathcal{P}_H$, note that

$$y_s^{\mathbb{P}} = y_t^{\mathbb{P}} - \int_s^t \hat{F}_r(y_r^{\mathbb{P}}, z_r^{\mathbb{P}})\,dr - \int_s^t z_r^{\mathbb{P}}\,dB_r, \qquad s \in [0,t], \mathbb{P}\text{-a.s.}$$

By Lemma 4.3, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the r.c.p.d. $\mathbb{P}^{t,\omega} \in \mathcal{P}_H^t$. Since solutions of BSDEs can be constructed via Picard iteration, one can easily check that

$$y_t^{\mathbb{P}}(\omega) = \mathcal{Y}_t^{\mathbb{P}^{t,\omega},t,\omega}(1,\xi) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.16}$$

Then by the definition of V_t we get

$$y_t^{\mathbb{P}}(\omega) \le V_t(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.17}$$

It follows from the comparison principle for BSDEs that $y_0^{\mathbb{P}} \le \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$. Since $\mathbb{P} \in \mathcal{P}_H$ is arbitrary, this shows that $v(\xi) \le \sup_{\mathbb{P}\in\mathcal{P}_H} \mathcal{Y}_0^{\mathbb{P}}(t, V_t)$.

(ii) It remains to prove the other inequality. Fix $\mathbb{P} \in \mathcal{P}_H$ and arbitrary $\varepsilon > 0$. Since Ω is separable, there exists a partition $E_t^i \in \mathcal{F}_t, i = 1, 2, \dots$ such that $\|\omega - \omega'\|_t \le \varepsilon$ for any i and any $\omega, \omega' \in E_t^i$. For each i, fix an $\hat{\omega}_i \in E_t^i$, and let $\mathbb{P}_t^i \in \mathcal{P}_H^t$ be an ε-optimizer of $V_t(\hat{\omega}_i)$, that is, $V_t(\hat{\omega}_i) \le \mathcal{Y}_t^{\mathbb{P}_t^i,t,\hat{\omega}_i} + \varepsilon$.

For each $n \ge 1$, define $\mathbb{P}^n := \mathbb{P}^{n,\varepsilon}$ by

$$\begin{aligned}\mathbb{P}^n(E) := \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_t^i}[(\mathbf{1}_E)^{t,\omega}]\mathbf{1}_{E_t^i}\right] + \mathbb{P}(E \cap \hat{E}_t^n)\\ \text{where } \hat{E}_t^n \triangleq \bigcup_{i>n} E_t^i.\end{aligned} \tag{4.18}$$

That is, $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, and its r.c.p.d. $(\mathbb{P}^n)^{t,\omega} = \mathbb{P}_t^i$ for $\omega \in E_t^i$, $1 \le i \le n$, and $(\mathbb{P}^n)^{t,\omega} = \mathbb{P}^{t,\omega}$ for $\omega \in \hat{E}_t^n$. We claim that

$$\mathbb{P}^n \in \mathcal{P}_H. \tag{4.19}$$

The proof is similar to Lemmas 4.1 and 4.3, and thus is also postponed to the Appendix.

Now for $1 \le i \le n$ and $\omega \in E_t^i$, by Lemma 4.6 and its proof we see that

$$\begin{aligned} V_t(\omega) &\le V_t(\hat{\omega}_i) + C\rho(\varepsilon) \le \mathcal{Y}_t^{\mathbb{P}_t^i,t,\hat{\omega}_i}(1,\xi) + \varepsilon + C\rho(\varepsilon)\\ &\le \mathcal{Y}_t^{\mathbb{P}_t^i,t,\omega}(1,\xi) + \varepsilon + C\rho(\varepsilon) = \mathcal{Y}_t^{(\mathbb{P}^n)^{t,\omega},t,\omega}(1,\xi) + \varepsilon + C\rho(\varepsilon). \end{aligned}$$

Here as usual the constant C varies from line to line. Then it follows from (4.16) that

$$V_t \le y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon), \qquad \mathbb{P}^n\text{-a.s. on } \bigcup_{i=1}^n E_t^i. \tag{4.20}$$

Let $(y^n, z^n) := (y^{n,\varepsilon}, z^{n,\varepsilon})$ denote the solution to the following BSDE on $[0, t]$:

$$y^n_s = [y^{\mathbb{P}^n}_t + \varepsilon + C\rho(\varepsilon)]\mathbf{1}_{\bigcup_{i=1}^n E^i_t} + V_t \mathbf{1}_{\hat{E}^n_t} - \int_s^t \hat{F}_r(y^n_r, z^n_r)\,dr - \int_s^t z^n_r\,dB_r, \qquad \mathbb{P}\text{-a.s.}$$

By the comparison principle of BSDEs we know $\mathcal{Y}^{\mathbb{P}}_0(t, V_t) \le y^n_0$. Since $\mathbb{P}^n = \mathbb{P}$ on $\mathcal{F}_t$, we have

$$y^{\mathbb{P}^n}_s = y^{\mathbb{P}^n}_t - \int_s^t \hat{F}_r(y^{\mathbb{P}^n}_r, z^{\mathbb{P}^n}_r)\,dr - \int_s^t z^{\mathbb{P}^n}_r\,dB_r, \qquad s \in [0, t], \mathbb{P}\text{-a.s.}$$

By the standard arguments in BSDE theory we get

$$|y^n_0 - y^{\mathbb{P}^n}_0|^2 \le C\mathbb{E}^{\mathbb{P}}[|\varepsilon + C\rho(\varepsilon)|^2 + |V_t - y^{\mathbb{P}^n}_t|^2 \mathbf{1}_{\hat{E}^n_t}].$$

By Lemma 4.6 and its proof we have $|V_t| \le C\Lambda_t$ and $|y^{\mathbb{P}^n}_t| \le C\Lambda_t$, $\mathbb{P}$-a.s. Then

$$\begin{aligned}\mathcal{Y}^{\mathbb{P}}_0(t, V_t) &\le y^n_0 \le y^{\mathbb{P}^n}_0 + C\big(\varepsilon + \rho(\varepsilon)\big) + C(\mathbb{E}^{\mathbb{P}}[|\Lambda_t|^2 \mathbf{1}_{\hat{E}^n_t}])^{1/2} \\ &\le v(\xi) + C\big(\varepsilon + \rho(\varepsilon)\big) + C(\mathbb{E}^{\mathbb{P}}[|\Lambda_t|^2 \mathbf{1}_{\hat{E}^n_t}])^{1/2}.\end{aligned}$$

Recall (4.10) and notice that $\hat{E}^n_t \downarrow \varnothing$. By sending $n \to \infty$ and applying the dominated convergence theorem we get

$$\mathcal{Y}^{\mathbb{P}}_0(t, V_t) \le v(\xi) + C\big(\varepsilon + \rho(\varepsilon)\big) \qquad \text{for all } \mathbb{P} \in \mathcal{P}_H.$$

Since $\varepsilon > 0$ is arbitrary, we complete the proof. □

We next introduce the right limit of the V which is defined for each (t, ω) and is clearly $\mathbb{F}^+$-progressively measurable:

$$V^+_t := \overline{\lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t}} V_r.$$

LEMMA 4.8. *Assume all the conditions in Theorem 4.5 hold. Then*

$$V^+_t = \lim_{r \in \mathbb{Q} \cap (t,1], r \downarrow t} V_r, \qquad \mathcal{P}_H\text{-q.s. and, thus, } V^+ \text{ is càdlàg } \mathcal{P}_H\text{-q.s.} \tag{4.21}$$

PROOF. For each $\mathbb{P} \in \mathcal{P}_H$, denote

$$\tilde{V}^{\mathbb{P}} := V - \mathcal{Y}^{\mathbb{P}}(1, \xi).$$

Then $\tilde{V}^{\mathbb{P}}_t \ge 0$, $\mathbb{P}$-a.s. For any $0 \le t_1 < t_2 \le 1$, let $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}))$. Note that $\mathcal{Y}^{\mathbb{P}}_{t_1}(t_2, V_{t_2})(\omega) = \mathcal{Y}^{\mathbb{P},t_1,\omega}_{t_1}(t_2, V^{t_1,\omega}_{t_2})$ for $\mathbb{P}$-a.s. ω. Then by Proposition 4.7 we get $V_{t_1} \ge y^{\mathbb{P},t_2}_{t_1}$, $\mathbb{P}$-a.s. Notice that $y^{\mathbb{P},1} = y^{\mathbb{P}}$. Denote

$$\tilde{y}^{\mathbb{P},t_2}_t := y^{\mathbb{P},t_2}_t - y^{\mathbb{P}}_t, \qquad \tilde{z}^{\mathbb{P},t_2}_t := \hat{a}^{-1/2}_t(z^{\mathbb{P},t_2}_t - z^{\mathbb{P}}_t).$$

Then $\tilde{V}^{\mathbb{P}}_{t_1} \geq \tilde{y}^{\mathbb{P},t_2}_{t_1}$, $\mathbb{P}$-a.s. and $(\tilde{y}^{\mathbb{P},t_2}, \tilde{z}^{\mathbb{P},t_2})$ satisfies the following BSDE on $[0, t_2]$:

$$\tilde{y}^{\mathbb{P},t_2}_t = \tilde{V}^{\mathbb{P}}_{t_2} - \int_t^{t_2} f^{\mathbb{P}}_s(\tilde{y}^{\mathbb{P},t_2}_s, \tilde{z}^{\mathbb{P},t_2}_s)\,ds - \int_t^{t_2} \tilde{z}^{\mathbb{P},t_2}_s\,dW^{\mathbb{P}}_s, \qquad \mathbb{P}\text{-a.s.},$$

where

$$f^{\mathbb{P}}_t(\omega, y, z) := \hat{F}_t\big(\omega, y + y^{\mathbb{P}}_t(\omega), \hat{a}_t^{-1/2}(\omega)\big(z + z^{\mathbb{P}}_t(\omega)\big)\big) - \hat{F}_t(\omega, y^{\mathbb{P}}_t(\omega), \hat{a}_t^{-1/2}(\omega) z^{\mathbb{P}}_t(\omega)).$$

Notice that $f^{\mathbb{P}}_t(0,0) = 0$, and $f^{\mathbb{P}}$ is uniformly Lipschitz continuous in (y, z). Following the definition in [15] and [4], $\tilde{V}^{\mathbb{P}}$ is a weak $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Now applying the downcrossing inequality Theorem 6 of [4], one can easily see that, for $\mathbb{P}$-a.e. ω, the limit $\lim_{r \in \mathbb{Q}\cap(t,1], r\downarrow t} \tilde{V}^{\mathbb{P}}_r(\omega)$ exists for all $t \in [0,1]$. Note that $y^{\mathbb{P}}$ is continuous, $\mathbb{P}$-a.s. We get that the $\overline{\lim}$ in the definition of V^+ is in fact the lim, $\mathbb{P}$-a.s. Then,

$$V^+_t = \lim_{r \in \mathbb{Q}\cap(t,1], r\downarrow t} V_r, t \in [0,1] \qquad \text{and, therefore, } V^+ \text{ is càdlàg, } \mathcal{P}_H\text{-q.s.} \qquad \square$$

We are now ready to prove our main duality result.

PROOF OF THEOREM 4.5. We proceed in several steps.

Step 1. We first show that V^+ is a strong $\hat{F}$-supermartingale under each $\mathbb{P} \in \mathcal{P}_H$. For any $\mathbb{P} \in \mathcal{P}_H$, denote $\tilde{V}^{+,\mathbb{P}} := V^+ - y^{\mathbb{P}}$. Given $0 \leq t_1 < t_2 < 1$, let $r^1_n \in \mathbb{Q} \cap (t_1, t_2]$, $r^1_n \downarrow t_1$ and $r^2_n \in \mathbb{Q} \cap (t_2, 1]$, $r^2_n \downarrow t_2$. We have $\tilde{V}^{\mathbb{P}}_{r^1_n} \geq \tilde{y}^{\mathbb{P},r^2_m}_{r^1_n}$, $\mathbb{P}$-a.s. for any $m, n \geq 1$. Sending $n \to \infty$, we get $\tilde{V}^{+,\mathbb{P}}_{t_1} \geq \tilde{y}^{\mathbb{P},r^2_m}_{t_1}$, $\mathbb{P}$-a.s. for any $m \geq 1$. Sending $m \to \infty$, by the stability of BSDEs we get $\tilde{V}^{+,\mathbb{P}}_{t_1} \geq \tilde{y}^{+,\mathbb{P},t_2}_{t_1}$, $\mathbb{P}$-a.s. where

$$\tilde{y}^{+,\mathbb{P},t_2}_t = \tilde{V}^{+,\mathbb{P}}_{t_2} - \int_t^{t_2} f^{\mathbb{P}}_s(\tilde{y}^{+,\mathbb{P},t_2}_s, \tilde{z}^{+,\mathbb{P},t_2}_s)\,ds - \int_t^{t_2} \tilde{z}^{+,\mathbb{P},t_2}_s\,dW^{\mathbb{P}}_s, \qquad \mathbb{P}\text{-a.s.}$$

That is, $\tilde{V}^{+,\mathbb{P}}$ is also a weak $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. Applying Theorem 7 of [4], $\tilde{V}^{+,\mathbb{P}}$ is a strong $f^{\mathbb{P}}$-supermartingale under $\mathbb{P}$. That is, recalling (2.5), for any $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$, we have $\tilde{V}^{+,\mathbb{P}}_{\tau_1} \geq \tilde{y}^{+,\mathbb{P},\tau_2}_{\tau_1}$, $\mathbb{P}$-a.s. where

$$\tilde{y}^{+,\mathbb{P},\tau_2}_t = \tilde{V}^{+,\mathbb{P}}_{\tau_2} - \int_t^{\tau_2} f^{\mathbb{P}}_s(\tilde{y}^{+,\mathbb{P},\tau_2}_s, \tilde{z}^{+,\mathbb{P},\tau_2}_s)\,ds - \int_t^{\tau_2} \tilde{z}^{+,\mathbb{P},\tau_2}_s\,dW^{\mathbb{P}}_s, \qquad t \in [0, \tau_2], \mathbb{P}\text{-a.s.}$$

This implies that $V^+_{\tau_1} \geq y^{+,\mathbb{P},\tau_2}_{\tau_1}$, $\mathbb{P}$-a.s. where $y^{+,\mathbb{P},\tau_2}_t := \tilde{y}^{+,\mathbb{P},\tau_2}_t + y^{\mathbb{P}}_t$, $z^{+,\mathbb{P},\tau_2}_t := \hat{a}_t^{1/2}(\tilde{z}^{+,\mathbb{P},\tau_2}_t + z^{\mathbb{P}}_t)$ satisfy

$$y^{+,\mathbb{P},\tau_2}_t = V^+_{\tau_2} - \int_t^{\tau_2} \hat{F}_s(\tilde{y}^{+,\mathbb{P},\tau_2}_s, \tilde{z}^{+,\mathbb{P},\tau_2}_s)\,ds - \int_t^{t_2} \tilde{z}^{+,\mathbb{P},\tau_2}_s\,dB_s, \qquad \mathbb{P}\text{-a.s.}$$

That is, V^+ is a strong $\hat{F}$-supermartingale under $\mathbb{P}$.

Step 2. For each $\mathbb{P} \in \mathcal{P}_H$, applying the nonlinear Doob–Meyer decomposition in [15], there exist unique ($\mathbb{P}$-a.s.) processes $\bar{\bar{Z}}^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P}, \mathbb{R}^d)$ and $K^{\mathbb{P}} \in \mathbb{I}^2(\mathbb{P}, \mathbb{R})$ such that

$$V_t^+ = V_0^+ + \int_0^t \hat{F}_s(V_s^+, \bar{\bar{Z}}_s^{\mathbb{P}})\, ds + \int_0^t \bar{\bar{Z}}_s^{\mathbb{P}}\, dB_s - K_t^{\mathbb{P}}, \qquad 0 \le t \le 1, \mathbb{P}\text{-a.s.} \tag{4.22}$$

Remark 4.9 below provides a simpler argument for this result. By Karandikar [12], since V^+ is a càdlàg semimartingale under each $\mathbb{P} \in \mathcal{P}_H$, we can define uniquely a universal process $\bar{\bar{Z}}$ by $d\langle V^+, B\rangle_t = \bar{\bar{Z}}_t d\langle B\rangle_t$, so that $\bar{\bar{Z}} = \bar{\bar{Z}}^{\mathbb{P}}$, $dt \times d\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$. Thus, we have

$$V_t^+ = V_0^+ + \int_0^t \hat{F}_s(V_s^+, \bar{\bar{Z}}_s)\, ds + \int_0^t \bar{\bar{Z}}_s\, dB_s - K_t^{\mathbb{P}}, \qquad 0 \le t \le 1, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_H. \tag{4.23}$$

Step 3. We remark that V_0^+ is $\mathcal{F}_0^+$-measurable and is not a constant in general. For each $\mathbb{P} \in \mathcal{P}_H \subset \overline{\mathcal{P}}_S$, and each $r \in \mathbb{Q} \cap (0, 1]$, we have $V_0 \ge y_0^{\mathbb{P},r}$, where $y_0^{\mathbb{P},r}$ is a constant, thanks to the Blumenthal zero–one law (2.5) under $\mathbb{P}$. It is clear that $\lim_{r\downarrow 0} y_0^{\mathbb{P},r} = V_0^+$, $\mathbb{P}$-a.s. Then $V_0 \ge V_0^+$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$. Now by the comparison of ODE and recalling (3.13) and (4.23), we see that $\bar{\bar{Y}}_1^{V_0,\bar{\bar{Z}}} \ge \bar{\bar{Y}}_1^{V_0^+,\bar{\bar{Z}}} \ge V_1^+ = \xi$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_H$. Now by the definition of $\hat{\mathcal{V}}(\xi)$, we get $\bar{\bar{\mathcal{V}}}(\xi) \le V_0 = v(\xi)$. This, together with (3.17), proves $\bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$. Moreover, the process $\bar{\bar{Z}}$ in (4.23) is clearly the optimal control for the problem $\bar{\bar{\mathcal{V}}}(\xi)$. Finally, when F has a progressively measurable optimizer, the existence of the optimal control for the problem $\bar{\mathcal{V}}(\xi)$ is obvious. □

REMARK 4.9. Following a suggestion of Nicole El Karoui, we derive the decomposition (4.22) by the following alternative argument. Consider the following reflected BSDE:

$$\begin{cases} \bar{\bar{Y}}_t^{\mathbb{P}} = \xi - \displaystyle\int_t^1 \hat{F}_s(\bar{\bar{Y}}_s^{\mathbb{P}}, \bar{\bar{Z}}_s^{\mathbb{P}})\, ds - \int_t^1 \bar{\bar{Z}}_s^{\mathbb{P}}\, dB_s + K_1^{\mathbb{P}} - K_t^{\mathbb{P}}, \\ \bar{\bar{Y}}_t^{\mathbb{P}} \ge V_t^+, [\bar{\bar{Y}}_{t-}^{\mathbb{P}} - V_{t-}^+]\, dK_t^{\mathbb{P}} = 0. \end{cases} \qquad 0 \le t \le 1, \mathbb{P}\text{-a.s.}$$

By Lepeltier and Xu [13], the above RBSDE has a unique solution and $\bar{\bar{Y}}^{\mathbb{P}}$ is càdlàg. Then it suffices to show that $\bar{\bar{Y}}^{\mathbb{P}} = V^+$, $\mathbb{P}$-a.s. In fact, if they are not equal, without loss of generality we assume $\bar{\bar{Y}}_0^{\mathbb{P}} > V_0^+$. For each $\varepsilon > 0$, denote $\tau_\varepsilon := \inf\{t : \bar{\bar{Y}}_t^{\mathbb{P}} \le V_t^+ + \varepsilon\}$. Then τ_ε is an $\overline{\mathbb{F}}^{\mathbb{P}}$-stopping time and $\bar{\bar{Y}}_{t-}^{\mathbb{P}} \ge V_{t-}^+ + \varepsilon > V_{t-}^+$ for all $t \le \tau_\varepsilon$. Then $K_t^{\mathbb{P}} = 0$, $t \le \tau_\varepsilon$, and thus

$$\bar{\bar{Y}}_t^{\mathbb{P}} = \bar{\bar{Y}}_{\tau_\varepsilon}^{\mathbb{P}} - \int_t^{\tau_\varepsilon} \hat{F}_s(\bar{\bar{Y}}_s^{\mathbb{P}}, \bar{\bar{Z}}_s^{\mathbb{P}})\,ds - \int_t^{\tau_\varepsilon} \bar{\bar{Z}}_s^{\mathbb{P}}\,dB_s.$$

Note that $\bar{\bar{Y}}_{\tau_\varepsilon}^{\mathbb{P}} \le V_{\tau_\varepsilon}^+ + \varepsilon$, by comparison theorem for BSDEs and following standard arguments we have $\bar{\bar{Y}}_0^{\mathbb{P}} \le y_0^{+,\mathbb{P},\tau_\varepsilon} + C\varepsilon \le V_0^+ + C\varepsilon$. Since ε is arbitrary, this contradicts with $\bar{\bar{Y}}_0^{\mathbb{P}} > V_0^+$.

We conclude this section by establishing a representation formula for V^+, which will be important for our accompanying paper [21]. For each $\mathbb{P} \in \mathcal{P}_H$ and $t \in [0, 1]$, denote

$$\begin{aligned} \mathcal{P}_H(t, \mathbb{P}) &:= \{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\} \quad \text{and} \\ \mathcal{P}_H(t+, \mathbb{P}) &:= \{\mathbb{P}' \in \mathcal{P}_H : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\}. \end{aligned} \tag{4.24}$$

Then we have the following.

PROPOSITION 4.10. *Assume all the conditions in Theorem 4.5 hold. Then, for each $\mathbb{P} \in \mathcal{P}_H$,*

$$V_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(1, \xi) \quad \textit{and}$$

$$V_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(1, \xi), \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Fix $\mathbb{P} \in \mathcal{P}_H$. Denote

$$V_t^{\mathbb{P}} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(1, \xi) \quad \text{and} \quad V_t^{\mathbb{P},+} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H(t+,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(1, \xi).$$

(i) We first prove the equality for V. For each $\mathbb{P}' \in \mathcal{P}_H(t, \mathbb{P}) \subset \mathcal{P}_H$, by (4.17) we have $y_t^{\mathbb{P}'} \le V_t$, $\mathbb{P}'$-a.s. Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t$, then $y_t^{\mathbb{P}'} \le V_t$, $\mathbb{P}$-a.s. and, thus, $V_t^{\mathbb{P}} \le V_t$, $\mathbb{P}$-a.s.

On the other hand, proceeding as in step (ii) of the proof of Proposition 4.7, we define $\mathbb{P}^n$ for each n, ε by (4.18). By (4.19), it is clear that $\mathbb{P}^n \in \mathcal{P}_H(t, \mathbb{P})$. Then it follows from (4.20) that

$$\mathbb{P}[V_t \le V_t^{\mathbb{P}} + \varepsilon + C\rho(\varepsilon)] \ge \mathbb{P}[V_t \le y_t^{\mathbb{P}^n} + \varepsilon + C\rho(\varepsilon)] \ge \mathbb{P}\Bigg[\bigcup_{1 \le i \le n} E_t^i\Bigg] \to 1 \quad \text{as } n \to \infty.$$

That is, $V_t \le V_t^{\mathbb{P}} + \varepsilon + C\rho(\varepsilon)$, $\mathbb{P}$-a.s. for all $\varepsilon > 0$. This implies that $V_t \le V_t^{\mathbb{P}}$, $\mathbb{P}$-a.s.

(ii) We now prove the equality for V^+. First, for each $\mathbb{P}' \in \mathcal{P}_H(t+, \mathbb{P}) \subset \mathcal{P}_H$ and $r \in \mathbb{Q} \cap (t, 1]$, we have $y_r^{\mathbb{P}'} \le V_r$, $\mathbb{P}'$-a.s. Sending $r \downarrow t$, we obtain $y_t^{\mathbb{P}'} \le V_t^+$, $\mathbb{P}'$-a.s. Since both $y_t^{\mathbb{P}'}$ and V_t^+ are $\mathcal{F}_t^+$-measurable and $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$, then $y_t^{\mathbb{P}'} \le V_t^+$, $\mathbb{P}$-a.s. and, thus, $V_t^{\mathbb{P},+} \le V_t$, $\mathbb{P}$-a.s.

On the other hand, for each $r \in \mathbb{Q} \cap (t, 1]$, since $V_r = V_r^{\mathbb{P}}$, $\mathbb{P}$-a.s. Following the same arguments in [21] Theorem 4.3, Step (iii) (we emphasize that there is no danger of cycle proof here!), we have

$$\text{there exist } \mathbb{P}_n \in \mathcal{P}(r, \mathbb{P}) \text{ such that } \mathcal{Y}_r^{\mathbb{P}_n}(1, \xi) \uparrow V_r, \mathbb{P}\text{-a.s.} \tag{4.25}$$

Then, it follows from the stability of BSDEs that

$$\mathcal{Y}_t^{\mathbb{P}}(r, V_r) = \mathcal{Y}_t^{\mathbb{P}}\Big(r, \lim_{n\to\infty} \mathcal{Y}_r^{\mathbb{P}_n}(1, \xi)\Big) = \lim_{n\to\infty} \mathcal{Y}_t^{\mathbb{P}}(r, \mathcal{Y}_r^{\mathbb{P}_n}(1, \xi)).$$

Since $\mathbb{P}_n \in \mathcal{P}(r, \mathbb{P}) \subset \mathcal{P}(t+, \mathbb{P})$, we have

$$\mathcal{Y}_t^{\mathbb{P}}(r, V_r) = \lim_{n\to\infty} \mathcal{Y}_t^{\mathbb{P}_n}(r, \mathcal{Y}_r^{\mathbb{P}_n}(1, \xi)) = \lim_{n\to\infty} \mathcal{Y}_t^{\mathbb{P}_n}(1, \xi) \le V_t^{\mathbb{P},+}, \qquad \mathbb{P}\text{-a.s.}$$

Sending $r \downarrow t$, by the stability of BSDEs again we obtain $V_t^+ \le V_t^{\mathbb{P},+}$, $\mathbb{P}$-a.s. $\square$

After the completion of this paper, Marcel Nutz provides us the following result which shows that, under our conditions that F and ξ are uniformly continuous in ω, actually $V^+ = V$. However, we decide to keep our original arguments because they are applicable to more general cases, for example, the case in Section 5 where we do not require the uniform continuity of F and ξ.

PROPOSITION 4.11 (M. Nutz). *Assume all the conditions in Theorem 4.5 hold. Then $V_t^+ = V_t$, $\mathcal{P}_H$-q.s.*

PROOF. First, by Lemma 4.6 V^+ is uniformly continuous in ω with the same modulus of continuity function ρ. Since V^+ is $\mathbb{F}^+$-progressively measurable, for any $\delta > 0$, we have $|V_t^+(\omega) - V_t^+(\omega')| \le C\rho(\|\omega - \omega'\|_{t+\delta})$. Sending $\delta \to 0$, we get $|V_t^+(\omega) - V_t^+(\omega')| \le C\rho(\|\omega - \omega'\|_t)$ and, thus, V^+ is $\mathbb{F}$-progressively measurable.

By Proposition 4.10, it is clear that $V_t^+ \le V_t$, $\mathcal{P}_H$-q.s. On the other hand, for any $\mathbb{P} \in \mathcal{P}_H$ and $\mathbb{P}' \in \mathcal{P}_H(t, \mathbb{P})$, by the second equality of Proposition 4.10 we have $\mathcal{Y}_t^{\mathbb{P}'}(1, \xi) \le V_t^+$, $\mathbb{P}'$-a.s. Since both sides of above are $\mathcal{F}_t$-measurable and $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t$, we have $\mathcal{Y}_t^{\mathbb{P}'}(1, \xi) \le V_t^+$, $\mathbb{P}$-a.s. Then the first equality of Proposition 4.10 implies that $V_t \le V_t^+$, $\mathbb{P}$-a.s. Therefore, $V^+ = V$, $\mathcal{P}_H$-q.s. $\square$

5. A weaker version of the second order target problem. The purpose of this section is to suggest a slight modification of the second order stochastic target problem so that its value is not affected by the relaxations of Section 3.3. The key tool for this is the aggregation approach developed in our accompanying paper [22]. The idea is to restrict our attention to an (uncountable) subset of $\mathcal{P}_H$, constructed out of a countable subset, so that a dominating measure is available.

As a consequence of this modified setup, we shall remove the continuity assumption on ξ. However, we still assume the nonlinearity H satisfies Assumption 3.1, in particular, H is uniformly continuous in ω and the domain D_{F_t} of its convex conjugate F is deterministic; see Section 6 for the general case.

5.1. *The dominating probability measure* $\hat{\mathbb{P}}$. Throughout this section we fix a *countable* subset $T_0 \subset [0,1]$ containing the end-points $\{0,1\}$, together with a *countable* sequence $\mathcal{A}_0 := \{\alpha^i, i \geq 1\}$ of deterministic integrable mappings $\alpha^i : [0,1] \to \mathbb{S}_d^{>0}$ satisfying the concatenation property:

$$\alpha^i \mathbf{1}_{[0,t)} + \alpha^j \mathbf{1}_{[t,1]} \in \mathcal{A}_0 \qquad \text{for all } i, j \geq 1, t \in T_0. \tag{5.1}$$

Note that α^i is deterministic, then by Lemma 2.2, $\hat{a} = \alpha^i$, $\mathbb{P}^{\alpha^i}$-a.s., and, thus, $\mathcal{A}_0$ is a generating class of diffusion coefficients in the sense of Definition 4.7 in [22]. Following Definition 4.8 in [22], let $\mathcal{A}$ be the separable class of diffusion coefficients generated by $(\mathcal{A}_0, T_0)$. Following Proposition 8.3 in [22], let $\mathcal{P}(\mathcal{A}) \subset \overline{\mathcal{P}}_S$ denote the corresponding measures. Then, by Definition 4.8 in [22],

$$\mathbb{P} \in \mathcal{P}(\mathcal{A}) \quad \text{if and only if} \quad \hat{a} = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \alpha^i \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}, \qquad \mathbb{P}\text{-a.s.} \tag{5.2}$$

for some

- sequence of $\mathbb{F}$-stopping times $\{\tau_n, n \geq 0\}$ with values in T_0, with $\tau_0 = 0$, $\tau_n < \tau_{n+1}$ on $\{\tau_n < 1\}$, and $\inf\{n : \tau_n = 1\} < \infty$,
- and some partition $\{E_i^n, i \geq 1\} \subset \mathcal{F}_{\tau_n}$ of Ω.

Finally, we assume $\mathbb{P}^i := \mathbb{P}^{\alpha^i} \in \mathcal{P}_H$ and denote $\mathcal{P}_H^{\mathcal{A}} := \mathcal{P}(\mathcal{A}) \cap \mathcal{P}_H$.

The dominating measure is now defined by

$$\hat{\mathbb{P}} := \hat{\mathbb{P}}^{\mathcal{A}_0, T_0} := \sum_{i=1}^{\infty} 2^{-i} \mathbb{P}^i. \tag{5.3}$$

Clearly, $\hat{\mathbb{P}}$ is a dominating measure of $\{\mathbb{P}^i, i \geq 1\}$. By Proposition 4.11 in [22], $\hat{\mathbb{P}}$ is in fact a dominating measure of $\mathcal{P}(\mathcal{A})$, and thus of $\mathcal{P}_H^{\mathcal{A}}$. Therefore, $\mathcal{P}_H^{\mathcal{A}}$-q.s. reduces to $\hat{\mathbb{P}}$-a.s.

5.2. *The second order target problem under* $\hat{\mathbb{P}}$. Recall the spaces defined in (3.7). Let $\hat{\mathbb{L}}_0^2(D) := \bigcap_{i \geq 1} \mathbb{L}^2(\mathbb{P}^i, D)$, and define the spaces $\hat{\mathbb{H}}_0^2(D)$, $\hat{\mathbb{D}}_0^2(D)$, $\hat{\mathbb{S}}_0^2(D)$, $\hat{\mathbb{I}}_0^2(D)$, $\hat{\mathbb{G}}_0^2(D_H)$ and $\widehat{\mathcal{SM}}_0^2(\mathbb{R}^d)$ similarly.

Now for an $\mathcal{F}_1$-measurable r.v. ξ, the modified second order target problem under $\hat{\mathbb{P}}$ is

$$\mathcal{V}_0(\xi) := \inf\{y \in \mathbb{R} : Y_1^{y,Z} \geq \xi, \hat{\mathbb{P}}\text{-a.s. for some } Z \in \widehat{\mathcal{SM}}_0^2(\mathbb{R}^d)\}, \tag{5.4}$$

where $Y^{y,Z} \in \hat{\mathbb{S}}^2_0(\mathbb{R})$ is defined by (3.9), except that $\mathcal{P}_H$-q.s. is replaced with $\hat{\mathbb{P}}$-a.s. (or, equivalently, $\mathcal{P}_H^{\mathcal{A}}$-q.s.).

Next, notice that the families of processes $\{\bar{Y}^{\mathbb{P}^i}, i \geq 1\}$ and $\{\bar{\bar{Y}}^{\mathbb{P}^i}, i \geq 1\}$, defined by (3.11) and (3.13) respectively, can be aggregated into processes $\bar{Y}$ and $\bar{\bar{Y}}$, thanks to Theorem 5.1 in [22]. We then define the following relaxations of (5.4):

$$\bar{\mathcal{V}}_0(\xi) := \inf\{y : \bar{Y}_1^{y,\bar{Z},\bar{\Gamma}} \geq \xi, \hat{\mathbb{P}}\text{-a.s. for some } (\bar{Z}, \bar{\Gamma}) \in \hat{\mathbb{H}}^2_0(\mathbb{R}^d) \times \hat{\mathbb{G}}^2_0(D_H)\}, \tag{5.5}$$

$$\bar{\bar{\mathcal{V}}}_0(\xi) := \inf\{y : \bar{\bar{Y}}_1^{y,\bar{\bar{Z}}} \geq \xi, \hat{\mathbb{P}}\text{-a.s. for some } \bar{\bar{Z}} \in \hat{\mathbb{H}}^2_0(\mathbb{R}^d)\}. \tag{5.6}$$

Finally, our modified dual formulation under $\hat{\mathbb{P}}$ is

$$v_0(\xi) := \sup_{\mathbb{P} \in \mathcal{P}_H^{\mathcal{A}}} \mathcal{Y}_0^{\mathbb{P}}(1, \xi), \tag{5.7}$$

where $\mathcal{Y}^{\mathbb{P}}$ is defined by means of the BSDE (3.15). Similar to (3.17), it is obvious that

$$\mathcal{V}_0(\xi) \geq \bar{\mathcal{V}}_0(\xi) = \bar{\bar{\mathcal{V}}}_0(\xi) \geq v_0(\xi). \tag{5.8}$$

5.3. *The main results*. In the present modified setting, we have the equality between the second order target problem and its first relaxation. For this, the following technical condition is needed.

ASSUMPTION 5.1. For any $\varepsilon > 0$, there is an $\mathbb{F}$-progressively measurable ε-maximizer $\gamma^\varepsilon := \gamma_t^\varepsilon(y, z)$ of (3.1) such that, for every $\delta > 0$,

$$|\gamma_t^\varepsilon(y, z)| \leq C_{\varepsilon,\delta}(1 + |y| + |z|), \qquad \hat{\mathbb{P}}\text{-a.s. on } \{\hat{a}_t \geq \delta I_d\}, \text{ for some } C_{\varepsilon,\delta} > 0.$$

Similar to (4.8), for each $i \geq 1$, define

$$\begin{aligned} \Lambda^i &:= \operatorname*{ess\,sup}_{0 \leq t \leq 1}{}^{\mathbb{P}^i} \Lambda_t^i, \\ \Lambda_t^i &:= \operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}_H^{\mathcal{A}}(t, \mathbb{P}^i)}{}^{\mathbb{P}^i} \left(\mathbb{E}_t^{\mathbb{P}}\left[|\xi|^2 + \int_t^1 |\hat{F}_s(0)|^2\right]\right)^{1/2}, \end{aligned} \tag{5.9}$$

where, as in Proposition 4.10,

$$\mathcal{P}_H^{\mathcal{A}}(t, \mathbb{P}^i) := \{\mathbb{P} \in \mathcal{P}_H^{\mathcal{A}} : \mathbb{P} = \mathbb{P}^i \text{ on } \mathcal{F}_t\}. \tag{5.10}$$

THEOREM 5.2. *Let Assumptions 3.1, 3.2 and 5.1 hold true. Assume further that*

$$\mathbb{E}^{\mathbb{P}^i}[|\Lambda^i|^2] < \infty \qquad \textit{for all } i \geq 1. \tag{5.11}$$

Then for any $\xi \in \hat{\mathbb{L}}^2_0(\mathbb{R})$, *we have* $\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi) = \bar{\bar{\mathcal{V}}}_0(\xi) = v_0(\xi)$, *and existence holds for the problem* $\bar{\bar{\mathcal{V}}}_0(\xi)$. *Moreover, if F has a progressively measurable optimizer, existence also holds for the problem* $\bar{\mathcal{V}}_0(\xi)$.

This main result $\mathcal{V}_0(\xi)=\bar{\mathcal{V}}_0(\xi)$ will be proved in the next subsection. The equality $\bar{\mathcal{V}}_0(\xi)=\bar{\bar{\mathcal{V}}}_0(\xi)$ was already stated in (5.8). The remaining statements are analogous to the proof of Theorem 4.5. We thus omit the proof and only comment on it:

- We first define for every $i\ge 1$ the dynamic problem:

$$V_t^i := \underset{\mathbb{P}\in\mathcal{P}_H^{\mathcal{A}}(t,\mathbb{P}^i)}{\operatorname{ess\,sup}^{\mathbb{P}^i}} \mathcal{Y}_t^{\mathbb{P}}(1,\xi), \tag{5.12}$$

  where $\mathcal{Y}^{\mathbb{P}}$ is defined by means of the BSDE (3.15) and $\mathcal{P}_H^{\mathcal{A}}(t,\mathbb{P}^i)$ is given in (5.10). In light of Proposition 4.10, this is the analogue of the process V in (3.10), except that this is defined $\mathbb{P}^i$-a.s. for every $i\ge 1$. However, using the aggregation Theorem 5.1 in [22], we can aggregate the family $\{V^i, i\ge 1\}$ into a universal process V, that is, $V=V^i$, $\mathbb{P}^i$-a.s. for all $i\ge 1$.
- Combining the arguments of Lemma 7.2 in [22] and Proposition 4.7, we have the dynamic programming principle:

$$V_{t_1} = \underset{\mathbb{P}\in\mathcal{P}_H^{\mathcal{A}}(t,\mathbb{P}^i)}{\operatorname{ess\,sup}^{\mathbb{P}^i}} \mathcal{Y}_{t_1}^{\mathbb{P}^i}(t_2, V_{t_2}), \qquad \mathbb{P}^i\text{-a.s. for all } i\ge 1.$$

- Exploiting the connection with reflected BSDEs, we then obtain the decomposition (4.23) under each $\mathbb{P}^i$, and we conclude by the definition of the problem $\bar{\bar{\mathcal{V}}}_0(\xi)$.

Our final result shows that, except for the initial second order target problem, under certain conditions all other problems are not altered by the modification of this section:

THEOREM 5.3. *Let Assumptions 3.1, 3.2, 4.4 hold, and assume further that*:

– *F is uniformly continuous in a for $a\in D_{F_t}$, and for all (t,ω,y,z) and all $a\in D_{F_t}$*:

$$\begin{gathered} |\xi(\omega)|\le C(1+\|\omega\|_1) \quad \text{and} \\ |F_t(\omega,y,z,a)|\le C(1+\|\omega\|_t+|y|+|z|+|a^{1/2}|), \end{gathered} \tag{5.13}$$

– *$\mathcal{P}_H^{\mathcal{A}}$ is dense in $\mathcal{P}_H$ in the sense that for any $\mathbb{P}=\mathbb{P}^\alpha\in\mathcal{P}_H$ and any $\varepsilon>0$*:

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^1 |(\alpha_t^\varepsilon)^{1/2}-\alpha_t^{1/2}|^2\,dt\right]\le\varepsilon \qquad \text{for some } \mathbb{P}^\varepsilon=\mathbb{P}^{\alpha^\varepsilon}\in\mathcal{P}_H^{\mathcal{A}}. \tag{5.14}$$

Then $v_0(\xi)=v(\xi)$ and, thus, $v_0(\xi)$ is independent from the choice of the sets $\mathcal{A}_0$ and T_0.

Assume further that Assumption 5.1 and (4.10) hold. Then

$$\mathcal{V}_0(\xi)=\bar{\mathcal{V}}_0(\xi)=\bar{\bar{\mathcal{V}}}_0(\xi)=v_0(\xi)=v(\xi)=\bar{\bar{\mathcal{V}}}(\xi)=\bar{\mathcal{V}}(\xi).$$

PROOF. By (3.17), Theorems 4.5 and 5.2, clearly it suffices to prove the first statement. Since $\mathcal{P}_H^{\mathcal{A}} \subset \mathcal{P}_H$, we have $v_0(\xi) \le v(\xi)$. Now for any $\mathbb{P} = \mathbb{P}^\alpha \in \mathcal{P}_H$ and any $\varepsilon > 0$, let $\mathbb{P}^\varepsilon = \mathbb{P}^{\alpha^\varepsilon} \in \mathcal{P}_H^{\mathcal{A}}$ satisfy (5.14). Recall the $W^{\mathbb{P}}$ defined in (2.2). Notice that

$$Y_t^{\mathbb{P}} = \xi(B.) + \int_t^1 F_s(B., Y_s^{\mathbb{P}}, Z_s^{\mathbb{P}}, \hat{a}_s)\, ds - \int_t^1 Z_s^{\mathbb{P}} \hat{a}_s^{1/2}\, dW_s^{\mathbb{P}}, \qquad 0 \le t \le 1, \mathbb{P}\text{-a.s.}$$

Let $(\tilde{Y}^{\mathbb{P}}, \tilde{Z}^{\mathbb{P}})$ denote the solution to the following BSDE under $\mathbb{P}_0$:

$$\tilde{Y}_t^{\mathbb{P}} = \xi(X_\cdot^\alpha) + \int_t^1 F_s(X_\cdot^\alpha, \tilde{Y}_s^{\mathbb{P}}, \tilde{Z}_s^{\mathbb{P}}, \alpha_s)\, ds - \int_t^1 \tilde{Z}_s^{\mathbb{P}} \alpha_s^{1/2}\, dB_s,$$
$$0 \le t \le 1, \mathbb{P}_0\text{-a.s.}$$

By Lemma 2.2, the $\mathbb{P}$-distribution of $Y^{\mathbb{P}}$ is equal to the $\mathbb{P}_0$-distribution of $\tilde{Y}^{\mathbb{P}}$. This, together with the Blumenthal zero–one law, implies that $Y_0^{\mathbb{P}} = \tilde{Y}_0^{\mathbb{P}}$. Similarly, $Y_0^{\mathbb{P}^\varepsilon} = \tilde{Y}_0^{\mathbb{P}^\varepsilon}$, where $(\tilde{Y}_0^{\mathbb{P}^\varepsilon}, \tilde{Z}_0^{\mathbb{P}^\varepsilon})$ is the solution of

$$\tilde{Y}_t^{\mathbb{P}^\varepsilon} = \xi(X_\cdot^{\alpha^\varepsilon}) + \int_t^1 F_s(X_\cdot^{\alpha^\varepsilon}, \tilde{Y}_s^{\mathbb{P}^\varepsilon}, \tilde{Z}_s^{\mathbb{P}^\varepsilon}, \alpha_s^\varepsilon)\, ds - \int_t^1 \tilde{Z}_s^{\mathbb{P}^\varepsilon} (\alpha_s^\varepsilon)^{1/2}\, dB_s,$$
$$0 \le t \le 1, \mathbb{P}_0\text{-a.s.}$$

By Proposition 2.1 from El Karoui, Peng and Quenez [9], we deduce that

$$\begin{aligned}
|Y_0^{\mathbb{P}} - Y_0^{\mathbb{P}^\varepsilon}|^2 &= |\tilde{Y}_0^{\mathbb{P}} - \tilde{Y}_0^{\mathbb{P}^\varepsilon}|^2 \\
&\le C\mathbb{E}^{\mathbb{P}_0}\biggl[|\xi(X_\cdot^\alpha) - \xi(X_\cdot^{\alpha^\varepsilon})|^2 + \int_0^1 |F_t(X_\cdot^\alpha, \tilde{Y}_t^{\mathbb{P}}, \tilde{Z}_t^{\mathbb{P}}, \alpha_t) \\
&\qquad - F_t(X_\cdot^{\alpha^\varepsilon}, \tilde{Y}_t^{\mathbb{P}}, \tilde{Z}_t^{\mathbb{P}}, \alpha_t^\varepsilon)|^2\, dt\biggr].
\end{aligned}$$

By (5.13) we have

$$\begin{aligned}
|\xi(X_\cdot^{\alpha^\varepsilon})| &\le C\|X^{\alpha^\varepsilon}\|_1 \le C\|X^\alpha\|_1 + C\|X^\alpha - X^{\alpha^\varepsilon}\|_1; \\
|F_t(X_\cdot^{\alpha^\varepsilon}, \tilde{Y}_t^{\mathbb{P}}, \tilde{Z}_t^{\mathbb{P}}, \alpha_t^\varepsilon)| &\le C(\|X^{\alpha^\varepsilon}\|_t + |\tilde{Y}_t^{\mathbb{P}}| + |\tilde{Z}_t^{\mathbb{P}}| + |\alpha_t^\varepsilon|^{1/2}) \\
&\le C(\|X^\alpha\|_1 + |\tilde{Y}_t^{\mathbb{P}}| + |\tilde{Z}_t^{\mathbb{P}}| + |\alpha_t|^{1/2}) \\
&\quad + C(\|X^{\alpha^\varepsilon} - X^\alpha\|_1 + |\alpha_t^\varepsilon - \alpha_t|^{1/2}).
\end{aligned}$$

It follows from (5.14) that $\mathbb{E}^{\mathbb{P}_0}[\sup_{0\le t\le 1} |X_t^\alpha - X_t^{\alpha^\varepsilon}|^2] \le \varepsilon$. Then $|\xi(X_\cdot^{\alpha^\varepsilon})|^2$ is uniformly integrable under $\mathbb{P}_0$ and $|F_t(X_\cdot^{\alpha^\varepsilon}, \tilde{Y}_t^{\mathbb{P}}, \tilde{Z}_t^{\mathbb{P}}, \alpha_t^\varepsilon)|^2$ is uniformly integrable under $dt \times d\mathbb{P}_0$. Now by the uniform continuity of ξ and F we get $\lim_{\varepsilon\to 0} |Y_0^{\mathbb{P}} - Y_0^{\mathbb{P}^\varepsilon}| = 0$. This implies that $Y_0^{\mathbb{P}} \le v_0(\xi)$ for all $\mathbb{P} \in \mathcal{P}_H$, and, therefore, $v(\xi) \le v_0(\xi)$. $\square$

A sufficient condition for the uniform continuity of F in terms of a is that D_H is bounded. We next provide a sufficient condition for the density condition (5.14).

PROPOSITION 5.4. *Let Assumption 3.1 hold and suppose that the domain D_F of F is independent of t. Assume further that T_0 is dense in* $[0, 1]$, *and there exists a countable dense subset $A \subset D_F$ such that, for all $a \in A$, the constant mapping a is in $\mathcal{A}_0$. Then $\mathcal{P}_H^A$ is dense in $\mathcal{P}_H$ in the sense of* (5.14).

PROOF. (i) We first prove that $\mathbb{P}^\alpha \in \mathcal{P}_H^A$ for any α taking the following form:

$$\text{There exist } 0 = t_0 < \cdots < t_n = 1 \text{ in } T_0 \text{ and a finite subset } A_n \subset A \text{ s.t.}$$
$$\alpha = \sum_{i=0}^{n-1} \alpha_{t_i} \mathbf{1}_{[t_i, t_{i+1})} + \alpha_{t_n} \mathbf{1}_{\{t_n\}} \text{ and } \alpha \text{ takes values in } A_n. \tag{5.15}$$

In fact, since $A_n \subset \mathbb{S}_d^{>0}$ is finite, then α has both lower (away from 0) and upper bounds, and thus $\mathbb{P}^\alpha$ is well defined. Using the notation in Lemma 2.2, we set $a := \alpha \circ \beta_\alpha$. Clearly, $a = \sum_{i=0}^{n-1} a_{t_i} \mathbf{1}_{[t_i, t_{i+1})} + a_{t_n} \mathbf{1}_{\{t_n\}}$ and a also takes values in A_n. By Lemma 2.2 we know $\hat{a} = a$, $dt \times d\mathbb{P}^\alpha$-a.s. and $\mathbb{P}^\alpha$ satisfies (3.4). Then it follows from (5.2) that $\mathbb{P}^\alpha \in \mathcal{P}(\mathcal{A})$. Moreover, by numerating $A_n = \{a^i, i = 1, \ldots, n\}$, we have $a = \sum_{i=1}^n a^i \mathbf{1}_{E_i}$, where $E_i := \{\omega : a_t(\omega) = a^i, 0 \le t \le 1\}$, $i = 1, \ldots, n$, form a partition of $\mathcal{F}_1$. By Lemma 5.2 in [22], we know $\mathbb{P}^\alpha = \mathbb{P}^{a^i}$ on E_i, that is, $\mathbb{P}^\alpha(E \cap E_i) = \mathbb{P}^{a^i}(E \cap E_i)$ for all $E \in \mathcal{F}_1$. Since each $\mathbb{P}^{a^i} \in \mathcal{P}_H^A$ satisfies (3.5), then so does $\mathbb{P}^\alpha$. This implies that $\mathbb{P}^\alpha \in \mathcal{P}_H$, and, therefore, $\mathbb{P}^\alpha \in \mathcal{P}_H^A$.

(ii) Now fix $\mathbb{P}^\alpha \in \mathcal{P}_H$. Since $\hat{a} \in D_F$, $dt \times d\mathbb{P}^\alpha$-a.s. by Lemma 2.2 we know $\alpha \in D_F$, $dt \times d\mathbb{P}_0$-a.s. For any $\varepsilon > 0$, since $\mathbb{E}^{\mathbb{P}_0}[\int_0^1 |\alpha_t|^2 \, dt] < \infty$, by standard arguments there exists $\mathbb{F}$-progressive measurable càdlàg process α^ε such that α^ε takes values in D_F and $\mathbb{E}^{\mathbb{P}_0}[\int_0^1 |(\alpha_t^\varepsilon)^{1/2} - (\alpha_t)^{1/2}|^2 \, dt] \le \varepsilon$. Now by the dense property of T_0 and A, there exists $\tilde{\alpha}^\varepsilon$ in the form (5.15) such that $\mathbb{E}^{\mathbb{P}_0}[\int_0^1 |(\tilde{\alpha}_t^\varepsilon)^{1/2} - (\alpha_t^\varepsilon)^{1/2}|^2 \, dt] \le \varepsilon$. Then $\mathbb{E}^{\mathbb{P}_0}[\int_0^1 |(\tilde{\alpha}_t^\varepsilon)^{1/2} - (\alpha_t)^{1/2}|^2 \, dt] \le C\varepsilon$. Since $\mathbb{P}^{\tilde{\alpha}^\varepsilon} \in \mathcal{P}_H^A$ by the above (i), the proof is complete. $\square$

5.4. *Proof of Theorem 5.2* $[\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi)]$. The proof requires the following extension of Bank and Baum [1] to the nonlinear case.

LEMMA 5.5. *Let $h_t(\omega, x, z) : [0, 1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathbb{F}$-progressively measurable, uniformly Lipschitz continuous in (x, z), and $h(0, 0) \in \hat{\mathbb{H}}_0^2(\mathbb{R})$. For a process $Z \in \hat{\mathbb{H}}_0^2(\mathbb{R}^d)$, let $X^Z \in \hat{\mathbb{S}}_0^2(\mathbb{R})$ denote the aggregating process of the solutions to the following ODE* (*with random coefficients*) *under each $\mathbb{P}^i$*:

$$X_t^Z = x + \int_0^t h_s(X_s^Z, Z_s) \, ds + \int_0^t Z_s \, dB_s, \qquad 0 \le t \le 1, \hat{\mathbb{P}}\text{-}a.s.$$

Then for any $\varepsilon > 0$, there exists $Z^\varepsilon \in \hat{\mathbb{H}}_0^2(\mathbb{R}^d)$ with finite variation, $\hat{\mathbb{P}}$-a.s. such that

$$\sup_{0\le t\le 1} |X_t^{Z^\varepsilon} - X_t^Z| \le \varepsilon, \qquad \hat{\mathbb{P}}\text{-}a.s.$$

PROOF. Recall (3.4). For $i \ge 1$, let $C_i = C_i(\underline{a}_{\mathbb{P}^i}, \overline{a}_{\mathbb{P}^i}) \ge 1$ be some constants which will be specified later. Note that (3.4) implies $\hat{a}^{1/2} Z \in \mathbb{H}^2(\mathbb{P}^i, \mathbb{R}^d)$. Define $\tilde{\mathbb{P}} := \sum_{i=1}^\infty \nu_i \mathbb{P}^i$, where $\nu_1 := 1 - \sum_{i=2}^\infty \nu_i > 0$, and

$$\frac{1}{\nu_i} := 2^i C_i \biggl[1 + \mathbb{E}^i\biggl\{\sup_{0\le t\le 1} |X_t^Z|^2 + \int_0^1 [|Z_t|^2 + |\hat{a}_t^{1/2} Z_t|^2 + |h_t(0,0)|^2]\,dt\biggr\}\biggr] \qquad \text{for } i \ge 2. \tag{5.16}$$

Then $\tilde{\mathbb{P}}$ is probability measure equivalent to $\hat{\mathbb{P}}$, $\mathbb{P}^i \le \nu_i^{-1}\tilde{\mathbb{P}}$, and

$$X^Z \in \mathbb{S}^2(\tilde{\mathbb{P}}, \mathbb{R}) \quad \text{and} \quad Z, \hat{a}^{1/2} Z \in \mathbb{H}^2(\tilde{\mathbb{P}}, \mathbb{R}^d). \tag{5.17}$$

Obviously, it suffices to find $Z^\varepsilon \in \mathbb{H}^2(\tilde{\mathbb{P}}, \mathbb{R})$ such that

$$Z^\varepsilon \text{ has finite variation and } \sup_{0\le t\le 1} |X_t^{Z^\varepsilon} - X_t^Z| \le \varepsilon, \qquad \tilde{\mathbb{P}}\text{-a.s.}$$

(1) Denote $X := X^Z$. As in Bank and Baum [1], we first prove that, for any $\mathbb{F}$-stopping time τ and any $\tilde{X}_\tau, \tilde{Z}_\tau \in \mathbb{L}^2(\tilde{\mathbb{P}}, \mathcal{F}_\tau)$, there exists a process $Z^{\varepsilon,\tau} \in \mathbb{H}^2(\tilde{\mathbb{P}}, \mathbb{R}^d)$ such that $Z_\tau^{\varepsilon,\tau} = \tilde{Z}_\tau$, $Z^{\varepsilon,\tau}$ is absolutely continuous in t with finite variation on $[\tau, 1]$, and

$$\tilde{\mathbb{P}}\Bigl[\sup_{\tau\le t\le 1} e^{-L(t-\tau)} |X_t^{\varepsilon,\tau} - X_t| \ge \varepsilon + |\tilde{X}_\tau - X_\tau|\Bigr] \le \varepsilon, \tag{5.18}$$

where L is the uniform Lipschitz constant of h with respect to x, and

$$X_t^{\varepsilon,\tau} = \tilde{X}_\tau + \int_\tau^t h_s(X_s^{\varepsilon,\tau}, Z_s^{\varepsilon,\tau})\,ds + \int_\tau^t Z_s^{\varepsilon,\tau}\,dB_s, \qquad t \ge \tau, \tilde{\mathbb{P}}\text{-a.s.} \tag{5.19}$$

For simplicity we assume $\tau = 0$ and $\tilde{X}_\tau = \tilde{x}$, $\tilde{Z}_\tau = \tilde{z}$. Set $Z_t := \tilde{z}$ for $t < 0$, and define $Z_t^n := n\int_{t-1/n}^t Z_s\,ds$ for every $n \ge 1$. Then $Z_0^n = \tilde{z}$, Z^n is continuous in t with finite variation, and, by (5.17),

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\biggl\{\int_0^1 [|Z_t^n - Z_t|^2 + |\hat{a}_t^{1/2}(Z_t^n - Z_t)|^2]\,dt\biggr\} = 0.$$

Let X^n and $\tilde{X}$ be defined by $X_0^n = \tilde{X}_0 = \tilde{x}$ and

$$dX_t^n = h_t(X_t^n, Z_t^n)\,dt + Z_t^n\,dB_t, \qquad d\tilde{X}_t = h_t(\tilde{X}_t, Z_t)\,dt + Z_t\,dB_t.$$

By the Lipschitz property of h, it follows from standard estimates on SDEs that

$$\lim_{n\to\infty} \mathbb{E}^{\tilde{\mathbb{P}}}\Bigl\{\sup_{0\le t\le 1} |X_t^n - \tilde{X}_t|^2\,dt\Bigr\} = 0 \quad \text{and} \quad e^{-Lt}|\tilde{X}_t - X_t| \le |\tilde{x} - x|.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}
&\tilde{\mathbb{P}}\Big[\sup_{0\le t\le 1} e^{-Lt}|X_t^n - X_t| \ge \varepsilon + |\tilde{x} - x|\Big] \\
&\quad\le \tilde{\mathbb{P}}\Big[\sup_{0\le t\le 1} e^{-Lt}|X_t^n - \tilde{X}_t| \ge \varepsilon\Big] \\
&\quad\le \tilde{\mathbb{P}}\Big[\sup_{0\le t\le 1} |X_t^n - \tilde{X}_t| \ge \varepsilon\Big] \longrightarrow 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

By setting $Z^{\varepsilon,\tau} := Z^n$ for n large enough so that the above probability is less than ε, we complete the proof of (5.18). By our construction, notice that

$$Z_{\tau'}^{\varepsilon,\tau} \in \mathbb{L}^2(\tilde{\mathbb{P}}, \mathcal{F}_{\tau'}) \qquad \text{for every } \mathbb{F}\text{-stopping time } \tau' \ge \tau. \tag{5.20}$$

(2) In this step, we construct a sequence of $\mathbb{F}$-stopping times $(\tau_i)_{i\ge 0}$ which yields the required approximation $(X^\varepsilon, Z^\varepsilon)$. We initialize our construction by $\tau_0 := 0$, $\tilde{X}_0 = X_0$ and $\tilde{Z}_0$ arbitrary. Let $\varepsilon > 0$ be fixed, and set $\varepsilon_n := 2^{-n}e^{-L}\varepsilon$.

Assume τ_i is defined and $(X^\varepsilon, Z^\varepsilon)$ have been defined over $[0, \tau_i]$ with $Z_{\tau_i}^\varepsilon \in \mathbb{L}^2(\tilde{\mathbb{P}}, \mathcal{F}_{\tau_i})$. By (5.18) there exists $\tilde{Z}^{i+1} \in \mathbb{H}^2(\tilde{\mathbb{P}}, \mathbb{R}^d)$ which is absolutely continuous in t and has finite variation on $[\tau_i, 1]$ such that $\tilde{Z}_{\tau_i}^{i+1} = Z_{\tau_i}^\varepsilon$ and

$$\tilde{\mathbb{P}}\Big\{\sup_{\tau_i\le t\le 1} e^{-L(t-\tau_i)}|\tilde{X}_t^{i+1} - X_t| \ge \varepsilon_{i+1} + |X_{\tau_i}^\varepsilon - X_{\tau_i}|\Big\} \le \varepsilon_{i+1},$$

where $\{\tilde{X}_t^{i+1}, t \in [\tau_i, 1]\}$ is the solution of the ODE (5.19) with initial condition $\tilde{X}_{\tau_i}^{i+1} = X_{\tau_i}^\varepsilon$. Denote

$$\tau_{i+1} := 1 \wedge \inf\big\{t \ge \tau_i : e^{-L(t-\tau_i)}|\tilde{X}_t^{i+1} - X_t| = \varepsilon_{i+1} + |X_{\tau_i}^\varepsilon - X_{\tau_i}|\big\},$$

and define

$$X_t^\varepsilon := \tilde{X}_t^{i+1}, \qquad Z_t^\varepsilon := \tilde{Z}_t^{i+1}, \qquad \forall t \in (\tau_i, \tau_{i+1}].$$

In particular, it follows from (5.20) that $Z_{\tau_{i+1}}^\varepsilon \in \mathbb{L}^2(\tilde{\mathbb{P}}, \mathcal{F}_{\tau_{i+1}})$.

We remark that, although the filtration $\mathbb{F}$ is not right continuous, since $\tilde{X}_t^{i+1} - X_t$ is continuous, the τ_{i+1} defined here is an $\mathbb{F}$-stopping time. Since $\sum_{i=1}^\infty \tilde{\mathbb{P}}(\tau_i < 1) \le \sum_{i=1}^\infty \varepsilon_i < 1$, it follows from the Borel–Cantelli Lemma that $\tilde{\mathbb{P}}(\tau_i < 1, \forall i) = 0$. That is, $(X^\varepsilon, Z^\varepsilon)$ is well defined on $[0, 1]$ and Z^ε is absolutely continuous in t and has finite variation on $[0, 1]$. Moreover, for $t \in [\tau_i, \tau_{i+1}]$,

$$\sup_{\tau_i\le t\le\tau_{i+1}} e^{-L(t-\tau_i)}|X_t^\varepsilon - X_t| \le \varepsilon_{i+1} + |X_{\tau_i}^\varepsilon - X_{\tau_i}|.$$

Then

$$\begin{aligned}
\sup_{\tau_i\le t\le\tau_{i+1}} e^{-Lt}|X_t^\varepsilon - X_t| &\le e^{-L\tau_i}\varepsilon_{i+1} + e^{-L\tau_i}|X_{\tau_i}^\varepsilon - X_{\tau_i}| \\
&\le \varepsilon_{i+1} + e^{-L\tau_i}|X_{\tau_i}^\varepsilon - X_{\tau_i}|.
\end{aligned}$$

By induction one can easily see that $\sup_{0\le t\le 1} e^{-Lt}|X^\varepsilon_t - X_t| \le \sum_{i=1}^\infty \varepsilon_i = e^{-L}\varepsilon$, and then

$$\sup_{0\le t\le 1} |X^\varepsilon_t - X_t| \le \varepsilon, \qquad \tilde{\mathbb{P}}\text{-a.s.}$$

(3) It remains to check that $Z^\varepsilon \in \mathbb{H}^2(\tilde{\mathbb{P}}, \mathbb{R}^d)$. For any $i, j \ge 1$, note that

$$X^\varepsilon_t = X^\varepsilon_{\tau_j} - \int_t^{\tau_j} h_s(X^\varepsilon_s, Z^\varepsilon_s)\,ds - \int_t^{\tau_j} Z^\varepsilon_s\,dB_s, \qquad t \le \tau_j, \mathbb{P}^i\text{-a.s.}$$

By the Lipschitz continuity of h and (3.4), and following standard arguments, one can easily see that, for some constant $C_i \ge 1$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^i}\left[\int_0^{\tau_j} |Z^\varepsilon_t|^2\,dt\right]\\
&\quad \le C_i \mathbb{E}^{\mathbb{P}^i}\left[|X^\varepsilon_{\tau_j}|^2 + \int_0^{\tau_j} |h_t(0,0)|^2\,dt\right]\\
&\quad \le C_i \mathbb{E}^{\mathbb{P}^i}\left[\sup_{0\le t\le 1} |X_t|^2 + \varepsilon^2 + \int_0^1 |h_t(0,0)|^2\,dt\right\} \qquad \text{for all } j \ge 1.
\end{aligned}$$

Set C_i in (5.16) to be the above constant C_i. Then by sending $j \to \infty$, we get

$$\mathbb{E}^{\mathbb{P}^i}\left[\int_0^1 |Z^\varepsilon_t|^2\,dt\right] \le \frac{1}{2^i \nu_i} \qquad \text{for all } i \ge 2.$$

Then

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_0^1 |Z^\varepsilon_t|^2\,dt\right] &= \sum_{i=1}^\infty \nu_i \mathbb{E}^{\mathbb{P}^i}\left[\int_0^1 |Z^\varepsilon_t|^2\,dt\right]\\
&\le \nu_1 \mathbb{E}^{\mathbb{P}^1}\left[\int_0^1 |Z^\varepsilon_t|^2\,dt\right] + \sum_{i=2}^\infty 2^{-i} < \infty.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 5.2 [$\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi)$]. In view of (5.8), we only need to show that $\mathcal{V}_0(\xi) \le \bar{\bar{\mathcal{V}}}_0(\xi)$ when $\bar{\bar{\mathcal{V}}}_0(\xi) < \infty$. For any $\varepsilon > 0$, there exist $\bar{\bar{y}} < \bar{\bar{\mathcal{V}}}_0(\xi) + \varepsilon$ and $\bar{\bar{Z}} \in \hat{\mathbb{H}}^2_0(\mathbb{R}^d)$ such that the corresponding $\bar{\bar{Y}}_1 := \bar{\bar{Y}}_1^{y,\bar{Z}} \ge \xi$, $\hat{\mathbb{P}}$-a.s. Set $\bar{y} := \bar{\bar{y}} + \varepsilon$ and $\bar{Z} := \bar{\bar{Z}}$. By Assumption 5.1, we may find $\bar{\Gamma} \in \hat{\mathbb{G}}^2_0(D_H) \cap \hat{\mathbb{H}}^2_0(D_H)$ such that the corresponding $\bar{Y}_1 := \bar{Y}_1^{\bar{y},\bar{Z},\bar{\Gamma}} \ge \xi$, $\hat{\mathbb{P}}$-a.s. Denote for $t \in [0,1]$,

$$Z^0_t := \int_0^t \bar{\Gamma}_s\,dB_s, \qquad \zeta_t := \bar{Z}_t - Z^0_t, \qquad Y^0_t := \int_0^t Z^0_s\,dB_s, \qquad X_t := \bar{Y}_t - Y^0_t,$$

and

$$h_t(\omega, x, z) := \tfrac{1}{2}\hat{a}_t(\omega) : \bar{\Gamma}_t(\omega) - H_t(\omega, x + Y^0_t(\omega), z + Z^0_t(\omega), \bar{\Gamma}_t(\omega)).$$

One easily checks that h satisfies the conditions of Lemma 5.5, and $X = X^{\zeta}$. Then, there exists $\zeta^{\varepsilon} \in \hat{\mathbb{S}}_0^2(\mathbb{R}^d)$ with finite variation over [0, 1] so that

$$\sup_{0\le t\le 1} |X_t^{\zeta^{\varepsilon}} - X_t| \le \varepsilon, \qquad \hat{\mathbb{P}}\text{-a.s.}$$

Set $Z^{\varepsilon} := \zeta^{\varepsilon} + Z^0$, $Y^{\varepsilon} := X^{\zeta^{\varepsilon}} + Y^0$, and observe that $d\langle Z^{\varepsilon}, B\rangle_t = d\langle Z^0, B\rangle_t = \bar{\Gamma}_t\,dt$, $\hat{\mathbb{P}}$-a.s. Therefore, $Z^{\varepsilon} \in \widehat{\mathcal{SM}}_0^2(\mathbb{R}^d)$. Setting $y := \bar{y}$, one can easily check that Y^{ε} satisfies (3.11) for given $(y, Z^{\varepsilon}, \bar{\Gamma})$. Notice that (3.11) coincides with (3.9) for given $\bar{\Gamma}$, we have $Y^{\varepsilon} = Y^{y,Z^{\varepsilon}}$. Then

$$Y^{y,Z^{\varepsilon}} - \bar{Y} = X^{\zeta^{\varepsilon}} - X \quad \text{and, thus,} \quad \sup_{0\le t\le 1} |Y_t^{y,Z^{\varepsilon}} - \bar{Y}_t| \le \varepsilon, \qquad \hat{\mathbb{P}}\text{-a.s.}$$

Let L denote the Lipschitz constant of H with respect to y, and set $y^{\varepsilon} := y + e^L\varepsilon$. Then

$$Y_t^{y^{\varepsilon},Z^{\varepsilon}} - Y_t^{y,Z^{\varepsilon}} = e^L\varepsilon + \int_0^t \lambda_s(Y_s^{y^{\varepsilon},Z^{\varepsilon}} - Y_s^{y,Z^{\varepsilon}})\,ds,$$

where $|\lambda_s| \le L$. This leads to $Y_1^{y^{\varepsilon},Z^{\varepsilon}} - Y_1^{y,Z^{\varepsilon}} = e^L\varepsilon e^{\int_0^1 \lambda_t\,dt} \ge \varepsilon$, and, thus,

$$Y_1^{y^{\varepsilon},Z^{\varepsilon}} \ge Y_1^{y,Z^{\varepsilon}} + \varepsilon \ge \bar{Y}_1 \ge \xi, \qquad \hat{\mathbb{P}}\text{-a.s.}$$

Therefore, $\mathcal{V}_0(\xi) \le y + e^L\varepsilon \le \bar{\bar{y}} + (1+e^L)\varepsilon \le \bar{\bar{\mathcal{V}}}_0(\xi) + (2+e^L)\varepsilon$. Since ε is arbitrary, this provides the required result. □

6. Extension. In this section we extend our setting in Section 3 by considering $\overline{\mathcal{P}}_S$ instead of $\mathcal{P}_H$ and by removing the constraints on the domains of H and F. In view of the length of this paper, we shall only formulate the extended problems heuristically and will not report the details. However, all the results in this paper can be extended to this new setting.

Let $H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ be a measurable mapping, and

$$F_t(\omega, y, z, a) := \sup_{\gamma\in\mathbb{R}^{d\times d}} \left\{\frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma)\right\}, \qquad a \in \mathbb{S}_d^{>0},$$

be the corresponding conjugate with respect to γ which takes values in $\mathbb{R} \cup \{\infty\}$. We assume D_{H_t}, the domain of H in γ, is independent of (y, z) and contains 0, H is uniformly Lipschitz continuous in (y, z) and lower-semicontinuous in γ for all $\gamma \in D_{H_t}$, and F is measurable. Then the domain D_{F_t} of F in a is also independent of (y, z), and F is uniformly Lipschitz continuous in (y, z), for all $a \in D_{F_t}$.

Recall the notation $\hat{F}_t^0 := \hat{F}_t(0, 0)$, and define the increasing sequence of $\mathbb{F}$-stopping times

$$\hat{\tau}_n := 1 \wedge \inf\left\{t \ge 0 : \int_0^t \hat{F}_s^0\,ds \ge n\right\}, \qquad n \ge 1; \quad \text{and} \quad \hat{\tau} := \lim_{n\to\infty} \hat{\tau}_n. \tag{6.1}$$

Notice that

$$\int_0^1 \hat{F}_s^0\,ds < \infty \qquad \text{on } \bigcup_{n\geq 1}\{\hat{\tau}_n = 1\} \quad \text{and}$$
$$(6.2)\qquad \int_0^1 \hat{F}_s^0\,ds = \infty \qquad \text{on } \bigcap_{n\geq 1}\{\hat{\tau}_n < 1\}.$$

We shall assume further that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^{\hat{\tau}_n} |\hat{F}_s^0|^2\,ds\right] < \infty \qquad \text{for all } \mathbb{P} \in \overline{\mathcal{P}}_S \text{ and } n \geq 1.$$

For the present extended setting, we introduce the space $\hat{\mathbb{L}}^2(\mathbb{R}) := \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \mathbb{L}^2(\mathbb{P}, \mathbb{R})$, together with

$$\hat{\mathbb{H}}^2(\mathbb{R}^d) := \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P}, \mathbb{R}^d) := \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \bigcap_{n\geq 1}\{Z : Z\mathbf{1}_{[0,\hat{\tau}_n]} \in \mathbb{H}^2(\mathbb{P}, \mathbb{R}^d)\},$$

$$\begin{aligned}\hat{\mathbb{G}}^2_H(D_H) &:= \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \mathbb{G}^2_{\mathrm{loc}}(\mathbb{P}, D_H)\\ &:= \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \bigcap_{n\geq 1}\left\{\Gamma : \left(\frac{1}{2}\hat{a} : \Gamma - H(0, 0, \Gamma)\right)\mathbf{1}_{[0,\hat{\tau}_n]} \in \mathbb{H}^2(\mathbb{P}, \mathbb{R})\right\},\end{aligned}$$

and the corresponding spaces for continuous processes (resp., semimartingales): $X \in \hat{\mathbb{S}}^2(\mathbb{R}) := \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \mathbb{S}^2_{\mathrm{loc}}(\mathbb{P}, \mathbb{R})$ [resp., $\widehat{\mathcal{SM}}^2(\mathbb{R}^d) := \bigcap_{\mathbb{P}\in\overline{\mathcal{P}}_S} \mathcal{SM}^2_{\mathrm{loc}}(\mathbb{P}, \mathbb{R}^d)$] iff for every $n \geq 1$ and $\mathbb{P} \in \overline{\mathcal{P}}_S$, $X_{\cdot\wedge\hat{\tau}_n} \in \mathbb{S}^2(\mathbb{P}, \mathbb{R})$ [resp., $\mathcal{SM}^2(\mathbb{P}, \mathbb{R}^d)$].

Now given $\xi \in \hat{\mathbb{L}}^2(\mathbb{R})$, the second order stochastic target problem is defined by

$$\mathcal{V}(\xi) := \inf\{y : Y_1^{y,Z} \geq \xi, \overline{\mathcal{P}}_S\text{-q.s. for some } Z \in \widehat{\mathcal{SM}}^2(\mathbb{R}^d)\},$$

where $Y := Y^{y,Z} \in \hat{\mathbb{S}}^2(\mathbb{R})$ is defined by the following ODE (with random coefficients):

$$\begin{aligned} Y_t &= y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)\,ds + \int_0^t Z_s \circ dB_s, \qquad t < \hat{\tau}, \overline{\mathcal{P}}_S\text{-q.s.}\\ Y_{\hat{\tau}} &:= \lim_{n\to\infty} Y_{\hat{\tau}_n} \qquad \text{on } \bigcup_{n\geq 1}\{\hat{\tau}_n = 1\},\\ Y_t &:= \infty \qquad \text{for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n\geq 1}\{\hat{\tau}_n < 1\}.\end{aligned}$$

Similarly, the extended relaxed problems are as follows:

$$\bar{\mathcal{V}}(\xi) := \inf\{y : \exists(\bar{Z}, \bar{\Gamma}) \in \hat{\mathbb{H}}^2(\mathbb{R}^d) \times \hat{\mathbb{G}}^2_H(D_H) \text{ s.t. } \bar{Y}_1^{\mathbb{P},y,\bar{Z},\bar{\Gamma}} \geq \xi,$$
$$\mathbb{P}\text{-a.s. for all } \mathbb{P} \in \overline{\mathcal{P}}_S\},$$

$$\bar{\bar{\mathcal{V}}}(\xi) := \inf\{y : \exists \bar{\bar{Z}} \in \hat{\mathbb{H}}^2(\mathbb{R}^d) \text{ s.t. } \bar{\bar{Y}}_1^{\mathbb{P},y,\bar{\bar{Z}}} \geq \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \overline{\mathcal{P}}_S\},$$

where $\bar{Y}^{\mathbb{P}} := \bar{Y}^{\mathbb{P},y,\bar{Z},\bar{\Gamma}}$ and $\bar{\bar{Y}}^{\mathbb{P}} := \bar{\bar{Y}}^{\mathbb{P},y,\bar{\bar{Z}}}$ are defined by

$$\begin{aligned}
\bar{Y}_t^{\mathbb{P}} &= y + \int_0^t \left(\frac{1}{2}\bar{\Gamma}_s : \hat{a}_s - H_s(\bar{Y}_s^{\mathbb{P}}, \bar{Z}_s, \bar{\Gamma}_s)\right) ds \\
&\qquad + \int_0^t \bar{Z}_s \, dB_s, \qquad t < \hat{\tau}, \mathbb{P}\text{-a.s.} \\
\bar{Y}_{\hat{\tau}}^{\mathbb{P}} &:= \lim_{n\to\infty} \bar{Y}_{\hat{\tau}_n}^{\mathbb{P}} \qquad \text{on } \bigcup_{n\ge 1}\{\hat{\tau}_n = 1\}, \\
\bar{Y}_t^{\mathbb{P}} &:= \infty \qquad \text{for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n\ge 1}\{\hat{\tau}_n < 1\}; \\
\bar{\bar{Y}}_t^{\mathbb{P}} &= y + \int_0^t \hat{F}_s(\bar{\bar{Y}}_s^{\mathbb{P}}, \bar{\bar{Z}}_s)\, ds + \int_0^t \bar{\bar{Z}}_s \, dB_s, \qquad t < \hat{\tau}, \mathbb{P}\text{-a.s.} \\
\bar{\bar{Y}}_{\hat{\tau}}^{\mathbb{P}} &:= \lim_{n\to\infty} \bar{\bar{Y}}_{\hat{\tau}_n}^{\mathbb{P}} \qquad \text{on } \bigcup_{n\ge 1}\{\hat{\tau}_n = 1\}, \\
\bar{\bar{Y}}_t^{\mathbb{P}} &:= \infty \qquad \text{for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n\ge 1}\{\hat{\tau}_n < 1\}.
\end{aligned} \tag{6.3}$$

Finally, we remark that $\mathbb{P}[\bigcup_n \{\hat{\tau}_n = 1\}] = 1$ for all $\mathbb{P} \in \mathcal{P}_H$. The dual formulation in this extended setting is the same as the original $v(\xi)$ defined in (3.16). That is, for dual formulation we still use $\mathcal{P}_H$, instead of $\overline{\mathcal{P}}_S$. Under certain technical conditions, again we can show that $\bar{\mathcal{V}}(\xi) = \bar{\bar{\mathcal{V}}}(\xi) = v(\xi)$. Moreover, if we extend the weaker version in Section 5 analogously, similar results will still hold.

APPENDIX

In this Appendix we prove Lemma 4.1 and claim (4.19). We shall use the notation of Lemma 2.2.

PROOF OF LEMMA 4.1. ($\mathbb{P}^{\tau,\omega} \in \overline{\mathcal{P}}_S^{\tau(\omega)}$). Let $\mathbb{P} = \mathbb{P}^{\alpha} \in \overline{\mathcal{P}}_S$ be given. We emphasize that we shall consider both the strong formulation (B, X^{α}) under $\mathbb{P}_0$ and the weak formulation $(W^{\mathbb{P}}, B)$ under $\mathbb{P}$. We prove the lemma in four steps.

Step 1. We first proceed in the strong formulation. Let $\tilde{\tau}$ be an arbitrary $\mathbb{F}$-stopping time. We claim that

$$(\mathbb{P}_0)^{\tilde{\tau},\omega} = \mathbb{P}_0^{\tilde{\tau}(\omega)} \qquad \text{for } \mathbb{P}_0\text{-a.e. } \omega \in \Omega. \tag{A.1}$$

Since $\int_0^1 |\alpha_s(\omega)|\, ds < \infty$, $\mathbb{P}_0$-a.s. clearly $\int_{\tilde{\tau}(\omega)}^1 |\alpha_s^{\tilde{\tau},\omega}(\tilde{\omega})|\, ds < \infty$ for $\mathbb{P}_0$-a.e. $\omega \in \Omega$ and $\mathbb{P}_0^{\tilde{\tau}(\omega)}$-a.e. $\tilde{\omega} \in \Omega^{\tilde{\tau}(\omega)}$. Then

$$\mathbb{P}^{\alpha^{\tilde{\tau},\omega}} \in \overline{\mathcal{P}}_S^{\tilde{\tau}(\omega)} \qquad \text{for } \mathbb{P}_0\text{-a.e. } \omega \in \Omega. \tag{A.2}$$

We now prove (A.1), which amounts to say, for $\mathbb{P}_0$-a.e. ω,

$$\mathbb{E}^{\mathbb{P}_0^{\tau,\omega}}[\xi] = \mathbb{E}^{\mathbb{P}_0^{\tau(\omega)}}[\xi] \qquad \text{for any bounded } \mathcal{F}_T^{\tau(\omega)}\text{-measurable r.v. } \xi. \tag{A.3}$$

By standard approximating arguments, it suffices to prove (A.3) by assuming

$$\xi = e^{\lambda_1 \tilde{B}_{t_1}^{\tau,\omega} + \cdots + \lambda_n \tilde{B}_{t_n}^{\tau,\omega}},$$
$$\text{where } \tilde{B}_t^{\tau,\omega} := \omega_t \mathbf{1}_{[0,\tau(\omega))}(t) + \big[\omega_{\tau(\omega)} + B_t^{\tau(\omega)}\big]\mathbf{1}_{[\tau(\omega),T]}$$

for all rational $0 < t_1 < \cdots < t_n \le T$ and $\lambda_1, \dots, \lambda_n \in \mathbb{Q}^d$. By the countability of rational numbers, we may allow the exceptional $\mathbb{P}_0$-null set to depend on ξ. Moreover, by backward induction, we may assume without loss of generality that $n = 1$ and $t_n = T$. That is, we want to prove, for any $\lambda \in \mathbb{Q}^d$,

$$\mathbb{E}^{\mathbb{P}_0^{\tau,\omega}}\big[e^{\lambda B_T^{\tau(\omega)}}\big] = \mathbb{E}^{\mathbb{P}_0^{\tau(\omega)}}\big[e^{\lambda B_T^{\tau(\omega)}}\big] \qquad \text{for } \mathbb{P}_0\text{-a.e. } \omega. \tag{A.4}$$

Note that

$$\mathbb{E}^{\mathbb{P}_0^{\tau(\omega)}}\big[e^{\lambda B_T^{\tau(\omega)}}\big] = e^{|\lambda|^2/2[T-\tau(\omega)]}.$$

Then, by (4.2) and the definition of r.c.p.d., (A.4) is equivalent to

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}_0}\big[e^{\lambda[B_T - B_\tau]}\eta_\tau\big] = \mathbb{E}^{\mathbb{P}_0}\big[e^{(|\lambda|^2/2)[T-\tau]}\eta_\tau\big] \\ &\qquad\qquad \text{where } \eta_t := \varphi(B_{s_1\wedge t}, \dots, B_{s_m \wedge t}), \end{aligned} \tag{A.5}$$

for any $0 < s_1 < \cdots < s_m \le T$ and any bounded and smooth function φ.

To see (A.5), we first assume τ takes only finitely many values, and by otherwise merging the partition points, we assume without loss of generality that τ takes only values $s_1, \dots, s_m$. Then, noting that $B_\cdot - B_{s_i}$ is a Brownian motion under $\mathbb{P}_0$,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0}\big[e^{\lambda[B_T - B_\tau]}\eta_\tau\big] &= \sum_{i=1}^m \mathbb{E}^{\mathbb{P}_0}\big[e^{\lambda[B_T - B_{s_i}]}\eta_{s_i}\mathbf{1}_{\{\tau = s_i\}}\big] \\ &= \sum_{i=1}^m \mathbb{E}^{\mathbb{P}_0}\big[\mathbb{E}^{\mathbb{P}_0}\big[e^{\lambda[B_T - B_{s_i}]}|\mathcal{F}_{s_i}\big]\eta_{s_i}\mathbf{1}_{\{\tau = s_i\}}\big] \\ &= \sum_{i=1}^m \mathbb{E}^{\mathbb{P}_0}\big[e^{(|\lambda|^2/2)(T-s_i)}\eta_{s_i}\mathbf{1}_{\{\tau = s_i\}}\big] = \mathbb{E}^{\mathbb{P}_0}\big[e^{(|\lambda|^2/2)[T-\tau]}\eta_\tau\big]. \end{aligned}$$

In the general case, we may find stopping times $\tau_n \downarrow \tau$ such that each τ_n takes finitely many values. Then

$$\mathbb{E}^{\mathbb{P}_0}\big[e^{\lambda[B_T - B_{\tau_n}]}\eta_{\tau_n}\big] = \mathbb{E}^{\mathbb{P}_0}\big[e^{(|\lambda|^2/2)[T-\tau_n]}\eta_{\tau_n}\big].$$

Send $n \to \infty$, and note that η is continuous in t, then by the Dominated Convergence Theorem we obtain (A.5), and hence prove (A.1).

Step 2. We construct the r.c.p.d. for $\mathbb{P}$ in weak formulation. Define

$$\tilde{\tau} := \tau \circ X^{\alpha} \quad \text{and} \quad \tilde{\alpha}^{\tau,\omega} := \alpha^{\tilde{\tau},\beta_\alpha(\omega)}. \tag{A.6}$$

One can easily see that $\tilde{\tau}$ is also an $\mathbb{F}$-stopping time. By the definition of $\mathbb{P}^{\alpha}$ and the definition of the mapping β_α in Lemma 2.2, we have $\tau = \tilde{\tau} \circ \beta_\alpha$, $\mathbb{P}^{\alpha}$-a.s. Then it follows from (A.2) that

$$\mathbb{P}^{\tilde{\alpha}^{\tau,\omega}} \in \overline{\mathcal{P}}_S^{\tau(\omega)} \qquad \text{for } \mathbb{P}^{\alpha}\text{-a.e. } \omega \in \Omega. \tag{A.7}$$

Step 3. We show that $\mathbb{P}^{\tau,\omega} = \mathbb{P}^{\tilde{\alpha}^{\tau,\omega}}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, by assuming the following claim which will be proved in Step 4 below:

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{\alpha}}[\varphi(B_{t_1\wedge\tau}, \ldots, B_{t_n\wedge\tau})\psi(B_{t_1}, \ldots, B_{t_n})] \\ &\qquad = \mathbb{E}^{\mathbb{P}^{\alpha}}[\varphi(B_{t_1\wedge\tau}, \ldots, B_{t_n\wedge\tau})\psi_\tau] \end{aligned} \tag{A.8}$$

for any $0 < t_1 < \cdots < t_n \le 1$ and bounded and continuous functions φ, ψ, where

$$\psi_\tau(\omega) := \mathbb{E}^{\mathbb{P}^{\tilde{\alpha}^{\tau,\omega}}}\big[\psi\big(\omega(t_1), \ldots, \omega(t_k), \omega(t) + B^t_{t_{k+1}}, \ldots, \omega(t) + B^t_{t_n}\big)\big] \qquad \text{for } t := \tau(\omega) \in [t_k, t_{k+1}).$$

Indeed, if (A.8) is true, then by the arbitrariness of φ and $(t_1, \ldots, t_n)$, it follows from the definition of r.c.p.d. that, for $\mathbb{P}^{\alpha}$-a.e. $\omega \in \Omega$ and for $t := \tau(\omega) \in [t_k, t_{k+1})$,

$$\psi_\tau(\omega) = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}\big[\psi\big(\omega(t_1), \ldots, \omega(t_k), \omega(t) + B^t_{t_{k+1}}, \ldots, \omega(t) + B^t_{t_n}\big)\big]. \tag{A.9}$$

We remark that the exceptional $\mathbb{P}^{\alpha}$-null set is supposed to depend on ψ and $t_1 < \cdots < t_n$. However, by standard approximating arguments, one can easily choose a common null set. That is, there exists a $\mathbb{P}^{\alpha}$-null set E_0 such that, for any $\omega \notin E_0$, (A.9) holds for all $(t_1, \ldots, t_n)$ and all bounded continuous functions ψ. This clearly implies that, for $\omega \notin E_0$,

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\eta] = \mathbb{E}^{\mathbb{P}^{\tilde{\alpha}^{\tau,\omega}}}[\eta] \qquad \text{for all bounded and } \mathcal{F}_1^{\tau(\omega)}\text{-measurable random variables } \eta.$$

Then $\mathbb{P}^{\tau,\omega} = \mathbb{P}^{\tilde{\alpha}^{\tau,\omega}}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$. This, together with (A.7), proves that $\mathbb{P}^{\tau,\omega} \in \mathcal{P}_S^{\tau(\omega)}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Step 4. We now prove (A.8). For $t := \tau(\omega) \in [t_k, t_{k+1})$, by definition of $\mathbb{P}^{\tilde{\alpha}^{\tau,\omega}}$ we have

$$\begin{aligned} \psi_\tau(\omega) = \mathbb{E}^{\mathbb{P}_0^{\tau(\omega)}}\bigg[\psi\bigg(&\omega(t_1), \ldots, \omega(t_k), \omega(t) + \int_t^{t_{k+1}} \big(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\big)^{1/2}\, dB_s^{\tau(\omega)}, \ldots, \\ &\omega(t) + \int_t^{t_n} \big(\alpha_s^{\tilde{\tau},\beta_\alpha(\omega)}\big)^{1/2}\, dB_s^{\tau(\omega)}\bigg)\bigg]. \end{aligned}$$

Then, for each $\omega \in \Omega$, when $t := \tilde{\tau}(\omega) = \tau(X^\alpha(\omega)) \in [t_k, t_{k+1})$,

$$
\begin{aligned}
&\psi_\tau(X^\alpha(\omega)) \\
&\quad = \mathbb{E}^{\mathbb{P}_0^{\tilde{\tau}(\omega)}}\biggl[\psi\biggl(X^\alpha_{t_1}(\omega), \dots, X^\alpha_{t_k}(\omega), X^\alpha_t(\omega) + \int_t^{t_{k+1}} (\alpha_s^{\tilde{\tau},\omega})^{1/2}\, dB_s^{\tilde{\tau}(\omega)}, \dots, \\
&\qquad\qquad X^\alpha_t(\omega) + \int_t^{t_n} (\alpha_s^{\tilde{\tau},\omega})^{1/2}\, dB_s^{\tilde{\tau}(\omega)}\biggr)\biggr];
\end{aligned}
$$

note that $(\mathbb{P}_0)^{\tau,\omega}$-distribution of $(B^{\tilde{\tau}(\omega)}, \alpha_s^{\tilde{\tau},\omega}(B^{\tilde{\tau}(\omega)})$ is equal to the $(\mathbb{P}_0)^\omega_\tau$-distribution of $(B_\cdot - B_{\tilde{\tau}(\omega)}, \alpha^{\tilde{\tau},\omega}(B_\cdot - B_{\tilde{\tau}(\omega)}))$. Recall (A.1), and note that, for each $\omega \in \Omega$,

$$
\alpha_s(\omega) = \alpha\bigl(\omega \otimes_{\tilde{\tau}(\omega)} \omega^{\tilde{\tau}(\omega)}\bigr) = \alpha_s^{\tilde{\tau},\omega}\bigl(\omega^{\tilde{\tau}(\omega)}\bigr).
$$

Then

$$
\begin{aligned}
&\psi_\tau(X^\alpha(\omega)) \\
&\quad = \mathbb{E}^{(\mathbb{P}_0)^\omega_{\tilde{\tau}}}\biggl[\psi\biggl(X^\alpha_{t_1}(\omega), \dots, X^\alpha_{t_k}(\omega), X^\alpha_t(\omega) + \int_t^{t_{k+1}} (\alpha_s)^{1/2}(B_\cdot)\, dB_s, \dots, \\
&\qquad\qquad X^\alpha_t(\omega) + \int_t^{t_n} (\alpha_s)^{1/2}(B_\cdot)\, dB_s\biggr)\biggr] \\
&\quad = \mathbb{E}^{(\mathbb{P}_0)^\omega_{\tilde{\tau}}}[\psi(X^\alpha_{t_1}, \dots, X^\alpha_{t_k}, X^\alpha_{t_{k+1}}, \dots, X^\alpha_{t_n})] \\
&\quad = \mathbb{E}^{\mathbb{P}_0}[\psi(X^\alpha_{t_1}, \dots, X^\alpha_{t_k}, X^\alpha_{t_{k+1}}, \dots, X^\alpha_{t_n})|\mathcal{F}_{\tilde{\tau}}](\omega), \qquad \mathbb{P}_0\text{-a.e. } \omega \in \Omega.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\alpha}[\varphi(B_{t_1\wedge\tau}, \dots, B_{t_n\wedge\tau})\psi_\tau] \\
&\quad = \mathbb{E}^{\mathbb{P}_0}[\varphi(X^\alpha_{t_1\wedge\tilde{\tau}}, \dots, X^\alpha_{t_n\wedge\tilde{\tau}})\psi_{\tilde{\tau}}(X^\alpha)] \\
&\quad = \mathbb{E}^{\mathbb{P}_0}\bigl[\varphi(X^\alpha_{t_1\wedge\tilde{\tau}}, \dots, X^\alpha_{t_n\wedge\tilde{\tau}})\mathbb{E}^{\mathbb{P}_0}[\psi(X^\alpha_{t_1}, \dots, X^\alpha_{t_k}, X^\alpha_{t_{k+1}}, \dots, X^\alpha_{t_n})|\mathcal{F}_{\tilde{\tau}}]\bigr] \\
&\quad = \mathbb{E}^{\mathbb{P}_0}[\varphi(X^\alpha_{t_1\wedge\tilde{\tau}}, \dots, X^\alpha_{t_n\wedge\tilde{\tau}})\psi(X^\alpha_{t_1}, \dots, X^\alpha_{t_k}, X^\alpha_{t_{k+1}}, \dots, X^\alpha_{t_n})] \\
&\quad = \mathbb{E}^{\mathbb{P}^\alpha}[\varphi(B_{t_1\wedge\tau}, \dots, B_{t_n\wedge\tau})\psi(B_{t_1}, \dots, B_{t_n})].
\end{aligned}
$$

This proves (A.8) and hence the lemma. $\square$

PROOF OF CLAIM (4.19). Let $\mathbb{P} = \mathbb{P}^\alpha$ and $\mathbb{P}^i_t = \mathbb{P}^{\alpha^i}$ for appropriate α and α^i, $i = 1, \dots, n$. Define

$$
\bar{\alpha}_s := \alpha_s \mathbf{1}_{[0,t)}(s) + \Biggl[\sum_{i=1}^n \alpha^i_s \mathbf{1}_{E^i_t}(X^\alpha) + \alpha_s \mathbf{1}_{\hat{E}^n_t}(X^\alpha)\Biggr]\mathbf{1}_{[t,1]}(s).
$$

Following similar arguments as in the proof of (A.8), one can easily show that, for any $0 < t_1 < \cdots < t_k = t < t_{k+1} < \cdots < t_n$ and any bounded continuous functions φ and ψ,

$$\mathbb{E}^{\mathbb{P}^\alpha}\left[\varphi(B_{t_1}, \ldots, B_{t_k}) \sum_{i=1}^n \mathbb{E}^{\mathbb{P}^{\alpha_t^i}}[\psi(B_{t_1}, \ldots, B_{t_k}, B_t + B^t_{t_{k+1}}, \ldots, B_t + B^t_{t_n})]\mathbf{1}_{E_t^i}\right]$$

$$= \mathbb{E}^{\mathbb{P}^{\bar{\alpha}}}[\varphi(B_{t_1}, \ldots, B_{t_k})\psi(B_{t_1}, \ldots, B_{t_n})].$$

Then $\mathbb{P}^n = \mathbb{P}^{\bar{\alpha}}$ and one sees immediately that $\mathbb{P}^n \in \overline{\mathcal{P}}_S$.

Moreover, since each $\mathbb{P}_t^i$ satisfies (4.7), one can easily check that $\mathbb{P}^n$ satisfies all the requirements in Definition 3.3, and thus $\mathbb{P}^n \in \mathcal{P}_H$. □

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