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The Law of Large Numbers for self-exciting correlated defaults

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Abstract

We consider a model of correlated defaults in which the default times of multiple entities depend not only on common and specific factors, but also on the extent of past defaults in the market, via the average loss process, including the average number of defaults as a special case. The paper characterizes the average loss process when the number of entities becomes large, showing that under some monotonicity conditions the limiting average loss process can be determined by a fixed point problem. We also show that the Law of Large Numbers holds under certain compatibility conditions.
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1. Introduction

Modeling of correlation between default probabilities of multiple "names" (individuals, firms, countries, etc.) has been one of the central issues in the theory and applications of managing and pricing credit risk in the past several years. There have been dozens of models in the literature.

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While each of these models has its own advantages and disadvantages, lax use of such models in practice could in part affect the understanding of the risk of the credit default and consequently contribute to the extent of a potential crisis in the market.

In this paper we propose a "bottom-up" model for correlated defaults within the standard "reduced form" framework. In particular, we assume that in a large collection of defaultable entities, the intensity of each individual default depends on factors specific to the individual entity, and on a common factor. The main novelty of our model is that we further allow a part of the common factor to have a *self-exciting* nature, reflecting the general "health" of the market. More precisely, we assume that the self-exciting factor takes the form of an "average loss process", including the average number of defaults to date as a special case. The self-exciting feature allows us, in the limiting case, to analyze the impact of such a "general health" index on the individual entities. However, it also generates a circular feedback phenomenon that is technically non-trivial.

The self-exciting structure of our model can be thought of as an example of the so-called "contagion" feature, which has been investigated by many authors using various approaches. These include Jarrow and Yu [16], Davis and Lo [5], Collin-Dufresne et al. [2,3], Dembo et al. [6], Giesecke and Goldberg [11], Giesecke and Weber [13,14], Frey and Backhaus [9,10], Horst [15], and Yu [22]. None of these models contains the circular nature presented in our model. In a recent work, Giesecke et al. [12] consider a model similar to ours. However, they impose a more special structure, which enables them to obtain large deviation type results, in addition to the Law of Large Numbers type results that we focus on. Using a rather different, interactive particles type model, Dai Pra et al. [4] obtain a rich set of limiting and large deviations results with many defaultable firms. The self-exciting feature is also present in Filipovic et al. [8], in a "top-down" model. For an overview of standard default risk models, one can consult, among many others, the texts Duffie and Singleton [7], Lando [20], and Frey et al. [21], and the references cited therein.

Assuming that all the factors are diffusion processes, we first show that the proposed self-exciting model is well-posed. This is not trivial because our model is "circular" due to its self-exciting nature. We next characterize the limit of the average default loss, if exists, via a fixed point problem. Under certain monotonicity conditions, which can be interpreted as the firms in the model are "partners" (or "competitors"), we prove the existence of the fixed point by using Zorn's lemma. In some special cases, the fixed point problem is reduced to an ordinary differential equation, which can be solved either explicitly or numerically.

Our main objective of the paper is to identify conditions under which the average number of defaults (or more generally the average default loss) does converge to the above fixed point, in the sense of the Law of Large Numbers, as the number of names tends to infinity. Besides the standard technical conditions, we need two types of assumptions: (i) the system is monotone in a certain sense; and (ii) the dependence on the average default loss is "weak". The first assumption is mainly for technical reasons and it simplifies our analysis significantly. The "weak" correlation assumption seems to be crucial for our results, and it will be interesting to understand the problem when the correlation is "strong". In a special case, we also prove a Central Limit Theorem for the average number of defaults.

It is worth remarking that these results, being of asymptotic nature, are not directly applicable to individual credit risk derivatives, because they require a large number of names to be involved in the limiting process. However, our results should be useful for the risk management at a level of an institution, or a country, with large portfolio of defaultable claims, when the aim is to analyze potential total losses. For example, it has been stated that the next crisis might come from

potentially numerous defaults of credit card holders. This paper provides a theoretical model which may prove useful for addressing such issues.

The rest of the paper is organized as follows. In Section 2 we formulate the problem and the model. In Section 3 we show that the self-exciting model that we are proposing is well-posed. In Section 4 we study the fixed point problem that determines the limiting process. In Section 5 we present some potential applications where the fixed point problem could be solved, and prove the Central Limit Theorem in a special case. Finally, Sections 6 and 7 are devoted to the main theorem involving the Law of Large Numbers and its proof.

2. Problem formulation

2.1. Average loss in correlated default models

We consider n "names", which could be individual investors, financial firms, loans, etc. We denote their default times by $\tau_1, \ldots, \tau_n$. Let us associate to each default time τ_i a "loss process" $L_t^i, t \geq 0$, so that the loss due to default at any time t is given by $L_{\tau^i}^i \mathbf{1}_{\{\tau^i \leq t\}}$. We define the "average loss" of all defaults at time t by

$$\bar{L}_t^n \triangleq \frac{1}{n} \sum_{i=1}^n L_{\tau_i}^i 1_{\{\tau_i \leq t\}}. \tag{2.1}$$

Clearly, one can have various interpretations for $\bar{L}$ by imposing various choices for L^i. For example, if we set $L^i \equiv 1$, then $\bar{L}^n$ is the average number of defaults (for example, the average number of foreclosures in a given region) among the n names.

Our main purpose is to investigate the limiting behavior of $\bar{L}^n$ as $n \to \infty$, namely,

$$\bar{L}_t^* \triangleq \lim_{n \to \infty} \bar{L}_t^n, \tag{2.2}$$

whenever the limit exists, and to characterize the limit $\bar{L}^*$. It is to be expected that $\bar{L}^*$ will depend substantially on the correlation of the default times and the loss processes. The following two examples are the extreme cases, whose limits are quite different in nature.

1°. Assume that the sequence $\{(\tau_i, L_{\tau_i}^i)\}_{i \geq 1}$ is i.i.d. Then, the Law of Large Numbers (LLN) implies that $\bar{L}_t^n \to \bar{L}_t^* = \mathbb{E}\{L_{\tau_1}^1 \mathbf{1}_{\{\tau_1 \leq t\}}\}$, $\mathbb{P}$-a.s.

2°. Assume that the times and the losses are fully correlated, that is, $\tau_1 = \cdots = \tau_n = \tau, L_{\tau_1}^1 = \cdots = L_{\tau_n}^n = L_\tau$. Then, $\bar{L}_t^n = \bar{L}_t^* = L_\tau \mathbf{1}_{\{\tau \leq t\}}$.

In this paper, we will provide quite a general model such that the default times $\tau_1, \ldots, \tau_n$ are correlated and the limit $\bar{L}^*$ exists. The main "self-exciting" feature of the model is that the correlation of $\tau_1, \ldots, \tau_n$ is built via the average loss $\bar{L}^n$.

2.2. The model

Throughout this paper we fix an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t \geq 0}$. We assume that the probability space is rich enough to support a sequence of independent standard Brownian motions $(B^0, B^1, \ldots, B^n, \ldots)$ and a sequence of exponential random variables $(E^1, \ldots, E^n, \ldots)$, all with rate 1 and are independent of the Brownian motions. We define the following sub-filtrations of $\mathbb{F}$:

$$\mathbb{F}^0 \triangleq \mathbb{F}^{B^0}, \qquad \mathbb{F}^i \triangleq \mathbb{F}^{B^0, B^i}, \quad i = 1, 2, \ldots, \tag{2.3}$$

the filtrations generated by the Brownian motions B^0 and (B^0, B^i), respectively, and augmented by the $\mathbb{P}$-null sets. For simplicity, let us assume that $\mathbb{F} = \bigvee_{i=1}^{\infty} \left(\mathbb{F}^i \vee \sigma(E^i) \right)$.

We now fix n and the loss processes L^i, $i = 1, \ldots, n$. As in reduced form models, see, e.g., [1,7,19], we define

$$\tau_i \triangleq \inf \left\{ t \geq 0 : Y_t^i \geq E^i \right\}, \tag{2.4}$$

where, for process $\bar{L}^n$ defined by (2.1), process Y^i denotes the "hazard process"

$$Y_t^i \triangleq \int_0^t \lambda_i(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^0, X_s^i, \bar{L}_s^n) ds, \tag{2.5}$$

and X^0, X^i, $i = 1, 2, \ldots, n$ are factor processes defined by

$$X_t^0 = x_0 + \int_0^t b_0(s, B_{\cdot \wedge s}^0, X_s^0, \bar{L}_s^n) ds + \int_0^t \sigma_0(s, B_{\cdot \wedge s}^0, X_s^0, \bar{L}_s^n) dB_s^0,$$

$$X_t^i = x_i + \int_0^t b_i(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^0, X_s^i, \bar{L}_s^n) ds \tag{2.6}$$

$$+ \int_0^t \sigma_i(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^0, X_s^i, \bar{L}_s^n) dB_s^i.$$

Throughout the paper, we assume the following *Standing Assumptions*.

Assumption 2.1. For each i, the process L^i is $\mathbb{F}^i$-adapted; the coefficients $b_0, \sigma_0 : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $b_i, \sigma_i, \lambda_i : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R})^2 \times \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ are Lebesgue measurable functions; and $\lambda_i \geq 0$.

We note that here X^0 denotes the common factor in the market, that is observable by everyone; X^i is the firm i's specific factor, observable only by firm i. It is possible that each individual firm has risk factors that are observable by others in the market, and we include such factors into the common factor X^0. It is clear that each τ_i is an $\mathbb{F}$-stopping time, but not necessarily an $\mathbb{F}^i$-stopping time. As pointed above, the main feature of our model is that the correlation among the defaults depends on, in addition to the common exogenous factor X^0, the past defaults through the process $\bar{L}^n$, so that it has a self-exciting nature. Moreover, since we model each τ_i rather than $\bar{L}^n$ directly, our model is "bottom-up".

When there is no confusion, for $\psi = b, \sigma, \lambda$ and $i = 1, 2, \ldots$, with a slight abuse of notation we denote

$$\begin{aligned} \psi_0(t, x_0, \alpha) &\triangleq \psi_0(t, \omega, x_0, \alpha) \triangleq \psi_0(t, B_{\cdot \wedge t}^0(\omega), x_0, \alpha), \\ \psi_i(t, x_0, x_i, \alpha) &\triangleq \psi_i(t, \omega, x_0, x_i, \alpha) \triangleq \psi_i(t, B_{\cdot \wedge t}^0(\omega), B_{\cdot \wedge t}^i(\omega), x_0, x_i, \alpha). \end{aligned} \tag{2.7}$$

Then clearly $\psi_0(\cdot, x_0, \alpha)$ is $\mathbb{F}^0$-adapted and $\psi_i(\cdot, x_0, x_i, \alpha)$ is $\mathbb{F}^i$-adapted.

Remark 2.2. (i) If $b_0, \sigma_0, b_i, \sigma_i, \lambda_i$ do not depend on $\bar{L}^n$, then our model becomes a standard reduced form model where the defaults are conditionally independent, conditional on the common factor X^0, and it is straightforward to check that in this case λ_i is the $\mathbb{F}^i$-intensity of τ_i, in the sense that $\mathbb{P}\{\tau_i > t | \mathcal{F}_t^i\} = \exp\{-\int_0^t \lambda_i(s, X_s^0, X_s^i) ds\}$, $t \geq 0$; see, e.g., [1,7].

(ii) In the general case when λ_i depends on $\bar{L}^n$, λ_i is obviously no longer an $\mathbb{F}^i$-adapted process (hence cannot be an "$\mathbb{F}^i$-intensity" of τ_i in the aforementioned sense). Due to the self-

exciting nature of our model, λ^i can be interpreted as the conditional intensity of τ^i, conditional on all the past defaults. See Proposition 3.3 for a more precise statement; see also [17,18] for more on construction of default times with given intensities. □

2.3. The main results

Notice that the system (2.1), (2.4)–(2.6) is "circular", and thus its well-posedness is by no means obvious. Our first result, Theorem 3.2, is that this system is indeed well-posed.

We next characterize the limit process $\bar{L}^*$ via a fixed point problem. We first conjecture that, if exists, $\bar{L}^*$ should be $\mathbb{F}^0$-adapted. Now, for an $\mathbb{F}^0$-adapted process α, by replacing $\bar{L}$ with α in the system (2.1), (2.4)–(2.6) we define (recall the convention (2.7))

$$
\begin{aligned}
X_t^{0,\alpha} &= x_0 + \int_0^t b_0(s, X_s^{0,\alpha}, \alpha_s)ds + \int_0^t \sigma_0(s, X_s^{0,\alpha}, \alpha_s)dB_s^0; \\
X_t^{i,\alpha} &= x_i + \int_0^t b_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds + \int_0^t \sigma_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)dB_s^i, \\
Y_t^{i,\alpha} &\triangleq \int_0^t \lambda_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds, \\
\tau_i^\alpha &\triangleq \inf\left\{t \geq 0 : Y_t^{i,\alpha} \geq E_i\right\}, \quad i = 1, \ldots, n; \\
\bar{L}_t^n(\alpha) &\triangleq \frac{1}{n}\sum_{i=1}^n L_{\tau_i^\alpha}^i 1_{\{\tau_i^\alpha \leq t\}}.
\end{aligned}
\tag{2.8}
$$

Clearly, given the information $\mathbb{F}^0$, processes $(X^{i,\alpha}, Y^{i,\alpha}, \tau_i^\alpha)$, $i = 1, \ldots, n$, are conditionally independent; see Remark 2.2. Thus, under conditional probability $\mathbb{P}\{\cdot |\mathbb{F}^0\}$, the standard Law of Large Numbers should imply, modulo some technical conditions, that

$$
\bar{L}_t^n(\alpha) - \mathbb{E}\{\bar{L}_t^n(\alpha)|\mathcal{F}_t^0\} \to 0, \quad t \geq 0.
\tag{2.9}
$$

Now if $\bar{L}^* = \alpha$, that is $\bar{L}^n \to \alpha$, one expects that the system (2.1), (2.4)–(2.6) converges to the system (2.8) in certain sense. In particular, $\bar{L}^n(\alpha)$ and $\bar{L}^n$ should have the same limit, that is, we should expect that the process α would have the following "fixed point" property:

$$
\alpha_t = \lim_{n\to\infty} \mathbb{E}\{\bar{L}_t^n(\alpha)|\mathcal{F}_t^0\}, \quad t \geq 0,
\tag{2.10}
$$

provided that the limit and the fixed point α both exist.

In Theorem 4.9, we will provide some sufficient conditions so that the fixed point problem (2.10) has a solution. Our main result of the paper, Theorem 6.3, proves the Law of Large Numbers in our model. That is, it shows that if α solves the fixed point problem (2.10), then under certain technical conditions, we have

$$
\lim_{n\to\infty} \mathbb{E}\{|\bar{L}_t^n - \alpha_t|\} = 0, \quad \forall t.
\tag{2.11}
$$

In particular, this implies that α is unique.

We finish this section by presenting a simple example in which $\bar{L}$ is the average number of defaults.

Example 2.3. Assume $L^i \equiv 1$, $\lambda_i = \lambda$, $\forall i$, and λ is independent of (B^i, X^i) (i.e., a "zero-factor" scenario). Then, conditioning on the values of (B^0, X^0), all τ_i^α's have the same distribution and

the right-hand side in (2.10) is equal to

$$\mathbb{P}\left\{\tau_1^\alpha \le t | \mathcal{F}_t^0\right\} = 1 - e^{-Y_t^\alpha} = 1 - e^{-\int_0^t \lambda(s, X_s^{0,\alpha}, \alpha_s)ds},$$

and Eq. (2.10) for α becomes

$$\alpha_t = 1 - e^{-\int_0^t \lambda(s, B_{\cdot \wedge s}^0, X_s^{0,\alpha}, \alpha_s)ds}.$$

A simple calculation implies that α should satisfy the following ODE:

$$\alpha_t' = (1 - \alpha_t)\lambda(t, B_{\cdot \wedge t}^0, X_t^0, \alpha_t), \quad \alpha_0 = 0. \quad \square \tag{2.12}$$

3. Well-posedness

In this section we verify that the system (2.1), (2.4)–(2.6) is indeed well-defined. In other words, we show that, for each $n \in \mathbb{N}$, there exists a unique solution $(X^0, \{X^i, Y^i\}_{i=1}^n)$ that satisfies (2.1), (2.4)–(2.6). For this purpose we impose the following technical conditions.

Assumption 3.1. (i) The mappings $x_0 \mapsto b_0(t, \omega, x_0, \alpha), \sigma_0(t, \omega, x_0, \alpha)$ are uniformly Lipschitz, uniformly in (t, ω, α); and the mappings $x_i \mapsto b_i(t, \omega, x_0, x_i, \alpha)$ and $\sigma(t, \omega, x_0, x_i, \alpha)$ are uniformly Lipschitz, uniformly in (t, ω, x_0, α).

(ii) Let $D_0 \subset \mathbb{R}$ denote domain of L^i, that is, L^i takes values in D_0. There exists a constant $K > 0$ such that, for any $\alpha \in D_0$, any $i = 1, \dots, n$, and any (t, ω, x_0, x_i),

$$|b_i(t, \omega, x_0, 0, \alpha) - b_i(t, \omega, 0, 0, \alpha)| + |\sigma_i(t, \omega, x_0, 0, \alpha) - \sigma_i(t, \omega, 0, 0, \alpha)|$$
$$\le K(1 + |x_0|),$$
$$|\lambda_i(t, \omega, x_0, x_i, \alpha) - \lambda_i(t, \omega, 0, 0, \alpha)| \le K(1 + |x_0| + |x_i|);$$
$$\mathbb{E}\left\{\left(\int_0^T \sup_{\alpha \in D_0} [|b_0| + |b_i|](t, 0, 0, \alpha)dt\right)^2 \right.$$
$$\left. + \int_0^T \sup_{\alpha \in D_0} [|\sigma_0|^2 + |\sigma_i|^2 + |\lambda_i|](t, 0, 0, \alpha)dt\right\} < \infty.$$

We then have the following theorem.

Theorem 3.2. *Assume Assumptions* 2.1 *and* 3.1 *hold. Then for each* $n \in \mathbb{N}$, *the system* (2.1) *and* (2.4)–(2.6) *admits a unique* $\mathbb{F}$-*adapted solution* $(X^0, \{X^i, Y^i\}_{i=1}^n)$.

Proof. In this proof and in the sequel we denote by $\tau_1^* \le \cdots \le \tau_n^*$ the order statistics of stopping times $\tau_1, \dots, \tau_n$. We construct a solution to the system in the following. It can be seen from the construction that the solution is unique.

Notice that, if there is a solution, one must have $\bar{L}_t^n = 0$ for $t < \tau_1^*$. We thus first consider the following system:

$$X_t^{0,1} = x_0 + \int_0^t b_0(s, X_s^{0,1}, 0)ds + \int_0^t \sigma_0(s, X_s^{0,1}, 0)dB_s^0;$$

$$X_t^{i,1} = x_i + \int_0^t b_i(s, X_s^{0,1}, X_s^{i,1}, 0)ds + \int_0^t \sigma_i(s, X_s^{0,1}, X_s^{i,1}, 0)dB_s^i, \quad i = 1, \dots, n.$$

This SDE obviously has a unique solution $(X^{0,1}, \{X^{i,1}\}_{i=1}^{n})$ under Assumptions 2.1 and 3.1. We can then define

$$Y_t^{i,1} \triangleq \int_0^t \lambda_i(s, X_s^{0,1}, X_s^{i,1}, 0) ds, \qquad \tau_i^1 \triangleq \inf\left\{ t \geq 0 : Y_t^{i,1} \geq E^i \right\}, \quad i = 1, \ldots, n;$$

$$\bar{L}_t^{n,1} \triangleq \frac{1}{n} \sum_{i=1}^n L_{\tau_i^1}^i \mathbf{1}_{\{\tau_i^1 \leq t\}}.$$

Suppose that we have defined processes $(X^{0,k}, X^{i,k}, Y^{i,k}, \bar{L}^{n,k})$ and stopping times τ_i^k for $i = 1, \ldots, n$. Now for $k + 1$, recalling that $\tau_k^{k,*}$ is the k-th order statistic of $\tau_1^k, \ldots, \tau_n^k$, we define for $i = 1, \ldots, n$

$$(X_t^{0,k+1}, X_t^{i,k+1}, Y_t^{i,k+1}, \bar{L}_t^{n,k+1}) \triangleq (X_t^{0,k}, X_t^{i,k}, Y_t^{i,k}, \bar{L}_t^{n,k}), \quad 0 \leq t \leq \tau_k^{k,*}, \tag{3.1}$$

and for $t \geq \tau_k^{k,*}$ and $i = 1, \ldots, n$,

$$X_t^{0,k+1} = X_{\tau_k^{k,*}}^{0,k} + \int_{\tau_k^{k,*}}^t b_0(s, X_s^{0,k+1}, \bar{L}_{\tau_k^{k,*}}^{n,k}) ds + \int_{\tau_k^{k,*}}^t \sigma_0(s, X_s^{0,k+1}, \bar{L}_{\tau_k^{k,*}}^{n,k}) dB_s^0;$$

$$X_t^{i,k+1} = X_{\tau_k^{k,*}}^{i,k} + \int_{\tau_k^{k,*}}^t b_i(s, X_s^{0,k+1}, X_s^{i,k+1}, \bar{L}_{\tau_k^{k,*}}^{n,k}) ds$$

$$+ \int_{\tau_k^{k,*}}^t \sigma_i(s, X_s^{0,k+1}, X_s^{i,k+1}, \bar{L}_{\tau_k^{k,*}}^{n,k}) dB_s^i;$$

$$Y_t^{i,k+1} \triangleq Y_{\tau_k^{k,*}}^{i,k} + \int_{\tau_k^{k,*}}^t \lambda_i(s, X_s^{0,k+1}, X_s^{i,k+1}, \bar{L}_{\tau_k^{k,*}}^{n,k}) ds;$$

$$\tau_i^{k+1} \triangleq \inf\left\{ t \geq 0 : Y_t^{i,k+1} \geq E^i \right\}, \qquad \bar{L}_t^{n,k+1} \triangleq \frac{1}{n} \sum_{i=1}^n L_{\tau_i^{k+1}}^i \mathbf{1}_{\{\tau_i^{k+1} \leq t\}}.$$

This defines τ_i^{k+1}, $i = 1, \ldots, n$. By (3.1), it is clear that

$$\tau_j^{k+1,*} = \tau_j^{k,*}, \quad j = 1, \ldots, k. \tag{3.2}$$

Repeating the same procedure, we may define $(X^{0,n}, X^{i,n}, Y^{i,n}, \bar{L}^{n,n})$ and τ_i^n for $i = 1, \ldots, n$. Finally, we define

$$(X_t^0, X_t^i, Y_t^i, \bar{L}_t^n) \triangleq (X_t^{0,n}, X_t^{i,n}, Y_t^{i,n}, \bar{L}_t^{n,n}), \quad 0 \leq t \leq \tau_n^{n,*}, \tag{3.3}$$

and for $t > \tau_n^{n,*}$,

$$X_t^0 = X_{\tau_n^{n,*}}^0 + \int_{\tau_n^{n,*}}^t b_0(s, X_s^0, \bar{L}_{\tau_n^{n,*}}^n) ds + \int_{\tau_n^{n,*}}^t \sigma_0(s, X_s^0, \bar{L}_{\tau_n^{n,*}}^n) dB_s^0;$$

$$X_t^i = X_{\tau_n^{n,*}}^i + \int_{\tau_n^{n,*}}^t b_i(s, X_s^0, X_s^i, \bar{L}_{\tau_n^{n,*}}^n) ds + \int_{\tau_n^{n,*}}^t \sigma_i(s, X_s^0, X_s^i, \bar{L}_{\tau_n^{n,*}}^n) dB_s^i;$$

$$Y_t^i \triangleq Y_{\tau_n^{n,*}}^i + \int_{\tau_n^{n,*}}^t \lambda_i(s, X_s^0, X_s^i, \bar{L}_{\tau_n^{n,*}}^n) ds;$$

$$\bar{L}_t^n \triangleq \bar{L}_{\tau_n^{n,*}}^n.$$

This defines $(X_t^0, X_t^i, Y_t^i, \bar{L}_t^n)$ for $t \geq 0$. Moreover, define τ_i by (2.4), $i = 1, \ldots, n$. One can check straightforwardly that $(X_t^0, X_t^i, Y_t^i, \bar{L}_t^n, \tau_i)$ satisfies the system (2.1), (2.4)–(2.6), and

$$\tau_j^* = \tau_j^{n,*} = \tau_j^{k,*}, \quad 1 \leq j \leq k \leq n. \quad \square \tag{3.4}$$

The next proposition gives the conditional distribution of stopping times τ_i^{k+1}, when the previous defaults are known. We say that random variables ξ_i are *conditionally independent on D* if $\xi_i \mathbf{1}_D$ are conditionally independent.

Proposition 3.3. *Assume Assumptions 2.1 and 3.1 hold, and let $i_1, \ldots, i_k$ be given. In the framework of Theorem 3.2, and recalling (3.4), denote*

$$D_k \triangleq \{\tau_1^* = \tau_{i_1}^k, \ldots, \tau_k^* = \tau_{i_k}^k\}, \qquad \mathcal{G}_t^k \triangleq \left(\bigvee_{l=1}^k \mathcal{F}_{\tau_k^*+t}^{i_l}\right) \bigvee \left(\bigvee_{j \neq i_1, \ldots, i_k} \mathcal{F}_{\tau_k^*}^j\right). \tag{3.5}$$

Then, for $j \neq i_1, \ldots, i_k$ and $t \geq 0$,

$$\mathbb{P}\left\{\tau_j^{k+1} > \tau_k^* + t | \mathcal{G}_t^k, D_k\right\} = \mathbb{E}\left\{\exp(Y_{\tau_k^*}^{j,k+1} - Y_{\tau_k^*+t}^{j,k+1}) | \mathcal{G}_t^k, D_k\right\} \quad \text{on } D_k. \tag{3.6}$$

Moreover, conditional on $\mathcal{G}_t^k \vee \sigma(D_k)$, the random vectors $(X_{\tau_k^*+t}^{j,k+1}, Y_{\tau_k^*+t}^{j,k+1}, \mathbf{1}_{\{\tau_j^{k+1} > \tau_k^*+t\}})$, $j \neq i_1, \ldots, i_k$, are conditionally independent on D_k, and consequently,

$$\mathbb{P}\left\{\tau_{k+1}^* > \tau_k^* + t | \mathcal{G}_t^k, D_k\right\}$$
$$= \mathbb{E}\left\{\exp\left(\sum_{j \neq i_1, \ldots, i_k} (Y_{\tau_k^*}^{j,k+1} - Y_{\tau_k^*+t}^{j,k+1})\right) | \mathcal{G}_t^k, D_k\right\} \quad \text{on } D_k. \tag{3.7}$$

Proof. (i) We first prove (3.6). For arbitrarily given $t_1 < \cdots < t_k$, denote

$$\tilde{D}_k \triangleq D_k \cap \left\{\tau_1^* = t_1, \ldots, \tau_k^* = t_k\right\}, \tag{3.8}$$

and define

$$(\tilde{X}^{0,1}, \tilde{X}^{i,1}, \tilde{Y}^{i,1}) \triangleq (X^{0,1}, X^{i,1}, Y^{i,1}), \quad \text{and} \quad \tilde{L}_{t_1}^{n,1} \triangleq \frac{1}{n} L_{t_1}^{i_1}.$$

For $j = 1, \ldots, k$, define $(\tilde{X}_t^{0,j+1}, \tilde{X}_t^{i,j+1}, \tilde{Y}_t^{i,j+1}) \triangleq (\tilde{X}_t^{0,j}, \tilde{X}_t^{i,j}, \tilde{Y}_t^{i,j})$ for $t \leq t_j$, and for $t \geq t_j$,

$$\tilde{X}_t^{0,j+1} = \tilde{X}_{t_j}^{0,j} + \int_{t_j}^t b_0(s, \tilde{X}_s^{0,j+1}, \tilde{L}_{t_j}^{n,j})ds + \int_{t_j}^t \sigma_0(s, \tilde{X}_s^{0,j+1}, \tilde{L}_{t_j}^{n,j})dB_s^0;$$

$$\tilde{X}_t^{i,j+1} = \tilde{X}_{t_j}^{i,j} + \int_{t_j}^t b_i(s, \tilde{X}_s^{0,j+1}, \tilde{X}_s^{i,j+1}, \tilde{L}_{t_j}^{n,j})ds$$
$$+ \int_{t_j}^t \sigma_i(s, \tilde{X}_s^{0,j+1}, \tilde{X}_s^{i,j+1}, \tilde{L}_{t_j}^{n,j})dB_s^i;$$

$$\tilde{Y}_t^{i,j+1} \triangleq \tilde{Y}_{t_j}^{i,j} + \int_{t_j}^t \lambda_i(s, \tilde{X}_s^{0,j+1}, \tilde{X}_s^{i,j+1}, \tilde{L}_{t_j}^{n,j})ds;$$

where, for $j > 1$,

$$\tilde{L}^{n,j}_{t_j} \triangleq \tilde{L}^{n,j-1}_{t_{j-1}} + \frac{1}{n} L^{i_j}_{t_j}.$$

Then, it is clear that

$$(X^{0,k+1}, X^{i,k+1}, Y^{i,k+1}) = (\tilde{X}^{0,k+1}, \tilde{X}^{i,k+1}, \tilde{Y}^{i,k+1}) \quad \text{on } \tilde{D}_k. \tag{3.9}$$

Note that, for any i and t,

$$\{\tau^{k+1}_i > t\} = \{E_i > Y^{i,k+1}_t\},$$
$$\{\tau^{k+1}_i = t\} = \Big\{Y^{i,k+1}_t = E_i \text{ and } Y^{i,k+1}_s < E_i \text{ for all } s < t\Big\}.$$

Then

$$\tilde{D}_k = \Big\{\tau^{k+1}_{i_1} = t_1, \ldots, \tau^{k+1}_{i_k} = t_k, Y^{i,k+1}_{t_k} < E_i, i \neq i_1, \ldots, i_k\Big\}$$
$$= \Big\{\tau^{k+1}_{i_1} = t_1, \ldots, \tau^{k+1}_{i_k} = t_k, E_j > Y^{j,k+1}_{t_k}, E_i > Y^{i,k+1}_{t_k}, i \neq i_1, \ldots, i_k, j\Big\}$$

and, for each j,

$$\mathcal{G}^k_t \vee \sigma\Big\{\tau^{k+1}_{i_1} = t_1, \ldots, \tau^{k+1}_{i_k} = t_k, E_i > Y^{i,k+1}_{t_k}, i \neq i_1, \ldots, i_k, j\Big\}$$
$$\subseteq \tilde{\mathcal{G}}^{k,j}_t \triangleq \Big(\bigvee_{i=1}^n \mathcal{F}^i_{t_k+t}\Big) \bigvee \Big(\bigvee_{i \neq j} \sigma(E_i)\Big).$$

Then, by (3.9), on $\tilde{D}_k$ we have

$$\mathbb{P}\Big\{\tau^{k+1}_j > \tau^*_k + t | \mathcal{G}^k_t, \tilde{D}_k\Big\} = \mathbb{E}\Big\{\mathbb{P}\Big\{E_j > Y^{j,k+1}_{t_k+t} | \tilde{\mathcal{G}}^{k,j}_t, E_j > Y^{j,k+1}_{t_k}\Big\} | \mathcal{G}^k_t, \tilde{D}_k\Big\}$$
$$= \mathbb{E}\Big\{\mathbb{P}\Big\{E_j > \tilde{Y}^{j,k+1}_{t_k+t} | \tilde{\mathcal{G}}^{k,j}_t, E_j > \tilde{Y}^{j,k+1}_{t_k}\Big\} | \mathcal{G}^k_t, \tilde{D}_k\Big\}. \tag{3.10}$$

Given $\tilde{\mathcal{G}}^{k,j}_t$ and $E_j > \tilde{Y}^{j,k+1}_{t_k}$, one can evaluate the conditional probability of the set $E_j > \tilde{Y}^{j,k+1}_{t_k+t}$ in (3.10) as

$$\mathbb{P}\Big\{E_j > \tilde{Y}^{j,k+1}_{t_k+t} | \tilde{\mathcal{G}}^{k,j}_t, E_j > \tilde{Y}^{j,k+1}_{t_k}\Big\} = \mathbb{E}\Big\{\exp(\tilde{Y}^{j,k+1}_{t_k} - \tilde{Y}^{j,k+1}_{t_k+t}) | \tilde{\mathcal{G}}^{k,j}_t, E_j > \tilde{Y}^{j,k+1}_{t_k}\Big\}.$$

Thus, by (3.9) again, we can continue from (3.10) to get

$$\mathbb{P}\Big\{\tau^{k+1}_j > \tau^*_k + t | \mathcal{G}^k_t, \tilde{D}_k\Big\} = \mathbb{E}\Big\{\mathbb{E}\Big\{\exp(\tilde{Y}^{j,k+1}_{t_k} - \tilde{Y}^{j,k+1}_{t_k+t}) | \tilde{\mathcal{G}}^{k,j}_t, E_j > \tilde{Y}^{j,k+1}_{t_k}\Big\} | \mathcal{G}^k_t, \tilde{D}_k\Big\}$$
$$= \mathbb{E}\Big\{\mathbb{E}\Big\{\exp(Y^{j,k+1}_{t_k} - Y^{j,k+1}_{t_k+t}) | \tilde{\mathcal{G}}^{k,j}_t, E_j > Y^{j,k+1}_{t_k}\Big\} | \mathcal{G}^k_t, \tilde{D}_k\Big\}$$
$$= \mathbb{E}\Big\{\exp(Y^{j,k+1}_{t_k} - Y^{j,k+1}_{t_k+t}) | \mathcal{G}^k_t, \tilde{D}_k\Big\}. \tag{3.11}$$

Since $t_1, \ldots, t_k$ are arbitrary, (3.6) follows.

(ii) By the arguments in (i), clearly $\bar{L}_{t_k} \mathbf{1}_{\tilde{D}_k}, X^{j,k+1}_{t_k}, Y^{j,k+1}_{t_k}$ are all $\mathcal{G}^k_0 \vee \sigma(\tilde{D}_k)$-measurable, $j \neq i_1, \ldots, i_k$. Then conditional on the filtration $\{\mathcal{G}^k_t \vee \sigma(\tilde{D}_k), t \geq 0\}$, the processes $\{X^{j,k+1}_{t_k+\cdot}, j \neq i_1, \ldots, i_k\}$ are conditionally independent on $\tilde{D}_k$. Thus so are $\{Y^{j,k+1}_{t_k+\cdot}, j \neq i_1, \ldots, i_k\}$ and therefore all τ^{k+1}_j's are conditionally independent on $\tilde{D}_k$. Since $t_1, \ldots, t_k$ are arbitrary, we see

that $(X^{j,k+1}_{\tau^*_k+}, Y^{j,k+1}_{\tau^*_k+}, \tau^{k+1}_j)$, $j \neq i_1, \ldots, i_k$, are conditionally independent on D_k, conditional on the filtration $\{\mathcal{G}^k_t \vee \sigma(D_k), t \geq 0\}$. Since $\tau^*_{k+1} = \tau^{k+1,*}_{k+1} = \min\{\tau^{k+1}_j : j \neq i_1, \ldots, i_k\}$ on the set D_k, (3.7) follows from (3.6) immediately. $\quad\square$

We conclude this section by some monotonicity properties of the system (2.8).

Assumption 3.4. b_0 is decreasing in α; for all i, b_i is increasing in x_0 and decreasing in α; λ_i is decreasing in x_0, x_i and increasing in α; $L^i \geq 0$ and is decreasing in t.

Lemma 3.5. *Assume that Assumptions 2.1, 3.1 and 3.4 hold. Then for any $\mathbb{F}^0$-adapted process α taking values in D_0, the system (2.8) is well-posed. Moreover, τ^α_i is decreasing in α, $i = 1, \ldots, n$, and $\bar{L}^n_t(\alpha)$ is increasing in t and α.*

Proof. Under Assumptions 2.1 and 3.1, it is clear that the system (2.8) is well-posed. Since $L^i \geq 0$, we see immediately that $\bar{L}^n_t(\alpha)$ is increasing in t.

We now assume $\alpha^1 \leq \alpha^2$. By the standard comparison theorem of SDEs one can easily show that

$$X^{0,\alpha_1} \geq X^{0,\alpha_2}, \qquad X^{i,\alpha_1} \geq X^{i,\alpha_2}, \qquad Y^{i,\alpha_1} \leq Y^{i,\alpha_2}.$$

It follows immediately that $\tau^{\alpha_1}_i \geq \tau^{\alpha_2}_i$. Since L^i is decreasing in t, we see that $\bar{L}^n_t(\alpha_1) \leq \bar{L}^n_t(\alpha_2)$. $\quad\square$

Remark 3.6. (i) If we interpret X^i as the performance of the i-th firm, then the monotonicity assumptions in Assumption 3.4 imply that the n firms are "partners" and are positively correlated to the common factor X^0, and thus they are all negatively correlated to the average past loss $\bar{L}$.

(ii) Assumption 3.4 can be replaced by

b_0 is increasing in α; and for all i, b_i is increasing in x^0 and α;

λ_i is decreasing in x^0, x^i and α; $L^i \geq 0$ and is decreasing in t.

In this case the firms are "competitors", and all the results in this paper will still hold true, after some obvious modifications. $\quad\square$

4. The fixed point theorem

Recall that the fixed point problem (2.10) provides the candidate for the limit process $\bar{L}^*$. We first have the following obvious result.

Proposition 4.1. *In the setting of Example 2.3, if λ is bounded and uniformly Lipschitz continuous in α, then ODE (2.12) has a unique solution α taking values in $[0, 1]$, and thus (2.10) has a unique fixed point.*

In the rest of this section we consider a more general and non-trivial case, in which the fixed point argument works. First, recall the coefficients in (2.5) and (2.6). For simplicity, we assume in this section that

$$x_i = x, \qquad b_i = b, \qquad \sigma_i = \sigma, \qquad \lambda_i = \lambda, \quad i \geq 1. \tag{4.1}$$

We next introduce assumptions on the loss processes L^i. Since L^i is $\mathbb{F}^i$-adapted, we can write

$$L^i_t = \varphi_i(t, B^0_{\cdot \wedge t}, B^i_{\cdot \wedge t}), \quad t \geq 0, \ i = 1, 2, \ldots \tag{4.2}$$

where each $\varphi_i : \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R})^2 \to \mathbb{R}$ is a measurable function. The simplest case is the one in which all φ_i's are identical. However, we may consider a more general case in which there is a classification over the possible level of losses. The basic idea is that there are different loss types, known to the public, and each firm's loss at default falls into a particular type with a certain "frequency". The following definition, albeit technical, reflects the essence of this idea in a general form.

Definition 4.2. Let $\varphi \triangleq \{\varphi(\theta)\}_{\theta \in [0,1]}$ be a family of measurable mappings $\varphi(\theta) : \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R})^2 \to \mathbb{R}$ and μ a probability measure on $[0, 1]$. We say the sequence $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ) if, for any $\varepsilon > 0$ and $T > 0$, there exist $k = k(\varepsilon, T)$, disjoint subsets $\Theta_1, \ldots, \Theta_k \subset [0, 1]$, and disjoint subsets $D_1, \ldots, D_k \subset \mathbb{N}$ such that

$$\mu\Big([0, 1] \setminus (\Theta_1 \cup \cdots \cup \Theta_k) \Big) < \varepsilon;$$

$$\sup_{i \in D_j} \left\| \varphi_i - \frac{1}{\mu(\Theta_j)} \int_{\Theta_j} \varphi(\theta) d\mu(\theta) \right\|_{T,\infty} < \varepsilon, \quad j = 1, \ldots, k; \tag{4.3}$$

$$\lim_{n \to \infty} \frac{\left| D_j \cap \{1, \ldots, n\} \right|}{n} = \mu(\Theta_j), \quad j = 1, \ldots, k.$$

Here $\|\varphi\|_{T,\infty} \triangleq \sup\Big\{ |\varphi(t, \mathbf{x}^0_{\cdot \wedge t}, \mathbf{x}_{\cdot \wedge t})| : 0 \leq t \leq T, \mathbf{x}^0, \mathbf{x} \in C(\mathbb{R}^+; \mathbb{R}) \Big\}$.

To illustrate the idea behind Definition 4.2, we provide several examples.

Example 4.3 (*Singleton Case*)**.** Let $\theta_0 \in [0, 1]$ and $\mu(\{\theta_0\}) = 1$. Then $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ) if and only if there exists a set $D \subset \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{\left| D \cap \{1, \ldots, n\} \right|}{n} = 1 \quad \text{and} \quad \lim_{i \in D, i \to \infty} \|\varphi_i - \varphi(\theta_0)\|_{T,\infty} = 0 \quad \text{for any } T > 0.$$

The simplest case for which $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ) in this case is of course when $\varphi_i = \varphi(\theta_0)$ for all $i \geq 1$. That is, there is only one type of loss.

Example 4.4 (*Discrete Case*)**.** Let $\{\theta_k, k \geq 1\} \subset [0, 1]$ and $\mu(\{\theta_k, k \geq 1\}) = 1$. Then $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ) if and only if there exist disjoint subsets $D_k \subset \mathbb{N}, k \geq 1$, such that

$$\lim_{n \to \infty} \frac{\left| D_k \cap \{1, \ldots, n\} \right|}{n} = \mu(\{\theta_k\}) \quad \text{and}$$
$$\lim_{i \in D_k, i \to \infty} \|\varphi_i - \varphi(\theta_k)\|_{T,\infty} = 0, \quad T > 0, \ k \geq 1.$$

In particular, if $k = 2, \mu(\theta_1) = \mu(\theta_2) = \frac{1}{2}$, then we could set $\varphi_i = \varphi(\theta_1)$ when i is odd and $\varphi_i = \varphi(\theta_2)$ when i is even, so that $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ).

Example 4.5 (*Continuous Case*)**.** Let μ be the Lebesgue measure on $[0, 1]$ and $\varphi(\theta) = \theta\varphi_0$, where φ_0 is a given mapping: $\mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R})^2 \to \mathbb{R}$. For each n and $2^{n-1} \leq i < 2^n$, assume $\varphi_i = (i2^{1-n} - 1)\varphi_0$. Then one can easily check that $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ).

We will need the following assumptions on the coefficients.

Assumption 4.6. (i) (4.1) holds and $\sigma_0(s, x_0, \alpha) = \sigma_0(s, x_0)$, $\sigma(s, x_0, x_i, \alpha) = \sigma(s, x_i)$;
(ii) (4.2) holds and $\{\varphi_i, i \geq 1\}$ has distribution (φ, μ), in the sense of Definition 4.2;
(iii) there exists a constant $K > 0$ such that $|\varphi_i| \leq K$ and $|\lambda| \leq K$.

We note that under Assumption 4.6(i), the system (2.8) now becomes:

$$X_t^{0,\alpha} = x_0 + \int_0^t b_0(s, B_{\cdot \wedge s}^0, X_s^{0,\alpha}, \alpha_s)ds + \int_0^t \sigma_0(s, B_{\cdot \wedge s}^0, X_s^{0,\alpha})dB_s^0;$$

$$X_t^{i,\alpha} = x + \int_0^t b(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds + \int_0^t \sigma(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^{i,\alpha})dB_s^i; \quad (4.4)$$

$$Y_t^{i,\alpha} \triangleq \int_0^t \lambda(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds; \quad \tau_i^\alpha \triangleq \inf\left\{t \geq 0 : Y_t^{i,\alpha} \geq E^i\right\}.$$

The following lemma is useful.

Lemma 4.7. *Assume Assumptions 2.1, 3.1 and 4.6 hold, and let α be an $\mathbb{F}^0$-adapted process taking values in $D_0 \triangleq [-K, K]$. Denote*

$$\bar{\varphi} \triangleq \int_0^1 \varphi(\theta)d\mu(\theta); \quad (4.5)$$

$$\Gamma_t(\alpha) \triangleq \mathbb{E}\left\{\int_0^t \bar{\varphi}(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^1)\lambda(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^1, X_s^{0,\alpha}, X_s^{1,\alpha}, \alpha_s)e^{-Y_s^{1,\alpha}}ds|\mathcal{F}_t^0\right\}. \quad (4.6)$$

Then

(i) τ_i^α *are conditionally i.i.d., conditional on $\mathbb{F}^0$, and*

$$\lim_{n \to \infty} \mathbb{E}\{|\bar{L}_t^n(\alpha) - \Gamma_t(\alpha)|\} = 0. \quad (4.7)$$

(ii) *Moreover, if Assumption 3.4 also holds, then $\Gamma(\alpha)$ is continuous and increasing in t, increasing in α, and satisfies $0 \leq \Gamma_t(\alpha) \leq K$, a.s.*

(iii) *The process $\Gamma(\alpha)$ can be written as*

$$\Gamma_t(\alpha) = \int_0^t \mathbb{E}\left\{\bar{\varphi}(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^1)\lambda(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^1, X_s^{0,\alpha}, X_s^{1,\alpha}, \alpha_s)e^{-Y_s^{1,\alpha}}|\mathcal{F}_s^0\right\}ds. \quad (4.8)$$

Proof. (i) By our assumptions, it is readily seen that $\{(B^i, X^{i,\alpha}, Y^{i,\alpha}, \tau_i^\alpha)\}_{i=1}^n$ are conditionally i.i.d., conditional on $\mathcal{F}_t^0$. So it suffices to prove (4.7).

For any $t > 0$ and $\varepsilon > 0$, let $k, \Theta_j, D_j, j = 1, \ldots k$, be as in Definition 4.2. Denote

$$\Theta_{k+1} \triangleq [0, 1] \setminus (\Theta_1 \cup \cdots \cup \Theta_k), \qquad D_{k+1} \triangleq \mathbb{N} \setminus (D_1 \cup \cdots \cup D_k),$$
$$D_j^n \triangleq D_j \cap \{1, \ldots, n\},$$

and

$$\bar{\varphi}_j \triangleq \frac{1}{\mu(\Theta_j)} \int_{\Theta_j} \varphi(\theta)d\mu(\theta).$$

Note that, by denoting $\varphi_i(s) \triangleq \varphi_i(s, B_{\cdot \wedge s}^0, B_{\cdot \wedge s}^i)$,

$$\bar{L}_t^n(\alpha) = \frac{1}{n}\sum_{i=1}^n L_{\tau_i^\alpha}^i \mathbf{1}_{\{\tau_i^\alpha \leq t\}} = \frac{1}{n}\sum_{i=1}^n \varphi_i(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha \leq t\}} = \frac{1}{n}\sum_{j=1}^{k+1}\sum_{i \in D_j^n} \varphi_i(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha \leq t\}}$$

$$= \frac{1}{n}\left[\sum_{j=1}^{k}\sum_{i\in D_j^n}\bar{\varphi}_j(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha\le t\}} + \sum_{j=1}^{k}\sum_{i\in D_j^n}[\varphi_i(\tau_i^\alpha)-\bar{\varphi}_j(\tau_i^\alpha)]\mathbf{1}_{\{\tau_i^\alpha\le t\}}\right.$$

$$\left.+ \sum_{i\in D_{k+1}^n}\varphi_i(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha\le t\}}\right]$$

and that

$$\Gamma_t(\alpha) = \mathbb{E}\left\{\bar{\varphi}(\tau_1^\alpha)\mathbf{1}_{\{\tau_1^\alpha\le t\}}\Big|\mathcal{F}_t^1\Big|\mathcal{F}_t^0\right\} = \mathbb{E}\left\{\bar{\varphi}(\tau_1^\alpha)\mathbf{1}_{\{\tau_1^\alpha\le t\}}|\mathcal{F}_t^0\right\}$$

$$= \mathbb{E}\left\{\left[\sum_{j=1}^{k+1}\bar{\varphi}_j(\tau_1^\alpha)\mu(\Theta_j)\right]\mathbf{1}_{\{\tau_1^\alpha\le t\}}|\mathcal{F}_t^0\right\}$$

$$= \sum_{j=1}^{k}\mu(\Theta_j)\mathbb{E}\left\{\bar{\varphi}_j(\tau_j^\alpha)\mathbf{1}_{\{\tau_j^\alpha\le t\}}|\mathcal{F}_t^0\right\} + \mu(\Theta_{k+1})\mathbb{E}\left\{\bar{\varphi}_{k+1}(\tau_1^\alpha)\mathbf{1}_{\{\tau_1^\alpha\le t\}}|\mathcal{F}_t^0\right\}.$$

Since $|\varphi_i|\le K$, it is obvious that $|\bar{\varphi}_i|\le K$. Then by (4.3) we have

$$\frac{1}{n}\sum_{j=1}^{k}\sum_{i\in D_j^n}|\varphi_i(\tau_i^\alpha)-\bar{\varphi}_j(\tau_i^\alpha)|\mathbf{1}_{\{\tau_i^\alpha\le t\}} \le \varepsilon;$$

$$\frac{1}{n}\sum_{i\in D_{k+1}^n}|\varphi_i(\tau_i^\alpha)| \le \frac{K|D_{k+1}^n|}{n} \to K\mu(\Theta_{k+1}) \le K\varepsilon;$$

$$\mu(\Theta_{k+1})|\bar{\varphi}_{k+1}(\tau_1^\alpha)| \le K\mu(\Theta_{k+1}) \le K\varepsilon.$$

Moreover, for each $j=1,\ldots,k$, by the standard Law of Large Numbers we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i\in D_j^n}\bar{\varphi}_j(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha\le t\}} = \lim_{n\to\infty}\frac{|D_j^n|}{n}\frac{1}{|D_j^n|}\sum_{i\in D_j^n}\bar{\varphi}_j(\tau_i^\alpha)\mathbf{1}_{\{\tau_i^\alpha\le t\}}$$

$$= \mu(\Theta_j)\mathbb{E}\left\{\bar{\varphi}_j(\tau_1^\alpha)\mathbf{1}_{\{\tau_1^\alpha\le t\}}|\mathcal{F}_t^0\right\}.$$

Thus

$$\overline{\lim_{n\to\infty}}\left|\bar{L}^n(\alpha)-\Gamma_t(\alpha)\right| \le (2K+1)\varepsilon.$$

Since ε is arbitrary, we prove (4.7).

(ii) It follows directly from Lemma 3.5 and (4.7) that $\Gamma(\alpha)$ is increasing in t and α, and $0\le\Gamma_t(\alpha)\le K$. Moreover, denote

$$\gamma_t(\alpha) \triangleq \bar{\varphi}(t,B^0_{\cdot\wedge t},B^1_{\cdot\wedge t})\lambda(t,B^0_{\cdot\wedge t},B^1_{\cdot\wedge t},X_t^{0,\alpha},X_t^{1,\alpha},\alpha_t)e^{-Y_t^{1,\alpha}}.$$

For any t and $\varepsilon>0$,

$$\left|\Gamma_{t+\varepsilon}(\alpha)-\Gamma_t(\alpha)\right| \le \left|\mathbb{E}\left\{\int_t^{t+\varepsilon}\gamma_s(\alpha)ds|\mathcal{F}_{t+\varepsilon}^0\right\}\right|$$

$$+ \left|\mathbb{E}\left\{\int_0^t\gamma_s(\alpha)ds|\mathcal{F}_{t+\varepsilon}^0\right\} - \mathbb{E}\left\{\int_0^t\gamma_s(\alpha)ds|\mathcal{F}_t^0\right\}\right|$$

$$\leq K^2 \varepsilon + \left| \mathbb{E}\left\{ \int_0^t \gamma_s(\alpha) ds \middle| \mathcal{F}_{t+\varepsilon}^0 \right\} - \mathbb{E}\left\{ \int_0^t \gamma_s(\alpha) ds \middle| \mathcal{F}_t^0 \right\} \right|.$$

Since the filtration $\mathbb{F}^0$ is continuous, we obtain immediately that $\lim_{\varepsilon \to 0} \Gamma_{t+\varepsilon}(\alpha) = \Gamma_t(\alpha)$. Similarly, one can show that $\lim_{\varepsilon \to 0} \Gamma_{t-\varepsilon}(\alpha) = \Gamma_t(\alpha)$. Therefore, $\Gamma(\alpha)$ is continuous in t.

(iii) First, by the Fubini theorem we can write (4.6) as

$$\Gamma_t(\alpha) = \mathbb{E}\left\{ \int_0^t \gamma_s(\alpha) ds \middle| \mathcal{F}_t^0 \right\} = \int_0^t \mathbb{E}\left\{ \gamma_s(\alpha) \middle| \mathcal{F}_t^0 \right\} ds.$$

Since for each $s \in [0, t]$, $\gamma_s(\alpha)$ is $\mathcal{F}_s$-measurable, and $\mathcal{F}_t^0 = \mathcal{F}_s^0 \vee \mathcal{F}_{s,t}^0$, where $\mathcal{F}_{s,t}^0 \triangleq \sigma(B_u^0 - B_s^0, s \leq u \leq t)$ is independent of $\mathcal{F}_s$, it can be fairly easily checked that

$$\mathbb{E}\{\gamma_s(\alpha) | \mathcal{F}_t^0\} = \mathbb{E}\{\gamma_s(\alpha) | \mathcal{F}_s^0 \vee \mathcal{F}_{s,t}^0\} = \mathbb{E}\{\gamma_s(\alpha) | \mathcal{F}_s^0\},$$

and (4.8) follows. $\quad \square$

Remark 4.8. The condition (4.1) is to ensure that τ_i^α are conditionally i.i.d. and thus one may apply the standard Law of Large Numbers. It can be weakened slightly if one applies the generalized Law of Large Numbers by using the Lindeberg condition. $\quad \square$

We conclude this section with the following important result.

Theorem 4.9. *Assume Assumptions 2.1, 3.1, 3.4 and 4.6 hold. Then there exists an $\mathbb{F}^0$-adapted process such that $\alpha = \Gamma(\alpha)$.*

Proof. We will apply Zorn's lemma to prove the theorem. First, denote

$$\mathscr{L} \triangleq \left\{ \alpha : \mathbb{F}^0\text{-adapted, increasing, càdlàg , and } 0 \leq \alpha \leq K \right\}.$$

By Lemma 4.7, we see that $\Gamma(\alpha) \in \mathscr{L}$ for any $\alpha \in \mathscr{L}$. We introduce a partial order "$\preceq$" in $\mathscr{L}$: $\alpha^1 \preceq \alpha^2$ if $\alpha_t^1 \leq \alpha_t^2, t \geq 0$, $\mathbb{P}$-a.s. Now consider the set

$$\mathscr{L}_0 \triangleq \{ \alpha \in \mathscr{L} : \alpha \leq \Gamma(\alpha) \}.$$

Obviously $0 \in \mathscr{L}_0$, so $\mathscr{L}_0$ is not empty.

Assume that $\{\alpha^\theta\}_{\theta \in \Theta}$ is a totally ordered subset of $\mathscr{L}_0$. Define $\hat{\alpha}_r \triangleq \operatorname{esssup}_{\theta \in \Theta} \alpha_r^\theta$ for all $r \in \mathbb{Q}_+$. Then clearly $\hat{\alpha}_r$ is increasing in r, a.s. Define

$$\hat{\alpha}_t \triangleq \overline{\lim_{r \in \mathbb{Q}_+ \cap (t,\infty), r \downarrow t}} \hat{\alpha}_r, \quad t \geq 0.$$

Then it is easy to check that $\hat{\alpha} \in \mathcal{L}$. Since α^θ is càdlàg , we have $\hat{\alpha}_t \geq \alpha_t^\theta, t \geq 0$, a.s. for all $\theta \in \Theta$. Furthermore, since Γ is increasing in α, $\Gamma(\hat{\alpha}) \geq \Gamma(\alpha^\theta) \geq \alpha^\theta$ for all θ. Then $\Gamma_r(\hat{\alpha}) \geq \hat{\alpha}_r, r \in \mathbb{Q}_+$, a.s. Since $\Gamma(\hat{\alpha})$ is continuous, we have $\Gamma_t(\hat{\alpha}) \geq \hat{\alpha}_t$ for all $t \geq 0$, a.s. Thus $\hat{\alpha} \in \mathscr{L}_0$, and therefore, $\hat{\alpha}$ is an upper bound of $\{\alpha^\theta\}_{\theta \in \Theta}$ in $\mathscr{L}_0$.

Now applying Zorn's lemma we conclude that $\mathscr{L}_0$ has a maximum point α^* in $\mathscr{L}_0$. We claim that $\alpha^* = \Gamma(\alpha^*)$. Indeed, suppose that the equality fails. Then there exists $\varepsilon > 0$ such that $\mathbb{P}(\tau_1 < \infty) > 0$, where $\tau_1 \triangleq \inf\left\{ t \geq 0 : \Gamma_t(\alpha^*) \geq \alpha_t^* + \varepsilon \right\}$ is an $\mathbb{F}^0$-stopping time. Let $\tau_2 \triangleq \inf\left\{ t \geq \tau_1 : \alpha_t^* \geq \alpha_{\tau_1}^* + \varepsilon \right\}$ be another $\mathbb{F}^0$-stopping time taking values in $[0, \infty]$, and define

$$\tilde{\alpha}_t^* \triangleq \begin{cases} \alpha_t^*, & t < \tau_1 \text{ or } t \geq \tau_2; \\ \alpha_{\tau_1}^* + \varepsilon, & \tau_1 \leq t < \tau_2. \end{cases}$$

Since α^* is càdlàg , we see that $\tau_2 > \tau_1$ on $\{\tau_1 < \infty\}$, thus

$$\alpha^* \preceq \tilde{\alpha}^* \quad \text{and} \quad \alpha^* \neq \tilde{\alpha}^*. \tag{4.9}$$

On the other hand, by the definition of τ_2 we see that $\tilde{\alpha}^*$ is still increasing, then it is clear that $\tilde{\alpha}^* \in \mathscr{L}$. Moreover, since Γ is increasing in both α and t, then for $t < \tau_1$ or $t \geq \tau_2$, we have $\Gamma_t(\tilde{\alpha}^*) \geq \Gamma_t(\alpha^*) \geq \alpha_t^*$, and for $t \in [\tau_1, \tau_2)$, $\Gamma_t(\tilde{\alpha}^*) \geq \Gamma_t(\alpha^*) \geq \Gamma_{\tau_1}(\alpha^*) \geq \alpha_{\tau_1}^* + \varepsilon = \tilde{\alpha}_t^*$. This implies that $\tilde{\alpha}^* \in \mathscr{L}_0$, in contradiction with (4.9) and the assumption that α^* is a maximum point of $\mathscr{L}_0$. $\quad \square$

5. Potential applications

In this section we present some potentially useful applications under the "i.i.d." framework. To the best of our knowledge, these cases have not been fully analyzed in the literature.

5.1. Pricing a single name credit derivative

Suppose we are interested in pricing a credit derivative written on one firm, but the default intensity of the firm, λ, depends on the average number of defaults of many firms, as in our model. If our assumptions hold and that number is approximated by the process α_t, then we can find the price by using $\lambda(t, B^0_{\cdot \wedge t}, X^0_t, \alpha_t)$.

Specifically, consider the setting of Example 2.3. Recall that in this case the fixed point can be determined by a randomized ODE (2.12):

$$\alpha_t = \int_0^t (1 - \alpha_s)\lambda(s, B^0_{\cdot \wedge s}, X^0_s, \alpha_s) ds. \tag{5.1}$$

Let us assume further that λ is linear in α, that is,

$$\lambda(t, X^0_t, \alpha_t) = A(t, B^0_{\cdot \wedge t}, X^0_t) + D(t, B^0_{\cdot \wedge t}, X^0_t)\alpha_t,$$

where A and D are continuous functions, and are uniformly Lipschitz in x. Then the ODE (5.1) becomes (path-by-path) a Riccati equation:

$$\begin{aligned}
\alpha_t' &= (1 - \alpha_t)\lambda(t, B^0_{\cdot \wedge t}, X^0_t, \alpha_t) \\
&= P(t, B^0_{\cdot \wedge t}, X^0_t) + Q(t, B^0_{\cdot \wedge t}, X^0_t)\alpha_t + R(t, B^0_{\cdot \wedge t}, X^0_t)\alpha_t^2,
\end{aligned}$$

where $P = A$, $Q = D - A$, and $R = D$. Since the equation clearly has a particular solution $\alpha_t \equiv 1$, the general solution can be written as

$$\alpha_t = 1 + 1/v_t$$

where v_t solves the linear equation

$$v_t' = [A(t, B^0_{\cdot \wedge t}, X^0_t) + D(t, B^0_{\cdot \wedge t}, X^0_t)]v_t + D(t, B^0_{\cdot \wedge t}, X^0_t).$$

Since $\alpha_0 = 0$, we have $v_0 = -1$. Solving this ODE we obtain

$$v_t = -e^{\int_0^t p_s ds} + \int_0^t e^{\int_s^t p_r dr} D(s, B^0_{\cdot \wedge s}, X^0_s) ds, \quad t \geq 0,$$

where $p \triangleq A + D$. The process α is thus explicitly found, as a functional of (B^0, X^0), and we then face a standard problem in credit derivatives pricing, in which the (limiting) intensity only depends on the factor (B^0, X^0).

If we further assume that A and D are constant, it then follows that

$$\alpha_t = 1 - \frac{A + D}{Ae^{(A+D)t} + D}.$$

Thus, the default intensity can be approximated by

$$\hat{\lambda}_t = A + D\alpha_t = (A + D)\left[1 - \frac{D}{Ae^{(A+D)t} + D}\right].$$

We have then shown the following. If the intensity is of the form $\lambda_t = A + D\bar{N}_t$ where $\bar{N}_t$ is the average number of defaults of many firms, then we can (approximately) price derivatives which depend on λ by replacing it by simple deterministic process $\hat{\lambda}$.

In this case we can compute the integral

$$\int_0^t \hat{\lambda}_s ds = \log\left(\frac{Ae^{(A+D)t} + D}{A + D}\right)$$

which corresponds to the (approximated) probability of default after t being

$$\hat{\mathbb{P}}\{\tau_1^\alpha > t\} = \frac{A + D}{Ae^{(A+D)t} + D}.$$

Moreover, by (3.7), the intensity of the next default event conditional on k defaults is equal to

$$n(1 - k/n)(A + Dk/n) = An + (D - A)k - k^2/n.$$

Thus, as the "strength of interaction parameter" D increases, the conditional intensity of the next default also increases. Furthermore, this intensity is increasing in the number of past defaults k for low values of k, but as there are fewer and fewer firms left (as k gets close to n), the intensity decreases.

5.2. Finding the expected loss

We now consider a problem of computing the expected loss of a portfolio of a large number of defaultable loans, for example credit card customers. We assume that the loss of entity i is given by (4.2). According to (4.6) and (4.8), we expect to have

$$\alpha_t = \mathbb{E}\left\{\int_0^t \bar{\varphi}(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s})\lambda(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s}, X^{0,\alpha}_s, X^{1,\alpha}_s, \alpha_s)e^{-Y^{1,\alpha}_s} ds \big| \mathcal{F}^0_t\right\}$$

$$= \int_0^t \mathbb{E}\{\bar{\varphi}(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s})\lambda(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s}, X^{0,\alpha}_s, X^{1,\alpha}_s, \alpha_s)e^{-Y^{1,\alpha}_s} | \mathcal{F}^0_s\} ds. \tag{5.2}$$

Let us assume further that

$$\lambda(\cdots) = \lambda_0(t, B^0_{\cdot \wedge t}, \alpha_t) + \lambda_1(t, B^0_{\cdot \wedge t}, B^i_{\cdot \wedge t}).$$

Then, we can write (5.2) as

$$\alpha_t = \int_0^t \left[F_s \lambda_0(s, B^0_{\cdot \wedge s}, \alpha_s) + G_s\right] e^{-\int_0^s \lambda_0(u, B^0_{\cdot \wedge u}, \alpha_u) du} ds,$$

where

$$F_s \triangleq \mathbb{E}\left\{\bar{\varphi}(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s})e^{-\int_0^s \lambda_1(u, B^0_{\cdot \wedge u}, B^1_{\cdot \wedge u})du}|\mathcal{F}^0_s\right\},$$

$$G_s \triangleq \mathbb{E}\left\{\bar{\varphi}(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s})\lambda_1(s, B^0_{\cdot \wedge s}, B^1_{\cdot \wedge s})e^{-\int_0^s \lambda_1(u, B^0_{\cdot \wedge u}, B^1_{\cdot \wedge u})du}|\mathcal{F}^0_s\right\}.$$

Or equivalently, denoting $\beta^0_t(\alpha) \triangleq e^{-\int_0^t \lambda_0(u, B^0_{\cdot \wedge u}, \alpha_u)du}$,

$$\alpha'_t = [F_t \lambda_0(t, B^0_{\cdot \wedge t}, \alpha_t) + G_t]\beta^0_t(\alpha). \tag{5.3}$$

If we assume, in addition, that

$$\lambda_0(t, B^0_{\cdot \wedge t}, \alpha_t) = \bar{\lambda}_0(t, B^0_{\cdot \wedge t}) + R(t, B^0_{\cdot \wedge t})\alpha_t = I_t + R_t \alpha_t,$$

then (5.3) becomes

$$\alpha'_t = [(F_t I_t + G_t) + F_t R_t \alpha_t]H_t \beta_t(\alpha) = [\tilde{F}_t + \tilde{R}_t \alpha_t]\beta_t(\alpha), \tag{5.4}$$

where $\beta_t(\alpha) = e^{-\int_0^t R_s \alpha_s ds}$, and

$$H_t = e^{-\int_0^t I_s ds}, \qquad \tilde{F}_t = (F_t I_t + G_t)H_t, \qquad \tilde{R}_t = F_t R_t H_t.$$

Differentiating on both sides of (5.4) and using (5.3) we obtain the ODE for α:

$$\begin{aligned}\alpha''_t &= \{[\tilde{F}'_t + \tilde{R}'_t \alpha_t + \tilde{R}_t \alpha'_t] - [\tilde{F}_t + \tilde{R}_t \alpha_t]R_t \alpha_t\}\beta_t(\alpha) \\ &= \frac{[\tilde{R}_t \alpha'_t + (\tilde{R}'_t + \tilde{F}_t R_t)\alpha_t + \tilde{R}_t R_t \alpha_t^2 + \tilde{F}'_t]\alpha'_t}{\tilde{F}_t + \tilde{R}_t \alpha_t}.\end{aligned} \tag{5.5}$$

Moreover, by (5.2) and (5.4) we have

$$\alpha_0 = 0, \qquad \alpha'_0 = \tilde{F}_0 = F_0 I_0 + G_0 = \bar{\varphi}(0, 0, 0)I_0 + G_0. \tag{5.6}$$

Eq. (5.5) with initial conditions (5.6) is a non-linear second order ODE, which in general can only be solved numerically.

To recap, we have shown that if we impose technical conditions to guarantee that the limiting average loss is indeed equal to α_t, then we should be able to compute this limiting loss, for all times t, in this fairly complex model for individual losses.

5.3. A central limit theorem

In the next section we will prove that $\bar{L}^n$ converges to the fixed point process α, under certain technical conditions. In this subsection we prove a stronger result, a central limit theorem in a special case. The central limit theorem in general case is beyond the scope of this paper, and we leave it for future study.

Theorem 5.1. *Consider the setting of Example 2.3, and we assume further that*

$$\lambda = \lambda(\alpha) > 0 \quad and \quad \lambda \in C^2([0, 1]).$$

Then the process $Z^n_t \triangleq \frac{\sqrt{n}}{\alpha'_t}[\bar{L}^n_t - \alpha_t]$ converges in distribution to $Z_t \triangleq \int_0^t (\alpha'_s)^{-\frac{1}{2}}dB_s$, where B is a standard Brownian motion.

Proof. We proceed in several steps.

Step 1. We first note that, in this case

$$\alpha'_t = \theta(\alpha_t) \quad \text{where } \theta(\alpha) \triangleq (1 - \alpha)\lambda(\alpha). \tag{5.7}$$

Denote $x_k \triangleq x_k^{(n)} \triangleq \frac{k}{n}, k = 0, \ldots, n$. By (3.7) we see that

$$\tau^*_{k+1} - \tau^*_k \text{ has exponential distribution with parameter } n\theta(x_k), \quad k = 0, \ldots, n-1$$
$$\text{and they are independent.} \tag{5.8}$$

In particular, this leads to the moment generating function of τ^*_k:

$$\mathbb{E}\left[e^{\gamma \tau^*_k}\right] = \prod_{i=0}^{k-1} \mathbb{E}\left[e^{\gamma(\tau^*_{i+1}-\tau^*_i)}\right] = \prod_{i=0}^{k-1} \frac{\theta(x_i)}{\theta(x_i) - \frac{\gamma}{n}}, \quad \gamma < \min_{0 \le i \le k-1}(n\theta(x_i)). \tag{5.9}$$

Step 2. Let $x \in (0, 1)$, $y \in \mathbb{R}$. Denote $k \triangleq [nx + \sqrt{n}y]$, the largest integer below $nx + \sqrt{n}y$. We note that, when n is large enough, $0 \le k \le \frac{1+x}{2}n < n$. Now for any $i \le k - 1$, we have $\theta(x_i) \ge \frac{1-x}{2} \min_{0 \le \alpha \le 1} \lambda(\alpha) > 0$. Then

$$
\begin{aligned}
\ln\left(\mathbb{E}\left[e^{\sqrt{n}\gamma \tau^*_k}\right]\right) &= \sum_{i=0}^{k-1} \ln\left(\frac{\theta(x_i)}{\theta(x_i) - \frac{\gamma}{\sqrt{n}}}\right) = \sum_{i=0}^{k-1} \ln\left(1 + \frac{\gamma}{\theta(x_i)\sqrt{n}} + \frac{\gamma^2}{n\theta^2(x_i)} + o\left(\frac{1}{n}\right)\right) \\
&= \sum_{i=0}^{k-1}\left[\frac{\gamma}{\theta(x_i)\sqrt{n}} + \frac{\gamma^2}{n\theta^2(x_i)} - \frac{1}{2}\frac{\gamma^2}{n\theta^2(x_i)} + o\left(\frac{1}{n}\right)\right] \\
&= \sum_{i=0}^{k-1}\left[\frac{\gamma}{\theta(x_i)\sqrt{n}} - \frac{1}{2}\frac{\gamma^2}{n\theta^2(x_i)} + o\left(\frac{1}{n}\right)\right] \\
&= \sqrt{n}\gamma \int_0^{x+\frac{y}{\sqrt{n}}} \frac{dz}{\theta(z)} + \frac{\gamma^2}{2} \int_0^{x+\frac{y}{\sqrt{n}}} \frac{dz}{\theta^2(z)} + o(1) \\
&= \sqrt{n}\gamma \int_0^{x} \frac{dz}{\theta(z)} + \frac{\gamma y}{\theta(x)} + \frac{\gamma^2}{2} \int_0^{x} \frac{dz}{\theta^2(z)} + o(1).
\end{aligned} \tag{5.10}
$$

Step 3. Note that, for any $t > 0$ and $x \in [0, 1]$,

$$\{\bar{L}^n_t \le x\} = \{\tau^*_{[nx]} \ge t\}. \tag{5.11}$$

Then, for any $y \in \mathbb{R}$, denoting $k \triangleq [n\alpha_t + \sqrt{n}\theta(\alpha_t)y]$,

$$\mathbb{P}\left(Z^n_t \le y\right) = \mathbb{P}\left(\bar{L}^n_t \le \alpha_t + \frac{\theta(\alpha_t)y}{\sqrt{n}}\right) = \mathbb{P}\left(\tau^*_k \ge t\right) = \mathbb{P}\left(\sqrt{n}[\tau^*_k - t] \ge 0\right). \tag{5.12}$$

By (5.10), we have

$$\ln\left(\mathbb{E}\left[e^{\sqrt{n}\gamma(\tau^*_k - t)}\right]\right) = -\sqrt{n}\gamma t + \sqrt{n}\gamma \int_0^{\alpha_t} \frac{dz}{\theta(z)} + \gamma y + \frac{\gamma^2}{2} \int_0^{\alpha_t} \frac{dz}{\theta^2(z)} + o(1).$$

Note that

$$\int_0^{\alpha_t} \frac{dz}{\theta(z)} = \int_0^t \frac{\alpha'_s ds}{\theta(\alpha_s)} = t, \qquad \int_0^{\alpha_t} \frac{dz}{\theta^2(z)} = \int_0^t \frac{\alpha'_s ds}{\theta^2(\alpha_s)} = \int_0^t \frac{ds}{\theta(\alpha_s)}.$$

Then

$$\ln\left(\mathbb{E}\left[e^{\sqrt{n}\gamma(\tau_k^*-t)}\right]\right) = \gamma y + \frac{\gamma^2}{2}\int_0^t \frac{ds}{\theta(\alpha_s)} + o(1). \tag{5.13}$$

This implies that $\sqrt{n}[\tau_k^* - t]$ converges in distribution to $N(y, \int_0^t \frac{ds}{\theta(\alpha_s)})$. Then, by (5.12),

$$\lim_{n\to\infty}\mathbb{P}\left(Z_t^n \le y\right) = \mathbb{P}\left(N\left(y, \int_0^t \frac{ds}{\theta(\alpha_s)}\right) \ge 0\right) = \mathbb{P}\left(N\left(0, \int_0^t \frac{ds}{\theta(\alpha_s)}\right) \le y\right).$$

That is, Z_t^n converges in distribution to $N(0, \int_0^t \frac{ds}{\theta(\alpha_s)})$. Or equivalently, $\bar{L}_t^n$ has asymptotic distribution $N\left(\alpha_t, \frac{\theta^2(\alpha_t)}{n}\int_0^t \frac{ds}{\theta(\alpha_s)}\right)$.

Step 4. We now fix $m \ge 2$, let $0 = t_0 < t_1 < \cdots < t_m, y_1, \ldots, y_m \in \mathbb{R}$, and set $k_j \triangleq [n\alpha_{t_j} + \sqrt{n}\theta(\alpha_{t_j})y_j], j = 1, \ldots, m$. Similarly to (5.12), we have

$$\mathbb{P}\left(Z_{t_j}^n \le y_j, j = 1, \ldots, m\right) = \mathbb{P}\left(\sqrt{n}[\tau_{k_j}^* - t_j] \ge 0\right). \tag{5.14}$$

Recall (5.8). Following the estimates in (5.10) and (5.13) one can easily show that,

$$\ln\left(\mathbb{E}\left[e^{\sqrt{n}\sum_{j=1}^m \gamma_j[(\tau_{k_j}^* - \tau_{k_{j-1}}^*) - (t_j - t_{j-1})]}\right]\right)$$
$$= \sum_{j=1}^m \ln\left(\mathbb{E}\left[e^{\sqrt{n}\gamma_j[(\tau_{k_j}^* - \tau_{k_{j-1}}^*) - (t_j - t_{j-1})]}\right]\right)$$
$$= \sum_{j=1}^m \left[\gamma_j[y_j - y_{j-1}] + \frac{\gamma_j^2}{2}\int_{t_{j-1}}^{t_j} \frac{ds}{\theta(\alpha_s)}\right] + o(1).$$

This implies that $(\sqrt{n}[\tau_{k_j}^* - t_j], j = 1, \ldots, m)$ converges in distribution to a multinormal distribution $(\xi_1, \ldots, \xi_m)$, where $(\xi_1, \xi_2 - \xi_1, \ldots, \xi_m - \xi_{m-1})$ are independent and $\xi_j - \xi_{j-1}$ has distribution $N(y_j - y_{j-1}, \int_{t_{j-1}}^{t_j} \frac{ds}{\theta(\alpha_s)})$. Then

$$\mathbb{E}[\xi_j] = y_j, \qquad \text{Cov}(\xi_{j_1}, \xi_{j_2}) = \text{Var}(\xi_{j_1}) = \int_0^{t_{j_1}} \frac{ds}{\theta(\alpha_s)}, \quad 1 \le j_1 \le j_2 \le m.$$

Recall the notation $Z_t \triangleq \int_0^t (\theta(\alpha_s))^{-\frac{1}{2}} dB_s$. One can easily see that $(Z_{t_1}, \ldots, Z_{t_m})$ has multinormal distribution with

$$\mathbb{E}[Z_{t_j}] = 0 \qquad \text{Cov}(Z_{t_{j_1}}, Z_{t_{j_2}}) = \text{Var}(Z_{t_{j_1}}) = \int_0^{t_{j_1}} \frac{ds}{\theta(\alpha_s)}, \quad 1 \le j_1 \le j_2 \le m.$$

Thus $(\xi_j, j = 1, \ldots, m)$ and $(y_j - Z_{t_j}, j = 1, \ldots, m)$ have the same distribution. Then by (5.14) we have

$$\lim_{n\to\infty}\mathbb{P}\left(Z_{t_j}^n \le y_j, j = 1, \ldots, m\right) = \mathbb{P}\left(Z_{t_j} \le y_j, j = 1, \ldots, m\right).$$

Since m and $(t_j, y_j), j = 1, \ldots, m$ are arbitrary, we conclude that the process Z^n converges in distribution to Z. $\quad\square$

By setting $y = 0$ in (5.10) and recalling (5.8), one can prove the following proposition in a straightforward manner.

Proposition 5.2. *Denote $\tilde{Z}_x^n \triangleq \sqrt{n}[\tau_{[nx]}^* - \int_0^x \frac{dz}{\theta(z)}]$, and $\tilde{Z}_x \triangleq \int_0^x (\theta(z))^{-1} dB_z$, $x \in [0, 1]$, where B_x is the value of a standard Brownian motion at time x. Then, the processes $(Z_x^n, 0 \leq x \leq 1)$ converge in distribution to $(Z_x, 0 \leq x \leq 1)$.*

6. The Law of Large Numbers

In this section we present our main result. The aim is to show that in our strongly correlated self-exciting model, the Law of Large Numbers still holds, and the limit will be a fixed point discussed in the previous sections. Since the proof is quite lengthy, we defer a part of the proof to the next section.

To begin with we strengthen the technical conditions.

Assumption 6.1.   (i) $\sigma_0(t, x_0, \alpha) = \sigma_0(t)$, $\sigma_i(t, x_0, x_i, \alpha) = \sigma_i(t)$;
 (ii) b_0, b_i, λ_i are Lipschitz continuous in x_0, x_i, uniformly in (t, ω, α), and L^i is Lipschitz continuous in t, with a common Lipschitz constant K;
(iii) b_0, b_i, λ_i are Lipschitz continuous in α, uniformly in (t, ω, x_0, x_i), with a common Lipschitz constant Λ_0;
(iv) $0 < \Lambda_1 \leq \lambda_i \leq \Lambda_2$; $0 \leq L^i \leq \Lambda_3$;
 (v) $\Lambda_0 \leq \frac{\Lambda_1^2}{3\Lambda_2\Lambda_3}$.

Remark 6.2. The condition (v) above implies that the system is "weakly" correlated to the average loss $\bar{L}$. $\square$

In this and the next section, we denote by C a generic constant which depends only on the constants K, Λ_i, $i = 0, 1, 2, 3$ in Assumption 6.1, and it may vary from line to line. We emphasize in particular that C is independent of n. Moreover, we denote by C_ε (resp. $C_{\varepsilon,T}$) if the constant depends additionally on ε (resp. ε, T).

The main result of this paper is the following.

Theorem 6.3. *Assume Assumptions 2.1, 3.1, 3.4 and 6.1 hold. If the fixed point problem (2.10) has an $\mathbb{F}^0$-adapted solution α satisfying*

$$\lim_{n \to \infty} \mathbb{E}\left\{ |\bar{L}_t^n(\alpha) - \alpha_t| \right\} = 0. \tag{6.1}$$

Then the Law of Large Numbers (2.11) holds.

As a direct consequence of Theorems 4.9 and 6.3, and (4.7), we have the following theorem.

Theorem 6.4. *Assume Assumptions 2.1, 3.1, 3.4, 4.6 and 6.1 hold. Then the fixed point problem: $\alpha = \Gamma(\alpha)$, has a unique solution α, and the Law of Large Numbers (2.11) holds.*

Before we prove Theorem 6.3, let us make a quick analysis. We fix some $T > 0$ and consider $t \leq T$. First recall $(X^{0,\alpha}, X^{i,\alpha}, Y^{i,\alpha}, \tau_i^\alpha, \bar{L}^n(\alpha))$ in (2.8). Since

$$|\bar{L}_t^n - \alpha_t| \leq |\bar{L}_t^n - \bar{L}_t^n(\alpha)| + |\bar{L}_t^n(\alpha) - \alpha_t|, \tag{6.2}$$

by (6.1) it suffices to analyze the convergence of $\mathbb{E}\left\{|\bar{L}_t^n - \bar{L}_t^n(\alpha)|\right\}$ as $n \to \infty$. Notice that

$$\mathbb{E}\left\{|\bar{L}_t^n - \bar{L}_t^n(\alpha)|\right\} \leq \frac{1}{n} \sum_{i=1}^{n} I_i \quad \text{where} \quad I_i \triangleq \mathbb{E}\left\{|L_{\tau_i} \mathbf{1}_{\{\tau_i \leq t\}} - L_{\tau_i^\alpha} \mathbf{1}_{\{\tau_i^\alpha \leq t\}}|\right\}. \tag{6.3}$$

Without loss of generality we only estimate I_n. Note that

$$I_n \leq C\mathbb{E}\left\{|\tau_n - \tau_n^\alpha| \mathbf{1}_{\{\tau_n \leq t, \tau_n^\alpha \leq t\}} + \mathbf{1}_{\{\tau_n < t < \tau_n^\alpha\}} + \mathbf{1}_{\{\tau_n^\alpha < t < \tau_n\}}\right\}. \tag{6.4}$$

Therefore a crucial step is then to estimate

$$\mathbb{E}\{\mathbf{1}_{\{\tau_n < t < \tau_n^\alpha\}}\} = \mathbb{P}\{\tau_n < t < \tau_n^\alpha\} = \mathbb{P}\{Y_t^n > E_n > Y_t^{n,\alpha}\},$$
$$\mathbb{E}\{\mathbf{1}_{\{\tau_n^\alpha < t < \tau_n\}}\} = \mathbb{P}\{\tau_n^\alpha < t < \tau_n\} = \mathbb{P}\{Y_t^{n,\alpha} > E_n > Y_t^n\}. \tag{6.5}$$

We notice that $Y^{n,\alpha}$ and E_n are independent. But the main difficulty here is that Y^n and E_n are *not* independent in general. Without knowing their joint distribution it is difficult to estimate these probabilities. We therefore introduce two approximating systems, which do not involve the n-th name and thus are independent of E_n, so that the probabilities in (6.5) can be estimated. To be more precise, let us consider the following approximating losses. For $i = 1, \ldots, n$,

$$\hat{X}_t^{0,1} = x_0 + \int_0^t b_0(s, \hat{X}_s^{0,1}, \hat{L}_s^1)ds + \int_0^t \sigma_0(s)dB_s^0;$$
$$\hat{X}_t^{i,1} = x_i + \int_0^t b_i(s, \hat{X}_s^{0,1}, \hat{X}_s^{i,1}, \hat{L}_s^1)ds + \int_0^t \sigma_i(s)dB_s^i;$$
$$\hat{Y}_t^{i,1} \triangleq \int_0^t \lambda_i(s, \hat{X}_s^{0,1}, \hat{X}_s^{i,1}, \hat{L}_s^1)ds;$$
$$\hat{\tau}_i^1 \triangleq \inf\{t : \hat{Y}_t^{i,1} \geq E_i\}, \qquad \hat{L}_t^1 \triangleq \frac{1}{n} \sum_{i=1}^{n-1} L_{\hat{\tau}_i^1}^i \mathbf{1}_{\{\hat{\tau}_i^1 \leq t\}}; \tag{6.6}$$

and

$$\hat{X}_t^{0,2} = x_0 + \int_0^t b_0(s, \hat{X}_s^{0,2}, \hat{L}_s^2)ds + \int_0^t \sigma_0(s)dB_s^0;$$
$$\hat{X}_t^{i,2} = x_i + \int_0^t b_i(s, \hat{X}_s^{0,2}, \hat{X}_s^{i,2}, \hat{L}_s^2)ds + \int_0^t \sigma_i(s)dB_s^i;$$
$$\hat{Y}_t^{i,2} \triangleq \int_0^t \lambda_i(s, \hat{X}_s^{0,2}, \hat{X}_s^{i,2}, \hat{L}_s^2)ds;$$
$$\hat{\tau}_i^2 \triangleq \inf\{t : \hat{Y}_t^{i,2} \geq E_i\}, \qquad \hat{L}_t^2 \triangleq \frac{\Lambda_3}{n} + \frac{1}{n} \sum_{i=1}^{n-1} L_{\hat{\tau}_i^2}^i \mathbf{1}_{\{\hat{\tau}_i^2 \leq t\}}. \tag{6.7}$$

We emphasize that $\hat{L}^1$ and $\hat{L}^2$ do not involve $\hat{\tau}_n^1, \hat{\tau}_n^2$. Consequently, except for $\hat{\tau}_n^1, \hat{\tau}_n^2$, the above systems are now independent of E_n. The following theorem is essential for our analysis. We defer its proof to the next section.

Theorem 6.5. *Assume Assumptions* 2.1, 3.1, 3.4 *and* 6.1 *hold. Then*
(i) *For* $i = 1, \ldots, n$, *it holds that,*

$$\hat{L}_t^1 \leq \bar{L}_t \leq \hat{L}_t^2, \qquad \hat{X}_t^{0,1} \geq X_t^0 \geq \hat{X}_t^{0,2}, \qquad \hat{X}_t^{i,1} \geq X_t^i \geq \hat{X}_t^{i,2},$$
$$\hat{Y}_t^{i,1} \leq Y_t^i \leq \hat{Y}_t^{i,2}, \qquad \hat{\tau}_i^1 \geq \tau_i \geq \hat{\tau}_i^2. \tag{6.8}$$

(ii) *For any $T > 0$, there exist constants $\varepsilon = \varepsilon_T \in [0, \frac{1}{2})$ and $C_T > 0$ such that*

$$\mathbb{E}\left\{\Delta X_t^0 + \Delta X_t^i + \Delta Y_t^i\right\} \leq \frac{C_T}{n^\varepsilon \ln n}, \quad \forall t \in [0, T], \ i = 1, \ldots, n, \tag{6.9}$$

where

$$\Delta X_t^0 \triangleq \hat{X}_t^{0,1} - \hat{X}_t^{0,2}, \qquad \Delta X_t^i \triangleq \hat{X}_t^{i,1} - \hat{X}_t^{i,2}, \qquad \Delta Y_t^i \triangleq \hat{Y}_t^{i,2} - \hat{Y}_t^{i,1}.$$

Proof. (i) follows immediately from the monotonicity assumptions and the construction of the solutions. The proof of (ii) is rather lengthy, and we postpone it to Section 7. $\quad\square$

We are now ready to prove Theorem 6.3.

Proof of Theorem 6.3. In light of the previous argument and (6.1)–(6.4), we need only to obtain uniform estimates for each term on the right hand side of (6.4) as n goes to ∞.

To this end we first note that, with a simple application of Gronwall's inequality and the uniform Lipschitz conditions on the coefficients, it is readily seen that

$$|X_t^0 - X_t^{0,\alpha}| + |X_t^i - X_t^{i,\alpha}| + |Y_t^i - Y_t^{i,\alpha}| \leq C \int_0^t |\bar{L}_s^n - \alpha_s| ds. \tag{6.10}$$

Now, for each n, if $\tau_n < \tau_n^\alpha < \infty$, then

$$Y_{\tau_n}^n = E_n = Y_{\tau_n^\alpha}^{n,\alpha} = Y_{\tau_n}^{n,\alpha} + \int_{\tau_n}^{\tau_n^\alpha} \lambda_n(s, X_s^{0,\alpha}, X_s^{n,\alpha}, \alpha_s) ds \geq Y_{\tau_n}^{n,\alpha} + \Lambda_1[\tau_n^\alpha - \tau_n].$$

Thus for $\tau_n < \tau_n^\alpha \leq t$, one has

$$\tau_n^\alpha - \tau_n \leq \frac{1}{\Lambda_1} |Y_{\tau_n}^n - Y_{\tau_n}^{n,\alpha}| \leq C \int_0^t |\bar{L}_s^n - \alpha_s| ds.$$

With a similar argument for the case $\tau_n^\alpha \leq \tau_n \leq t$ we then obtain

$$|\tau_n^\alpha - \tau_n| \leq \frac{1}{\Lambda_1} |Y_{\tau_n}^n - Y_{\tau_n}^{n,\alpha}| \leq C \int_0^t |\bar{L}_s^n - \alpha_s| ds. \tag{6.11}$$

Next, recall (6.5). By Theorem 6.5(i) one has

$$\mathbb{P}\{\tau_n < t < \tau_n^\alpha\} = \mathbb{P}\{Y_t^n > E_n > Y_t^{n,\alpha}\} \leq \mathbb{P}\{\hat{Y}_t^{n,2} > E_n > Y_t^{n,\alpha}\}.$$

However, since E_n is now *independent* of $\hat{Y}_t^{n,2}$ and $Y_t^{n,\alpha}$, we can use the fact that $E_n \sim \exp(1)$ to get

$$\begin{aligned}
\mathbb{P}\{\tau_n < t < \tau_n^\alpha\} &\leq \mathbb{E}\{|e^{-Y_t^{n,\alpha}} - e^{-\hat{Y}_t^{n,2}}|\} \leq \mathbb{E}\{|Y_t^{n,\alpha} - \hat{Y}_t^{n,2}|\} \\
&\leq \mathbb{E}\left\{|Y_t^{n,\alpha} - Y_t^n| + |Y_t^n - \hat{Y}_t^{n,2}|\right\} \\
&\leq C\mathbb{E}\left\{\int_0^t |\bar{L}_s^n - \alpha_s| ds + \Delta Y_t^n\right\},
\end{aligned} \tag{6.12}$$

thanks to (6.10) and (6.8). Similarly we can also derive that

$$\mathbb{P}\{\tau_n^\alpha < t < \tau_n\} \leq C\mathbb{E}\left\{\int_0^t |\bar{L}_s^n - \alpha_s| ds + \Delta Y_t^n\right\}. \tag{6.13}$$

This, together with (6.4) and (6.10)–(6.12), as well as (6.10), leads to that

$$I_n \leq C \int_0^t \mathbb{E}\{|\bar{L}_s^n - \alpha_s|\}ds + C\mathbb{E}\left\{|L_t^n(\alpha) - \alpha_t| + \Delta Y_t^n\right\}.$$

Next, fix $T > 0$. For all $0 \leq t \leq T$, by Theorem 6.5 we have

$$I_n \leq C \int_0^t \mathbb{E}\{|\bar{L}_s^n - \alpha_s|\}ds + C\mathbb{E}\left\{|L_t^n(\alpha) - \alpha_t|\right\} + \frac{C_T}{\ln n}.$$

Similarly, for $i = 1, \ldots, n$, we have

$$I_i \leq C \int_0^t \mathbb{E}\{|\bar{L}_s^n - \alpha_s|\}ds + C\mathbb{E}\left\{|L_t^n(\alpha) - \alpha_t|\right\} + \frac{C_T}{\ln n}.$$

Then (6.2) and (6.3) lead to

$$\mathbb{E}\left\{|\bar{L}_t^n - \alpha_t|\right\} \leq C \int_0^t \mathbb{E}\{|\bar{L}_s^n - \alpha_s|\}ds + C\mathbb{E}\left\{|\bar{L}_t^n(\alpha) - \alpha_t|\right\} + \frac{C_T}{\ln n}.$$

Applying Gronwall's inequality we obtain

$$\mathbb{E}\{|\bar{L}_t^n - \alpha_t|\} \leq C_T\mathbb{E}\left\{|\bar{L}_t^n(\alpha) - \alpha_t|\right\} + \frac{C_T}{\ln n}, \quad 0 \leq t \leq T.$$

The theorem then follows immediately from (6.1). $\quad\square$

7. Proof of Theorem 6.5(ii)

In this section we prove Theorem 6.5(ii). We begin with two technical lemmas. The first one is a refinement of Gronwall's inequality.

Lemma 7.1. *Let $\{a_k^n\}_{k,n}^{n,\infty}$ be a two indices sequence of nonnegative numbers. Assume that the following recursive relation holds for some constant C:*

$$a_0^n = 0 \quad and \quad a_{k+1}^n \leq \left[1 + \frac{\ln k}{n} + \frac{C}{n}\right]a_k^n + \frac{C}{n(n-k)}, \quad k = 0, 1, \ldots, n-1.$$

Then there exists $\tilde{C} \geq C$, such that for any $\varepsilon > 0$,

$$\sup_{1 \leq k \leq (1-\varepsilon)n} a_k^n \leq \frac{\tilde{C}}{\varepsilon n^\varepsilon \ln n} \to 0, \quad as \ n \to \infty.$$

Proof. For any $0 < \varepsilon < 1$, and $k \leq (1-\varepsilon)n$, we have

$$a_{k+1}^n \leq [1 + \theta_n]a_k^n + \frac{C}{\varepsilon n^2}, \quad \text{where } \theta_n := \frac{\ln n}{n} + \frac{C}{n}.$$

This implies that

$$a_{k+1}^n \leq \frac{C}{\varepsilon n^2} \sum_{j=0}^k [1 + \theta_n]^j \leq \frac{C[1 + \theta_n]^{k+1}}{\varepsilon n^2 \theta_n}.$$

Therefore, for n large enough and for some $\tilde{C} \geq C$ which may vary from line to line,

$$
\sup_{1 \leq k \leq (1-\varepsilon)n} a_k^n \leq \frac{C[1+\theta_n]^{(1-\varepsilon)n}}{\varepsilon n^2 \theta_n} = \frac{C}{\varepsilon n^2 \theta_n} e^{(1-\varepsilon)n \ln(1+\theta_n)}
$$

$$
\leq \frac{\tilde{C}}{\varepsilon n \ln n} e^{(1-\varepsilon)n \theta_n} = \frac{\tilde{C}}{\varepsilon n \ln n} e^{(1-\varepsilon)(\ln n + C)}
$$

$$
\leq \frac{\tilde{C}}{\varepsilon n^\varepsilon \ln n} \to 0, \quad \text{as } n \to \infty.
$$

The proof is complete. $\square$

Lemma 7.2. *Let ξ and η be two random variables and ψ an increasing (resp. decreasing) function with $\mathbb{E}|\psi(\xi)| < \infty$ and $\mathbb{E}|\psi(\eta)| < \infty$. Assume $\mathbb{P}\{\xi > x\} \leq \mathbb{P}\{\eta > x\}$ for any $x \in \mathbb{R}$. Then $\mathbb{E}\{\psi(\xi)\} \leq (resp. \geq)\mathbb{E}\{\psi(\eta)\}$.*

Proof. We prove only the case in which ψ is increasing. Denote $G_\xi(x) \triangleq \mathbb{P}(\xi > x)$ and $G_\eta(x) \triangleq \mathbb{P}(\eta > x)$, $x \in \mathbb{R}$. Since ψ is increasing, we have

$$
\psi(x)G_\xi(x) = E\left\{\psi(x)\mathbf{1}_{\{\xi > x\}}\right\} \leq E\left\{\psi(\xi)\mathbf{1}_{\{\xi > x\}}\right\} \to 0 \quad \text{as } x \to \infty.
$$

Similarly,

$$
\lim_{x \to \infty} \psi(x)G_\eta(x) = 0, \qquad \lim_{x \to -\infty} \psi(x)[1 - G_\xi(x)] = 0,
$$

$$
\lim_{x \to -\infty} \psi(x)[1 - G_\eta(x)] = 0.
$$

Then

$$
\lim_{x \to \infty} \psi(x)[G_\xi(x) - G_\eta(x)] = 0,
$$

$$
\lim_{x \to -\infty} \psi(x)[G_\xi(x) - G_\eta(x)] = \lim_{x \to -\infty} \psi(x)\Big[[1 - G_\eta(x)] - [1 - G_\xi(x)]\Big] = 0.
$$

Integrating by parts, we get

$$
\mathbb{E}\{\psi(\xi) - \psi(\eta)\} = -\int_{-\infty}^{\infty} \psi(t)d[G_\xi(t) - G_\eta(t)] = \int_{-\infty}^{\infty} [G_\xi(t) - G_\eta(t)]d\psi(t).
$$

The result follows immediately. $\square$

Proof of Theorem 6.5 (ii). Denote $\{\tau_k^{1,*}\}$ and $\{\tau_k^{2,*}\}$ to be the order statistics of $\{\hat{\tau}_k^1\}$ and $\{\hat{\tau}_k^2\}$, respectively. Denote

$$
J_t^i \triangleq \Delta X_t^0 + \Delta X_t^i + \Delta Y_t^i, \qquad \tau_0^{1,*} \triangleq \tau_0^{2,*} \triangleq 0, \qquad \Delta \tau_k^* \triangleq \tau_k^{1,*} - \tau_k^{2,*},
$$

$$
\Delta \tau_k \triangleq \hat{\tau}_k^1 - \hat{\tau}_k^2.
$$

Then clearly, $J_0^i = \Delta \tau_0^* = 0$. The main idea of the proof is to estimate $J_{\tau_k^{*,2}}^i$ and $\Delta \tau_k^*$ by induction on k. We proceed in several steps.

Step 1. Fix $k \geq 0$, and assume $t \in [\tau_k^{2,*}, \tau_{k+1}^{2,*}]$. Note that in this interval the system (6.7) has a simple structure:

$$
\hat{X}_t^{0,2} = \hat{X}_{\tau_k^{2,*}}^{0,2} + \int_{\tau_k^{2,*}}^{t} b_0(s, \hat{X}_s^{0,2}, \hat{L}_{\tau_k^{2,*}}^2)ds + \int_{\tau_k^{2,*}}^{t} \sigma_0(s)dB_s^0,
$$

$$\hat{X}_t^{i,2} = \hat{X}_{\tau_k^{2,*}}^{i,2} + \int_{\tau_k^{2,*}}^t b_i(s, \hat{X}_s^{0,2}, \hat{X}_s^{i,2}, \hat{L}_{\tau_k^{2,*}}^2) ds + \int_{\tau_k^{2,*}}^t \sigma_i(s) dB_s^i,$$

$$\hat{Y}_t^{i,2} = \hat{Y}_{\tau_k^{2,*}}^{i,2} + \int_{\tau_k^{2,*}}^t \lambda_i(s, \hat{X}_s^{0,2}, \hat{X}_s^{i,2}, \hat{L}_{\tau_k^{2,*}}^2) ds.$$

To understand the system (6.6) in this interval, we denote $\tilde{\tau}_l \triangleq (\tau_l^{1,*} \vee \tau_k^{2,*}) \wedge \tau_{k+1}^{2,*}$ for $l = 0, \ldots, k+1$. By (6.8) we deduce that $\tau_k^{1,*} \geq \tau_k^{2,*}$, for all k, thus we must have

$$\tau_k^{2,*} = \tilde{\tau}_0 \leq \tilde{\tau}_1 \leq \cdots \leq \tilde{\tau}_{k+1} = \tau_{k+1}^{2,*}.$$

Let us now consider the sub-intervals $[\tilde{\tau}_l, \tilde{\tau}_{l+1}]$, on which we have

$$\hat{X}_t^{0,1} = \hat{X}_{\tilde{\tau}_l}^{0,1} + \int_{\tilde{\tau}_l}^t b_0(s, \hat{X}_s^{0,1}, \hat{L}_{\tilde{\tau}_l}^1) ds + \int_{\tilde{\tau}_l}^t \sigma_0(s) dB_s^0,$$

$$\hat{X}_t^{i,1} = \hat{X}_{\tilde{\tau}_l}^{i,1} + \int_{\tilde{\tau}_i}^t b_i(s, \hat{X}_s^{0,1}, \hat{X}_s^{i,1}, \hat{L}_{\tilde{\tau}_l}^1) ds + \int_{\tilde{\tau}_l}^t \sigma_i(s) dB_s^i,$$

$$\hat{Y}_t^{i,1} = \hat{Y}_{\tilde{\tau}_l}^{i,1} + \int_{\tilde{\tau}_l}^t \lambda_i(s, \hat{X}_s^{0,1}, \hat{X}_s^{i,1}, \hat{L}_{\tilde{\tau}_l}^1) ds.$$

Note that on the set $\{\tilde{\tau}_l < \tilde{\tau}_{l+1}\}$, we must have $\tau_l^{1,*} < \tau_{k+1}^{2,*}$ and $\tau_{l+1}^{1,*} > \tau_k^{2,*}$. Assume that for each $j = 1, \ldots, l$, the ordered statistics is attained at $\tau_j^{1,*} = \hat{\tau}_{i_j}^1$. Then, in light of (6.8) we have $\tau_{i_j}^2 \leq \hat{\tau}_{i_j}^1 < \tau_{k+1}^{2,*}$, and thus

$$\hat{\tau}_{i_j}^2 \leq \tau_k^{2,*}. \tag{7.1}$$

Then, we have

$$0 \leq \hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1 = \frac{1}{n} \sum_{j=1}^l [L_{\hat{\tau}_{i_j}^2}^{i_j} - L_{\hat{\tau}_{i_j}^1}^{i_j}] + \frac{1}{n} \sum_{1 \leq i \leq k, \tau_i^{2,*} \neq \hat{\tau}_{i_j}^2} L_{\tau_i^{2,*}}^i$$

$$\leq \frac{K}{n} \sum_{j=1}^l \Delta \tau_{i_j} + \frac{k-l+1}{n} \Lambda_3. \tag{7.2}$$

By the Lipschitz continuity, we have

$$d\Delta X_t^0 \leq \left[K \Delta X_t^0 + \Lambda_0(\hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1) \right] dt;$$

$$d\Delta X_t^i \leq \left[K \Delta X_t^0 + K \Delta X_t^i + \Lambda_0(\hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1) \right] dt;$$

$$d\Delta Y_t^i \leq \left[K \Delta X_t^0 + K \Delta X_t^i + \Lambda_0(\hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1) \right] dt.$$

Then,

$$dJ_t^i \leq 3 \left[K J_t^i + \Lambda_0(\hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1) \right] dt,$$

and thus

$$e^{-3Kt} J_t^i \leq e^{-3K\tilde{\tau}_l} J_{\tilde{\tau}_l}^i + \frac{\Lambda_0}{K} [e^{-3K\tilde{\tau}_l} - e^{-3Kt}][\hat{L}_{\tau_k^{2,*}}^2 - \hat{L}_{\tilde{\tau}_l}^1].$$

Let us define

$$A_0 \triangleq 0, \qquad A_t \triangleq A_{\tilde{\tau}_l} + \frac{\Lambda_0}{K}[e^{-3K\tilde{\tau}_l} - e^{-3Kt}][\hat{L}^2_{\tau_k^{2,*}} - \hat{L}^1_{\tilde{\tau}_l}], \quad t \in [\tilde{\tau}_l, \tilde{\tau}_{l+1}]. \tag{7.3}$$

Then, A is increasing, and by induction one can easily see that

$$e^{-3Kt} J_t^i \le A_t, \quad t \ge 0, \ i = 1, \dots, n. \tag{7.4}$$

Step 2. The process A plays a very important role in our proof. We next show that

$$\Delta \tau_k^* \le \frac{1}{\Lambda_1} e^{3K\tau_k^{2,*}} A_{\tau_k^{2,*}}. \tag{7.5}$$

Indeed, note that for any i,

$$\hat{Y}^{i,2}_{\hat{\tau}_i^2} = E_i = \hat{Y}^{i,1}_{\hat{\tau}_i^1} = \hat{Y}^{i,1}_{\hat{\tau}_i^2} + \int_{\hat{\tau}_i^2}^{\hat{\tau}_i^1} \lambda_i(s, \hat{X}^{0,1}_s, \hat{X}^{i,1}_s, \hat{L}^1_s) ds \ge \hat{Y}^{i,1}_{\hat{\tau}_i^2} + \Lambda_1 \Delta \tau_i.$$

This, together with the monotonicity properties in (6.8) (for Y^i), shows that

$$\Delta \tau_i \le \frac{1}{\Lambda_1} \Delta Y^i_{\hat{\tau}_i^2} \le \frac{1}{\Lambda_1} e^{3K\hat{\tau}_i^2} A_{\hat{\tau}_i^2}. \tag{7.6}$$

Assume that the order statistics $\tau^{2,*}$'s are attained at $\tau_1^{2,*} = \hat{\tau}_{i_1}^2, \dots, \tau_k^{2,*} = \hat{\tau}_{i_k}^2$. Then for $j = 1, \dots, k$, one has

$$\hat{\tau}_{i_j}^1 = \hat{\tau}_{i_j}^2 + \Delta \tau_{i_j} \le \hat{\tau}_{i_j}^2 + \frac{1}{\Lambda_1} e^{3K\hat{\tau}_{i_j}^2} A_{\hat{\tau}_{i_j}^2} \le \tau_k^{2,*} + \frac{1}{\Lambda_1} e^{3K\tau_k^{2,*}} A_{\tau_k^{2,*}}.$$

Since $\tau_k^{1,*} \le \max_{1 \le j \le k} \hat{\tau}_{i_j}^1$, we obtain (7.5).

Step 3. We now estimate $A_{\tau_{k+1}^{2,*}} - A_{\tau_k^{2,*}}$ in terms of $\tau_l^{2,*}$, $l \le k+1$. Plugging (7.2) and (7.6) into (7.3) and recalling (7.1), we see that

$$A_{\tilde{\tau}_{l+1}} \le A_{\tilde{\tau}_l} + \frac{\Lambda_0}{K}[e^{-3K\tilde{\tau}_l} - e^{-3K\tilde{\tau}_{l+1}}]\left[\frac{K}{n\Lambda_1} \sum_{j=1}^l e^{3K\hat{\tau}_{i_j}^2} A_{\hat{\tau}_{i_j}^2} + \frac{k-l+1}{n}\Lambda_3\right]$$

$$\le A_{\tilde{\tau}_l} + 3\Lambda_0[\tilde{\tau}_{l+1} - \tilde{\tau}_l]\left[\frac{Kl}{n\Lambda_1} A_{\tau_k^{2,*}} + \frac{k-l+1}{n}\Lambda_3 e^{-3K\tau_k^{2,*}}\right].$$

Summing over $l = 0, \dots, k$, we obtain

$$A_{\tau_{k+1}^{2,*}} - A_{\tau_k^{2,*}}$$

$$\le \frac{C}{n} A_{\tau_k^{2,*}} \sum_{l=0}^k l[\tilde{\tau}_{l+1} - \tilde{\tau}_l] + \frac{3\Lambda_0\Lambda_3}{n} e^{-3K\tau_k^{2,*}} \sum_{l=0}^k (k-l+1)[\tilde{\tau}_{l+1} - \tilde{\tau}_l]$$

$$= \frac{C}{n} A_{\tau_k^{2,*}} \sum_{l=1}^k [\tau_{k+1}^{2,*} - \tilde{\tau}_l] + \frac{3\Lambda_0\Lambda_3}{n} e^{-3K\tau_k^{2,*}} \sum_{l=1}^k [\tilde{\tau}_l - \tau_k^{2,*}]$$

$$\le \frac{Ck}{n} A_{\tau_k^{2,*}}[\tau_{k+1}^{2,*} - \tau_k^{2,*}] + \frac{3\Lambda_0\Lambda_3}{n} e^{-3K\tau_k^{2,*}} \sum_{l=1}^k [(\tau_l^{1,*} - \tau_k^{2,*})^+ \wedge (\tau_{k+1}^{2,*} - \tau_k^{2,*})]$$

$$= \frac{Ck}{n} A_{\tau_k^{2,*}}[\tau_{k+1}^{2,*} - \tau_k^{2,*}] + \frac{3\Lambda_0\Lambda_3}{n} e^{-3K\tau_k^{2,*}}$$

$$\times \sum_{l=1}^{k+1}[(\Delta\tau_l^* + \tau_l^{2,*} - \tau_k^{2,*})^+ \wedge (\tau_{k+1}^{2,*} - \tau_k^{2,*})].$$

Note that, for any $x, \alpha, \beta > 0$, $(x - \alpha)^+ \wedge \beta \le \frac{\beta}{\alpha+\beta}x$. Then, by (7.5), we deduce from the above

$$A_{\tau_{k+1}^{2,*}} - A_{\tau_k^{2,*}}\left[1 + \frac{Ck}{n}[\tau_{k+1}^{2,*} - \tau_k^{2,*}]\right]$$

$$\le \frac{3\Lambda_0\Lambda_3}{n} e^{-3K\tau_k^{2,*}}\left[\sum_{l=1}^{k} \Delta\tau_l^* \frac{\tau_{k+1}^{2,*} - \tau_k^{2,*}}{\tau_{k+1}^{2,*} - \tau_k^{2,*} + \tau_k^{2,*} - \tau_l^{2,*}} + \tau_{k+1}^{2,*} - \tau_k^{2,*}\right]$$

$$\le \frac{3\Lambda_0\Lambda_3}{n\Lambda_1} \sum_{l=1}^{k} A_{\tau_l^{2,*}} \frac{\tau_{k+1}^{2,*} - \tau_k^{2,*}}{\tau_{k+1}^{2,*} - \tau_k^{2,*} + \tau_k^{2,*} - \tau_l^{2,*}} + \frac{C}{n}[\tau_{k+1}^{2,*} - \tau_k^{2,*}]. \tag{7.7}$$

Step 4. We shall take expectation on both sides of (7.7). For that purpose, we apply Lemma 7.2 repeatedly to prove

$$\mathbb{E}\left\{\tau_{k+1}^{2,*} - \tau_k^{2,*}|\tilde{\mathcal{G}}_k\right\} \le \frac{1}{(n-k)\Lambda_1},$$

$$\mathbb{E}\left\{\frac{\tau_{k+1}^{2,*} - \tau_k^{2,*}}{\tau_{k+1}^{2,*} - \tau_k^{2,*} + \tau_k^{2,*} - \tau_l^{2,*}}|\tilde{\mathcal{G}}_l\right\} \le \frac{(n-l)\Lambda_2}{(n-k)\Lambda_1}\frac{1}{k-l}, \tag{7.8}$$

where $1 \le l \le k - 1$ and

$$\tilde{\mathcal{G}}_k := \sigma\left(\tau_l^{2,*}, A_{\tau_l^{2,*}}, l = 1, \dots, k\right), \quad k \ge 1.$$

Indeed, for any $t_1 < \cdots < t_k$ and $i_1, \dots, i_k$, recall (3.5) and (3.8). By Assumption 6.1 (iv) we derive from (3.7) that

$$e^{-(n-k)\Lambda_2 t} \le \mathbb{P}\left\{\tau_{k+1}^{2,*} > t_k + t|\mathcal{G}_t^k, \tau_l^{2,*} = \hat{\tau}_{i_l}^2 = t_l, l = 1, \dots, k\right\} \le e^{-(n-k)\Lambda_1 t}. \tag{7.9}$$

By (7.3), one can easily check that

$$A_{\tau_j^{2,*}}\mathbf{1}_{\{\tau_l^{2,*}=\hat{\tau}_{i_l}^2=t_l, l=1,\dots,k\}} \text{ is } \mathcal{G}_t^k \bigvee\left(\bigvee_{l=1}^{k} \sigma(\tau_l^{2,*} = \hat{\tau}_{i_l}^2 = t_l)\right)\text{-measurable}, \quad j = 1, \dots, k.$$

Then (7.9) implies that

$$e^{-(n-k)\Lambda_2 t} \le \mathbb{P}\left\{\tau_{k+1}^{2,*} - \tau_k^{2,*} > t|\tilde{\mathcal{G}}_k\right\} \le e^{-(n-k)\Lambda_1 t}. \tag{7.10}$$

Now, using the second inequality in (7.10) and applying Lemma 7.2 (by setting $\xi = \tau_{k+1}^{2,*} - \tau_k^{2,*}$, $\eta \sim \exp\{(n-k)\Lambda_1\}$, and $\psi(x) = x$) we obtain the first estimate in (7.8).

Next, since $\frac{x}{a+x}$ is concave in x, applying Jensen's inequality we get

$$\mathbb{E}\left\{\frac{\tau_{k+1}^{2,*} - \tau_k^{2,*}}{\tau_{k+1}^{2,*} - \tau_k^{2,*} + \tau_k^{2,*} - \tau_l^{2,*}}|\tilde{\mathcal{G}}_k\right\} \le \frac{\frac{1}{(n-k)\Lambda_1}}{\frac{1}{(n-k)\Lambda_1} + \tau_k^{2,*} - \tau_l^{2,*}}. \tag{7.11}$$

Let $\tilde{E}_1, \ldots, \tilde{E}_k$ be i.i.d. exponential random variables with rate 1 and independent of $\mathbb{F}$. Since $\frac{a}{a+x}$ is decreasing in x for $a > 0$, we can apply Lemma 7.2 repeatedly by using the first inequality in (7.10) and setting $\xi \sim \exp\{(n-k)\Lambda_2\}$, $\eta = \tau_{j+1}^{2,*} - \tau_j^{2,*}$, and $\psi(x) = \frac{a}{a+x}$ to get

$$\mathbb{E}\left\{\frac{\frac{1}{(n-k)\Lambda_1}}{\frac{1}{(n-k)\Lambda_1} + \tau_k^{2,*} - \tau_l^{2,*}} \Big| \tilde{\mathcal{G}}_l\right\} \leq \mathbb{E}\left\{\frac{\frac{1}{(n-k)\Lambda_1}}{\frac{1}{(n-k)\Lambda_1} + \sum_{j=l}^{k-1} \frac{\tilde{E}_j}{(n-j)\Lambda_2}}\right\}$$

$$= \mathbb{E}\left\{\frac{1}{1 + \frac{(n-k)\Lambda_1}{(n-l)\Lambda_2} \sum_{j=l}^{k-1} \tilde{E}_j}\right\}.$$

For $k - l \geq 1$, noticing that $\sum_{j=l}^{k-1} \tilde{E}_j$ has exponential distribution with rate $k - l$, we have

$$\mathbb{E}\left\{\frac{\frac{1}{(n-k)\Lambda_1}}{\frac{1}{(n-k)\Lambda_1} + \tau_k^{2,*} - \tau_l^{2,*}} \Big| \tilde{\mathcal{G}}_l\right\} \leq \frac{(n-l)\Lambda_2}{(n-k)\Lambda_1} \mathbb{E}\left\{\frac{1}{\sum_{j=l}^{k-1} \tilde{E}_j}\right\} = \frac{(n-l)\Lambda_2}{(n-k)\Lambda_1} \frac{1}{k-l}.$$

Plug this into (7.11) and notice that $\tilde{\mathcal{G}}_l \subset \tilde{\mathcal{G}}_k$, we prove the second estimate in (7.8).

Step 5. We now take expectation on both sides of (7.7). Denote

$$a_k \triangleq \mathbb{E}\{A_{\tau_k^{2,*}}\}, \qquad a_k^* \triangleq \max_{0 \leq i \leq k} a_i. \tag{7.12}$$

By (7.8) we have

$$a_{k+1} \leq \left[1 + \frac{Ck}{n(n-k)}\right] a_k + \frac{3\Lambda_0\Lambda_3}{n\Lambda_1}\left[\sum_{l=1}^{k-1} a_l \frac{(n-l)\Lambda_2}{(n-k)\Lambda_1} \frac{1}{k-l} + a_k\right] + \frac{C}{n(n-k)}$$

$$\leq a_k^*\left[1 + \frac{Ck}{n(n-k)} + \frac{C}{n} + \frac{3\Lambda_0\Lambda_2\Lambda_3}{n\Lambda_1^2} \sum_{l=1}^{k-1}\left[\frac{1}{k-l} + \frac{1}{n-k}\right]\right] + \frac{C}{n(n-k)}$$

$$\leq a_k^*\left[1 + \frac{Ck}{n(n-k)} + \frac{C}{n} + \frac{3\Lambda_0\Lambda_2\Lambda_3}{\Lambda_1^2} \frac{\ln k}{n}\right] + \frac{C}{n(n-k)}.$$

For $k \leq (1-\varepsilon)n$, thanks to Assumption 6.1(v), we have

$$a_{k+1} \leq \left[1 + \frac{\ln k}{n} + \frac{C_\varepsilon}{n}\right] a_k^* + \frac{C_\varepsilon}{n^2}.$$

This implies that

$$a_{k+1}^* \leq \left[1 + \frac{\ln k}{n} + \frac{C_\varepsilon}{n}\right] a_k^* + \frac{C_\varepsilon}{n^2}, \quad k \leq (1-\varepsilon)n.$$

Since $a_0 = 0$, applying Lemma 7.1 we obtain, for any $\varepsilon > 0$,

$$a_{[(1-\varepsilon)n]}^* \leq \frac{C_\varepsilon}{n^\varepsilon \ln n}. \tag{7.13}$$

Step 6. We finally prove (6.9). Recall that A is increasing. For any $\varepsilon > 0$ and denoting $k_\varepsilon := [(1 - \varepsilon)n]$, the largest integer below $(1 - \varepsilon)n$, by (7.4) and (7.13) we have

$$
\begin{aligned}
\mathbb{E}\{J_t^i\} &= \mathbb{E}\Big\{ J_t^i [\mathbf{1}_{\{\tau_{k_\varepsilon}^{2,*} \geq t\}} + \mathbf{1}_{\{\tau_{k_\varepsilon}^{2,*} < t\}}] \Big\} \leq e^{3Kt} \mathbb{E}\Big\{ A_t \mathbf{1}_{\{\tau_{k_\varepsilon}^{2,*} \geq t\}} \Big\} + \mathbb{E}\Big\{ J_t^i \mathbf{1}_{\{\tau_{k_\varepsilon}^{2,*} < t\}} \Big\} \\
&\leq C_T \mathbb{E}\{A_{\tau_{k_\varepsilon}^{2,*}}\} + \mathbb{E}\{J_t^i \mathbf{1}_{\{\tau_{k_\varepsilon}^{2,*} < t\}}\} \leq \frac{C_{\varepsilon,T}}{n^\varepsilon \ln n} + \mathbb{E}^{\frac{1}{2}}\{|J_t^i|^2\} \mathbb{P}^{\frac{1}{2}}\{\tau_{k_\varepsilon}^{2,*} < t\} \\
&\leq \frac{C_{\varepsilon,T}}{n^\varepsilon \ln n} + C_T \mathbb{P}^{\frac{1}{2}}\{\tau_{k_\varepsilon}^{2,*} < t\}.
\end{aligned}
$$

However, from (7.10) and applying Lemma 7.2 we see that

$$
\mathbb{P}\{\tau_{k_\varepsilon}^{2,*} < t\} \leq \mathbb{P}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} < t \Big\}, \quad \text{where } \hat{\tau}_{k_\varepsilon}^{2,*} \triangleq \sum_{i=1}^{k_\varepsilon} \frac{\tilde{E}_i}{(n-i)\Lambda_2}.
$$

Observe that

$$
\mathbb{E}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} \Big\} = \sum_{i=1}^{k_\varepsilon} \frac{1}{(n-i)\Lambda_2} \geq \frac{1}{2\Lambda_2} \ln \frac{1}{\varepsilon};
$$

$$
\mathrm{Var}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} \Big\} = \sum_{i=1}^{k_\varepsilon} \frac{1}{(n-i)^2 \Lambda_2^2} \leq \frac{2(1-\varepsilon)}{\varepsilon \Lambda_2^2 n}.
$$

Choosing $\varepsilon \triangleq \varepsilon_T > 0$ so that $\ln \frac{1}{\varepsilon} = 2\Lambda_2(T+1)$, we then have

$$
\begin{aligned}
\mathbb{P}\{\tau_{k_\varepsilon}^{2,*} < t\} &\leq \mathbb{P}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} < t \Big\} \leq \mathbb{P}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} - \mathbb{E}\{\hat{\tau}_{k_\varepsilon}^{2,*}\} < t - \frac{1}{2\Lambda_2} \ln \frac{1}{\varepsilon} \Big\} \\
&\leq \mathbb{P}\Big\{ \hat{\tau}_{k_\varepsilon}^{2,*} - \mathbb{E}\{\hat{\tau}_{k_\varepsilon}^{2,*}\} < -1 \Big\} \leq \mathrm{Var}(\hat{\tau}_{k_\varepsilon}^{2,*}) \leq \frac{2(1-\varepsilon_T)}{\varepsilon_T \Lambda_2^2 n} = \frac{C_T}{n}.
\end{aligned}
$$

Thus,

$$
\mathbb{E}\{J_t^i\} \leq \frac{C_{\varepsilon,T}}{n^\varepsilon \ln n} + \frac{C_T}{\sqrt{n}}.
$$

This proves (6.9) immediately. □

Acknowledgments

References

[1] T.R. Bielecki, M. Rutkowski, Credit Risk: Modeling, Valuation and Hedging, in: Springer Finance, Springer-Verlag, Berlin, 2002.

[2] P. Collin-Dufresne, R. Goldstein, J. Helwege, Is credit event risk priced? Modeling contagion via updating of beliefs, Working Paper, Univ. California Berkeley, 2003.

[3] P. Collin-Dufresne, R. Goldstein, J. Hugonnier, A general formula for valuing defaultable securities, Econometrica 72 (2004) 1377–1407.

 [4] P. Dai Pra, W.J. Runggaldier, E. Sartori, M. Tolotti, Large portfolio losses: a dynamic contagion model, The Annals of Applied Probability 19 (2009) 347–394.
 [5] M. Davis, V. Lo, Infectious default, Quantitative Finance 1 (2001) 382–387.
 [6] A. Dembo, J.D. Deuschel, D. Duffie, Large portfolio losses, Finance and Stochastics 8 (2004) 3–16.
 [7] D. Duffie, K. Singleton, Credit Risk: Pricing, Measurement, and Management, Princeton University Press, 2003.
 [8] D. Filipovic, L. Overbeck, T. Schmidt, Dynamic CDO term structure modeling, Mathematical Finance 21 (2011) 53–71.
 [9] R. Frey, J. Backhaus, Credit derivatives in models with interacting default intensities: a Markovian approach, Preprint, Dept. of Mathematics, Universitat Leipzig, 2006.
[10] R. Frey, J. Backhaus, Dynamic hedging of syntentic CDO tranches with spread risk and default contagion, Journal of Economic Dynamics and Control 34 (2010) 710–724.
[11] K. Giesecke, L.R. Goldberg, Sequential defaults and incomplete information, Journal of Risk 7 (2004) 1–26.
[12] K. Giesecke, K. Spiliopoulos, R. Sowers, Default clustering in large portfolios: typical and atypical events, Preprint, 2010.
[13] K. Giesecke, S. Weber, Cyclical correlations, credit contagion and portfolio losses, Journal of Banking and Finance 28 (2005) 3009–3036.
[14] K. Giesecke, S. Weber, Credit contagion and aggregate losses, Journal of Economic Dynamics and Control 30 (2006) 741–767.
[15] U. Horst, Stochastic cascades, contagion and large portfolio losses, Journal of Economic Behaviour and Organization 63 (2007) 25–54.
[16] R.A. Jarrow, F. Yu, Counterparty risk and the pricing of defaultable securities, Journal of Finance 53 (2001) 2225–2243.
[17] M. Jeanblanc, S. Song, An explicit model of default time with given survival probability, Stochastic Processes and their Applications 21 (2011) 1678–1704.
[18] M. Jeanblanc, S. Song, Random times with given survival probability and their $\mathbb{F}$-martingale decomposition formula, Preprint, 2011.
[19] M. Jeanblanc, M. Yor, M. Chesney, Mathematical Methods for Financial Markets, in: Springer Finance, Springer-Verlag London, Ltd., London, 2009.
[20] D. Lando, Credit Risk Modeling: Theory and Applications, Princeton University Press, 2004.
[21] A. Mc Neil, R. Frey, P. Embrechts, Quantitative Risk Management: Concepts, Techniques, and Tools., Princeton University Press, 2005.
[22] F. Yu, Correlated Defaults in Intensity-Based Models, Mathematical Finance 17 (2007) 155–173.