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THE CONTINUOUS TIME NONZERO-SUM DYNKIN GAME PROBLEM AND APPLICATION IN GAME OPTIONS*

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Abstract. In this paper we study the nonzero-sum Dynkin game in continuous time, which is a two-player noncooperative game on stopping times. We show that it has a Nash equilibrium point for general stochastic processes. As an application, we consider the problem of pricing American game contingent claims by the utility maximization approach.

Key words. nonzero-sum game, Dynkin game, Snell envelope, stopping time, utility maximization, American game contingent claim

AMS subject classifications. 91A15, 91A10, 91A30, 60G40, 91A60

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1. Introduction.

Dynkin games of zero-sum or nonzero-sum, continuous or discrete time types, are games on stopping times. Since their introduction by Dynkin in [10], they have attracted much research activity (see, e.g., [1, 2, 4, 5, 6, 7, 8, 11, 12, 14, 15, 19, 20, 21, 22, 23, 24, 25, 26] and the references therein).

To begin with let us describe briefly those game problems. Assume we have a system controlled by two players or agents, a_1 and a_2. The system works or is alive up to the time when one of the agents decides to stop the control at a stopping time τ_1 for a_1 and τ_2 for a_2. An example of that system is a *recallable option* in a financial market (see [15, 17] for more details). When the system is stopped the payment for a_1 (resp., a_2) amounts to a quantity $J_1(\tau_1, \tau_2)$ (resp., $J_2(\tau_1, \tau_2)$), which could be negative, in which case it is a cost. We say that the nonzero-sum Dynkin game associated with J_1 and J_2 has a Nash equilibrium point (NEP) if there exists a pair of stopping times (τ_1^*, τ_2^*) such that for any (τ_1, τ_2) we have

$$J_1(\tau_1^*, \tau_2^*) \geq J_1(\tau_1, \tau_2^*) \quad \text{and} \quad J_2(\tau_1^*, \tau_2^*) \geq J_2(\tau_1^*, \tau_2).$$

The particular case where $J_1 + J_2 = 0$ corresponds to the zero-sum Dynkin game. In this case, when the pair (τ_1^*, τ_2^*) exists it satisfies

$$J_1(\tau_1, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*) \leq J_1(\tau_1^*, \tau_2) \text{ for any } \tau_1,\ \tau_2.$$

We call such a (τ_1^*, τ_2^*) a saddle point for the game. Additionally this existence implies in particular that

$$\inf_{\tau_2} \sup_{\tau_1} J_1(\tau_1, \tau_2) = \sup_{\tau_1} \inf_{\tau_2} J_1(\tau_1, \tau_2),$$

i.e., the game has a value.

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Mainly, in the zero-sum setting, authors aim at proving the existence of the value and/or a saddle point for the game, while in the nonzero-sum framework they focus on the issue of the existence of an NEP for the game.

In continuous time, for decades there has been much research on zero-sum Dynkin games [1, 2, 5, 6, 8, 10, 11, 12, 15, 19, 20, 21, 25, 26]. Recently this type of game has attracted new interest since it has been applied in mathematical finance (see, e.g., [3, 15, 16, 17]) in connection with the pricing of American game options introduced by Kifer in [17]. Compared with the zero-sum setting, there are much fewer results on nonzero-sum Dynkin games in the literature. Nevertheless in the Markovian framework, one can quote, among other papers, [4, 7, 23, 24], which deal with the nonzero-sum Dynkin game. In the non-Markovian framework, Etourneau [14] showed that the game has an NEP if some of the processes which define the game (Y^1 and Y^2 of (2.1) below) are supermartingales. Note that even in the Markovian setting, an equivalent condition is supposed. On the other hand, there are some other works which study the existence of approximate equilibrium points (see, e.g., [21]).

The main objective of this work is to study the existence of NEPs for nonzero-sum Dynkin games in a non-Markovian framework. For very general processes, we construct an NEP, and thus it always exists. This removes Etourneau-type conditions and, to the best of our knowledge, is novel in the literature. Our approach is based on the Snell envelope theory. We next apply our general existence result to price American game contingent claim by the utility maximization approach. Kuhn [18] studied a similar problem by assuming that agents a_1 and a_2 use only discrete stopping times and exponential utilities. We remove these constraints.

The rest of the paper is organized as follows. In section 2, we make the setting of the problem precise and give some preliminary results related to the Snell envelope notion. In section 3, we construct a sequence of pairs of decreasing stopping times and show that their limit pair is an NEP for the game. Finally, in section 4, we apply the result of section 3 to price American game contingent claim by the utility maximization approach. $\square$

2. Formulation of the problem. Throughout this paper T is a real positive constant which stands for the horizon of the problem, and $(\Omega, \mathcal{F}, P)$ is a fixed probability space on which is defined a filtration $\mathbf{F} := (\mathcal{F}_t)_{0\le t\le T}$ which satisfies the usual hypotheses, i.e., it is complete and right continuous.

Next,

- for any $\mathbf{F}$-stopping time θ, let $\mathcal{T}_\theta$ denote the set of $\mathbf{F}$-stopping times τ such that $\tau \in [\theta, T]$, P-a.s.;
- let $\mathcal{D}$ denote the space of $\mathbf{F}$-adapted $\mathbb{R}$-valued right continuous with left limits (RCLL) processes X such that the set of random variables $\{X_\tau,\ \tau \in \mathcal{T}_0\}$ are uniformly integrable.

We consider a game problem with two players, a_1 and a_2. For $i = 1, 2$, player a_i can choose a stopping time $\tau_i \in \mathcal{T}_0$ to stop the game. So the game actually ends at $\tau_1 \wedge \tau_2$. Each player a_i is associated with two payoff/cost processes X^i, Y^i. Their expected utilities $J_i(\tau_1, \tau_2)$, $i = 1, 2$, are defined as follows:

$$
\begin{aligned}
J_1(\tau_1,\tau_2) &\stackrel{\triangle}{=} E\Big\{X^1_{\tau_1} 1_{\{\tau_1\le\tau_2\}} + Y^1_{\tau_2} 1_{\{\tau_2<\tau_1\}}\Big\},\\
J_2(\tau_1,\tau_2) &\stackrel{\triangle}{=} E\Big\{X^2_{\tau_2} 1_{\{\tau_2<\tau_1\}} + Y^2_{\tau_1} 1_{\{\tau_1\le\tau_2\}}\Big\}.
\end{aligned} \tag{2.1}
$$

That is, if player a_i is the one who actually stops the game (i.e., $\tau_i < \tau_j$ for $j \neq i$), then he receives $X^i_{\tau_i}$; if the game is stopped by the other player a_j (i.e., $\tau_j < \tau_i$), then

a_i receives $Y^i_{\tau_j}$. In the case that $\tau_1 = \tau_2$ we take the convention that a_1 is responsible for stopping the game. We can of course assume instead that a_2 is responsible in this case, and thus the corresponding payoffs/costs inside the expectations in (2.1) become

$$X^1_{\tau_1} 1_{\{\tau_1<\tau_2\}} + Y^1_{\tau_2} 1_{\{\tau_2\leq\tau_1\}} \quad \text{and} \quad X^2_{\tau_2} 1_{\{\tau_2\leq\tau_1\}} + Y^2_{\tau_1} 1_{\{\tau_1<\tau_2\}}.$$

Throughout the paper we shall use the following assumptions.

Assumption A1. The processes $X^1, X^2, Y^1, Y^2 \in \mathcal{D}$ and X^1, X^2 have only positive jumps.

Assumption A2. For $i = 1, 2$, $X^i_t \leq Y^i_t$ for any $t \leq T$, P-a.s.

Assumption A3. For any $\tau \in \mathcal{T}_0$, $P(\{X^1_\tau < Y^1_\tau\}\backslash\{X^2_\tau < Y^2_\tau\}) = 0$.

Assumption A1 is more or less the minimum requirement for the problem. Assumption A2 implies that there is a penalty for stopping the game early. We can study similarly the situation with a reward for early stopping, namely, to replace Assumption A2 with $X^i_t \geq Y^i_t$. Moreover, if we assume $X^2 < Y^2$, then Assumption A3 is redundant.

Our main goal is to study the NEP of the game.

DEFINITION 2.1. *We say that* $(\tau^*_1, \tau^*_2) \in \mathcal{T}_0 \times \mathcal{T}_0$ *is a Nash equilibrium point of the nonzero-sum Dynkin game associated with* J_1 *and* J_2 *if*

$$J_1(\tau_1, \tau^*_2) \leq J_1(\tau^*_1, \tau^*_2), \quad J_2(\tau^*_1, \tau_2) \leq J_2(\tau^*_1, \tau^*_2) \quad \forall \tau_1, \tau_2 \in \mathcal{T}_0. \tag{2.2}$$

As pointed out previously, this problem has been studied by several authors in the Markovian framework [4, 7, 23, 24], i.e., when in addition to Assumptions A1–A3, the processes X^i and Y^i are deterministic functions of a Markov process $(m_t)_{t\leq T}$. If this latter condition is not satisfied, Etourneau showed in [14] that the game has an NEP when Y^1 and Y^2 are supermartingales. Note that even in the Markovian framework authors assume an equivalent condition to Etourneau's.

Our main result is the following theorem, which assumes only Assumptions A1–A3 but without any regularity assumption on Y^1, Y^2.

THEOREM 2.2. *Under Assumptions* A1, A2, *and* A3, *the nonzero-sum Dynkin game associated with* J_1 *and* J_2 *has an NEP* (τ^*_1, τ^*_2).

We shall construct (τ^*_1, τ^*_2) in the next section. Our construction is based on the Snell envelope of processes, which we introduce briefly now. For more details on this subject one can refer, e.g., to El-Karoui [13] or Dellacherie and Meyer [9].

LEMMA 2.3 (see [9, p. 431] or [13, p. 140]). *For any* $U \in \mathcal{D}$, *there exists an* $\mathbf{F}$*-adapted* $\mathbb{R}$*-valued RCLL process* W *such that* W *is the smallest supermartingale which dominates* U; *i.e., if* $\bar{W}$ *is another RCLL supermartingale such that* $\bar{W}_t \geq U_t$ *for all* $0 \leq t \leq T$, *then* $\bar{W}_t \geq W_t$ *for any* $0 \leq t \leq T$. *The process* W *is called the Snell envelope of* U. *Moreover, the following properties hold:*

(i) *For any* $\mathbf{F}$*-stopping time* θ *we have*

$$W_\theta = \operatorname*{esssup}_{\tau\in\mathcal{T}_\theta} E[U_\tau|\mathcal{F}_\theta] \quad (\textit{and then } W_T = U_T), \quad P\textit{-a.s.} \tag{2.3}$$

(ii) *Assume that* U *has only positive jumps. Then the stopping time*

$$\tau^* \stackrel{\triangle}{=} \inf\{s \geq 0, W_s = U_s\} \wedge T$$

is optimal, i.e.,

$$E[W_0] = E[W_{\tau^*}] = E[U_{\tau^*}] = \sup_{\tau\in\mathcal{T}_0} E[U_\tau]. \tag{2.4}$$

Remark 2.4. As a by-product of (2.4) we have $W_{\tau^*} = U_{\tau^*}$, and the process W is a martingale on the time interval $[0, \tau^*]$.

3. Construction of a Nash equilibrium point. In this section we shall construct a sequence of pairs of decreasing stopping times $(\tau_{2n+1}, \tau_{2n+2})$ and show that their limits (τ_1^*, τ_2^*) are an NEP. First, notice that Y^1 is required only to be RCLL, and that Y_T^1 is never used in (2.1); without loss of generality we will also assume the following.

Assumption A4. P-a.s., $Y_T^1 = X_T^1$.

We emphasize that this is just for notational simplicity. If we assume Assumption A4 does not hold, we may replace the integrands in (3.1) below with

$$X_\tau^1 1_{\{\tau<\tau_{2n}\}} + \Big[X_T^1 1_{\{\tau_{2n}=T\}} + Y_{\tau_{2n}}^1 1_{\{\tau_{2n}<T\}}\Big] 1_{\{\tau\ge\tau_{2n}\}},$$

and all the arguments will be the same.

We start by defining $\tau_1 \triangleq T$ and $\tau_2 \triangleq T$. For $n = 1, \dots,$ assume τ_{2n-1} and τ_{2n} have been defined; we then define τ_{2n+1} and τ_{2n+2} as follows. First, let

$$W_t^{2n+1} \triangleq \operatorname*{esssup}_{\tau\in\mathcal{T}_t} E_t\Big\{X_\tau^1 1_{\{\tau<\tau_{2n}\}} + Y_{\tau_{2n}}^1 1_{\{\tau\ge\tau_{2n}\}}\Big\}, \quad t \le T, \tag{3.1}$$

where here and in what follows $E_t\{\cdot\} \triangleq E\{\cdot|\mathcal{F}_t\}$, and

$$\tilde\tau_{2n+1} \triangleq \inf\{t \ge 0 : W_t^{2n+1} = X_t^1\} \wedge \tau_{2n}; \quad \tau_{2n+1} \triangleq \begin{cases} \tilde\tau_{2n+1} & \text{if } \tilde\tau_{2n+1} < \tau_{2n}; \\ \tau_{2n-1} & \text{if } \tilde\tau_{2n+1} = \tau_{2n}. \end{cases} \tag{3.2}$$

Next, let

$$W_t^{2n+2} \triangleq \operatorname*{esssup}_{\tau\in\mathcal{T}_t} E_t\Big\{X_\tau^2 1_{\{\tau<\tau_{2n+1}\}} + Y_{\tau_{2n+1}}^2 1_{\{\tau\ge\tau_{2n+1}\}}\Big\}, \quad t \le T, \tag{3.3}$$

and

$$\tilde\tau_{2n+2} \triangleq \inf\{t \ge 0 : W_t^{2n+2} = X_t^2\} \wedge \tau_{2n+1}; \quad \tau_{2n+2} \triangleq \begin{cases} \tilde\tau_{2n+2} & \text{if } \tilde\tau_{2n+2} < \tau_{2n+1}; \\ \tau_{2n} & \text{if } \tilde\tau_{2n+2} = \tau_{2n+1}. \end{cases} \tag{3.4}$$

We note that the integrand in (3.1) is slightly different from that of $J_1(\tau, \tau_{2n})$ in (2.1). The main reason is that, in order to apply Lemma 2.3, we need the process U^{2n+1} in (3.6) below to be RCLL. But nevertheless we will prove later in Lemma 3.3 that W^{2n+1} serves our purpose well.

LEMMA 3.1. *Assume Assumptions* A1 *and* A2 *hold. For* $n = 1, 2, \dots,$ τ_n *is a stopping time and* $\tau_{n+2} \le \tau_n$.

Proof. We prove the following stronger results by induction on n:

$$\tau_n \in \mathcal{T}_0, \quad \{\tau_n < \tau_{n+1}\} \subset \{\tilde\tau_{n+2} \le \tau_n\}, \quad \tau_{n+2} \le \tau_n. \tag{3.5}$$

Obviously (3.5) holds for $n = 1, 2$. Assume it is true for $2n-1$ and $2n$. We shall prove it for $2n+1$ and $2n+2$.

First, define

$$U_t^{2n+1} \triangleq X_t^1 1_{\{t<\tau_{2n}\}} + Y_{\tau_{2n}}^1 1_{\{t\ge\tau_{2n}\}}. \tag{3.6}$$

Since τ_{2n} is a stopping time, by Assumptions A1 and A2 we know $U^{2n+1} \in \mathcal{D}$, and it has only positive jumps. Applying Lemma 2.3, we see that W^{2n+1} is the Snell envelope of U^{2n+1} and $\tilde\tau_{2n+1}$ is the minimal optimal stopping time.

To prove $\tau_{2n+1} \in \mathcal{T}_0$, we note that

$$\{\tilde{\tau}_{2n+1} = \tau_{2n}\} \subset \{\tau_{2n-1} \geq \tau_{2n}\}. \tag{3.7}$$

In fact, if $\tau_{2n-1} < \tau_{2n}$, then by the second claim of (3.5) for $2n-1$ we have $\tilde{\tau}_{2n+1} \leq \tau_{2n-1}$. This implies $\tilde{\tau}_{2n+1} < \tau_{2n}$, and then (3.7) follows immediately. Now recall definition (3.2); for any $t \leq T$ we have

$$\begin{aligned}\{\tau_{2n+1} \leq t\} &= \Big[\{\tilde{\tau}_{2n+1} \leq t\} \cap \{\tilde{\tau}_{2n+1} < \tau_{2n}\}\Big] \bigcup \Big[\{\tau_{2n-1} \leq t\} \cap \{\tilde{\tau}_{2n+1} = \tau_{2n}\}\Big] \\ &= \Big[\{\tilde{\tau}_{2n+1} \leq t\} \cap \{\tilde{\tau}_{2n+1} < \tau_{2n} \leq t\}\Big] \bigcup \Big[\{\tilde{\tau}_{2n+1} \leq t\} \cap \{\tau_{2n} > t\}\Big] \\ &\quad \bigcup \Big[\{\tau_{2n-1} \leq t\} \cap \{\tilde{\tau}_{2n+1} = \tau_{2n} \leq t\}\Big].\end{aligned}$$

Then $\{\tau_{2n+1} \leq t\} \in \mathcal{F}_t$ for any t and thus τ_{2n+1} is a stopping time.

Next we focus on the inclusion of (3.5). On $\{\tau_{2n+1} < \tau_{2n+2}\}$, by the definition of τ_{2n+2} in (3.4), we have $\tau_{2n+2} = \tau_{2n}$. Then $U_t^{2n+3} = U_t^{2n+1}$ for $t \geq \tau_{2n+1}$ and thus

$$W^{2n+3}_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}} = W^{2n+1}_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}}. \tag{3.8}$$

On the other hand, if $\tilde{\tau}_{2n+1} = \tau_{2n}$, by the third claim of (3.5) for $2n$, (3.7), and the definition of (3.2), we have $\tau_{2n+2} \leq \tau_{2n} \leq \tau_{2n-1} = \tau_{2n+1}$. Thus $\{\tau_{2n+1} < \tau_{2n+2}\} \subset \{\tau_{2n+1} = \tilde{\tau}_{2n+1} < \tau_{2n}\}$, and therefore, by Remark 2.4,

$$W^{2n+1}_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}} = X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}}.$$

This, together with (3.8), implies that

$$W^{2n+3}_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}} = X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n+2}\}}.$$

Therefore we deduce, by the definition of $\tilde{\tau}_{2n+3}$ in (3.2), that

$$\{\tau_{2n+1} < \tau_{2n+2}\} \subset \{\tilde{\tau}_{2n+3} \leq \tau_{2n+1}\}. \tag{3.9}$$

Let us now show the inequality in (3.5). Actually if $\tau_{2n+3} > \tau_{2n+1}$, by definition (3.2) we have $\tau_{2n+3} = \tilde{\tau}_{2n+3} < \tau_{2n+2}$. Then $\tau_{2n+1} < \tilde{\tau}_{2n+3} < \tau_{2n+2}$. This contradicts (3.9). Therefore, $\tau_{2n+3} \leq \tau_{2n+1}$.

Finally, one can prove (3.5) for $2n+2$ similarly. $\square$

The following is another important property of the stopping times τ_n.

LEMMA 3.2. *Assume Assumptions* A1 *and* A2 *hold. On* $\{\tau_n = \tau_{n-1}\}$, *we have* $\tau_m = T$ *for all* $m \leq n$.

Proof. The result is obvious for $n = 2$. Assume it is true for n. Now for $n+1$, on $\{\tau_{n+1} = \tau_n\}$, by the definition of τ_{n+1} in (3.2) or (3.4) we have $\tau_{n+1} = \tau_{n-1}$. Then $\tau_n = \tau_{n-1}$ and thus by induction assumption we get the result. $\square$

The following lemma shows that τ_n is an optimal stopping time for some approximating problem.

LEMMA 3.3. *Assume Assumptions* A1, A2, *and* A4 *hold. For any* $\tau \in \mathcal{T}_0$ *and any* n *we have*

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n}) \quad \textit{and} \quad J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}). \tag{3.10}$$

Proof. First, by the definition of W^{2n+1} in (3.1) we have $W^{2n+1}_{\tau_{2n}} = Y^1_{\tau_{2n}}$. Next, by Lemma 2.3 we have $W^{2n+1}_t \geq X^1_t$ for any $t \in [0, \tau_{2n}]$, and W^{2n+1} is a supermartingale over $[0, \tau_{2n}]$. Then, for any $\tau \in \mathcal{T}_0$,

$$\begin{aligned} J_1(\tau, \tau_{2n}) &= E\Big\{X^1_\tau 1_{\{\tau \leq \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} < \tau\}}\Big\} \\ &\leq E\Big\{W^{2n+1}_\tau 1_{\{\tau \leq \tau_{2n}\}} + W^{2n+1}_{\tau_{2n}} 1_{\{\tau_{2n} < \tau\}}\Big\} = E\{W^{2n+1}_{\tau_{2n}\wedge\tau}\} \leq W^{2n+1}_0. \end{aligned} \tag{3.11}$$

On the other hand, by Lemma 3.2 and Assumption A4 we have

$$\begin{aligned} J_1(\tau_{2n+1}, \tau_{2n}) &= E\Big\{X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} \leq \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} < \tau_{2n+1}\}}\Big\} \\ &= E\Big\{X^1_{\tau_{2n+1}} 1_{\{\tau_{2n+1} < \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} \leq \tau_{2n+1}\}}\Big\}. \end{aligned}$$

By (3.2), (3.7), and Remark 2.4, we get

$$J_1(\tau_{2n+1}, \tau_{2n}) = E\Big\{X^1_{\tilde\tau_{2n+1}} 1_{\{\tilde\tau_{2n+1} < \tau_{2n}\}} + W^{2n+1}_{\tau_{2n}} 1_{\{\tilde\tau_{2n+1} = \tau_{2n}\}}\Big\} = E\{W^{2n+1}_{\tilde\tau_{2n+1}}\} = W^{2n+1}_0.$$

This, together with (3.11), proves $J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$.

Similarly we can prove $J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2})$. □

Now define

$$\tau_1^* \triangleq \lim_{n\to\infty} \tau_{2n+1} \quad \text{and} \quad \tau_2^* \triangleq \lim_{n\to\infty} \tau_{2n}. \tag{3.12}$$

We shall prove that (τ_1^*, τ_2^*) is an NEP. We divide the proof into several lemmas.

LEMMA 3.4. *Assume Assumptions* A1 *and* A2 *hold.*

(i) *For any* $\tau \in \mathcal{T}_0$, *we have* $\lim_{n\to\infty} J_1(\tau, \tau_{2n}) = J_1(\tau, \tau_2^*)$.

(ii) *For any* $\tau \in \mathcal{T}_0$ *such that* $P(\tau = \tau_1^* < T) = 0$, *we have* $\lim_{n\to\infty} J_2(\tau_{2n+1}, \tau) = J_2(\tau_1^*, \tau)$.

Proof. (i) By Assumption A1, we have

$$\begin{aligned} \lim_{n\to\infty} J_1(\tau, \tau_{2n}) &= \lim_{n\to\infty} E\Big\{X^1_\tau 1_{\{\tau \leq \tau_{2n}\}} + Y^1_{\tau_{2n}} 1_{\{\tau_{2n} < \tau\}}\Big\} \\ &= E\Big\{X^1_\tau 1_{\{\tau \leq \tau_2^*\}} + Y^1_{\tau_2^*} 1_{\{\tau_2^* < \tau\}}\Big\} = J_1(\tau, \tau_2^*). \end{aligned}$$

(ii) Since $\{\tau < \tau_{2n+1}\} \subset \{\tau < T\}$, we have

$$\lim_{n\to\infty} E\Big\{X^2_\tau 1_{\{\tau < \tau_{2n+1}\}}\Big\} = \lim_{n\to\infty} E\Big\{X^2_\tau 1_{\{\tau < \tau_{2n+1}, \tau \neq \tau_1^*\}}\Big\} = E\Big\{X^2_\tau 1_{\{\tau < \tau_1^*\}}\Big\}.$$

Moreover, note that $\tau_1^* \leq \tau_{2n+1}$; then $\{\tau_1^* = T\} \subset \{\tau_{2n+1} = T\}$. Applying the assumption $P(\tau = \tau_1^* < T) = 0$ twice, we have

$$\begin{aligned} &\lim_{n\to\infty} E\Big\{Y^2_{\tau_{2n+1}} 1_{\{\tau_{2n+1} \leq \tau\}}\Big\} = \lim_{n\to\infty} E\Big\{Y^2_{\tau_1^*} 1_{\{\tau_{2n+1} \leq \tau\}} \Big[1_{\{\tau \neq \tau_1^*\}} + 1_{\{\tau = \tau_1^*\}}\Big]\Big\} \\ &= E\Big\{Y^2_{\tau_1^*} 1_{\{\tau_1^* < \tau\}} + Y^2_{\tau_1^*} 1_{\{\tau = \tau_1^* = T\}}\Big\} = E\Big\{Y^2_{\tau_1^*} 1_{\{\tau_1^* \leq \tau\}}\Big\}. \end{aligned}$$

Then

$$\begin{aligned} \lim_{n\to\infty} J_2(\tau_{2n+1}, \tau) &= \lim_{n\to\infty} E\Big\{X^2_\tau 1_{\{\tau < \tau_{2n+1}\}} + Y^2_{\tau_{2n+1}} 1_{\{\tau_{2n+1} \leq \tau\}}\Big\} \\ &= E\Big\{X^2_\tau 1_{\{\tau < \tau_1^*\}} + Y^2_{\tau_1^*} 1_{\{\tau_1^* \leq \tau\}}\Big\} = J_2(\tau_1^*, \tau). \end{aligned}$$

The proof is complete. □

LEMMA 3.5. *Assume Assumptions* A1–A4 *hold. Then it holds that*

$$\lim_{n\to\infty} J_1(\tau_{2n+1},\tau_{2n}) = J_1(\tau_1^*,\tau_2^*); \quad \lim_{n\to\infty} J_2(\tau_{2n-1},\tau_{2n}) = J_2(\tau_1^*,\tau_2^*).$$

Proof. (i) We first show that

$$\lim_{n\to\infty} J_2(\tau_{2n-1},\tau_{2n}) = J_2(\tau_1^*,\tau_2^*). \tag{3.13}$$

Note that

$$J_2(\tau_{2n-1},\tau_{2n}) = E\Big\{\Big[X^2_{\tau_{2n}} 1_{\{\tau_{2n}<\tau_{2n-1}\}} + Y^2_{\tau_{2n-1}} 1_{\{\tau_{2n-1}\le\tau_{2n}\}}\Big]\Big[1_{\{\tau_1^*\neq\tau_2^*\}} + 1_{\{\tau_1^*=\tau_2^*\}}\Big]\Big\}.$$

Since $X^2, Y^2 \in \mathcal{D}$, sending $n\to\infty$ we have

$$\begin{aligned}
&\lim_{n\to\infty} J_2(\tau_{2n-1},\tau_{2n})\\
&= \lim_{n\to\infty} E\Big\{X^2_{\tau_2^*} 1_{\{\tau_2^*<\tau_1^*\}} + Y^2_{\tau_1^*} 1_{\{\tau_1^*<\tau_2^*\}} + \Big[X^2_{\tau_2^*} 1_{\{\tau_{2n}<\tau_{2n-1}\}} + Y^2_{\tau_1^*} 1_{\{\tau_{2n-1}\le\tau_{2n}\}}\Big] 1_{\{\tau_1^*=\tau_2^*\}}\Big\}\\
&= E\Big\{X^2_{\tau_2^*} 1_{\{\tau_2^*<\tau_1^*\}} + Y^2_{\tau_1^*} 1_{\{\tau_1^*\le\tau_2^*\}}\Big\} + I = J_2(\tau_1^*,\tau_2^*) + I,
\end{aligned} \tag{3.14}$$

where

$$I \triangleq \lim_{n\to\infty} E\Big\{\Big[X^2_{\tau_1^*} - Y^2_{\tau_1^*}\Big] 1_{\{\tau_{2n}<\tau_{2n-1},\tau_1^*=\tau_2^*\}}\Big\}. \tag{3.15}$$

On the other hand, set

$$\tau \triangleq \begin{cases} \tau_2^* & \text{if } \tau_2^* < \tau_1^*;\\ T & \text{if } \tau_2^* \ge \tau_1^*. \end{cases}$$

Then $\tau \in \mathcal{T}_0$ and $P(\tau = \tau_1^* < T) = 0$. By Lemma 3.4(ii) we have

$$\begin{aligned}
&\lim_{n\to\infty} J_2(\tau_{2n-1},\tau) = J_2(\tau_1^*,\tau) = E\Big\{X^2_\tau 1_{\{\tau<\tau_1^*\}} + Y^2_{\tau_1^*} 1_{\{\tau\ge\tau_1^*\}}\Big\}\\
&= E\Big\{X^2_{\tau_2^*} 1_{\{\tau_2^*<\tau_1^*\}} + Y^2_{\tau_1^*} 1_{\{\tau_1^*\le\tau_2^*\}}\Big\} = J_2(\tau_1^*,\tau_2^*).
\end{aligned}$$

By Lemma 3.3, we get $I \ge 0$. Now by Assumption A2 we have

$$I = 0. \tag{3.16}$$

Then (3.14) implies (3.13).

(ii) It remains to prove

$$\lim_{n\to\infty} J_1(\tau_{2n+1},\tau_{2n}) = J_1(\tau_1^*,\tau_2^*). \tag{3.17}$$

Similarly to (3.14) we have

$$\begin{aligned}
&\lim_{n\to\infty} J_1(\tau_{2n+1},\tau_{2n})\\
&= \lim_{n\to\infty} E\Big\{X^1_{\tau_1^*} 1_{\{\tau_1^*<\tau_2^*\}} + Y^1_{\tau_2^*} 1_{\{\tau_2^*<\tau_1^*\}} + \Big[X^1_{\tau_1^*} 1_{\{\tau_{2n+1}\le\tau_{2n}\}} + Y^1_{\tau_1^*} 1_{\{\tau_{2n+1}>\tau_{2n}\}}\Big] 1_{\{\tau_1^*=\tau_2^*\}}\Big\}\\
&= \lim_{n\to\infty} E\Big\{X^1_{\tau_1^*} 1_{\{\tau_1^*\le\tau_2^*\}} + Y^1_{\tau_2^*} 1_{\{\tau_2^*<\tau_1^*\}} + \Big[Y^1_{\tau_1^*} - X^1_{\tau_1^*}\Big] 1_{\{\tau_{2n+1}>\tau_{2n},\tau_1^*=\tau_2^*\}}\Big\}\\
&= J_1(\tau_1^*,\tau_2^*) + \lim_{n\to\infty} E\Big\{\Big[Y^1_{\tau_1^*} - X^1_{\tau_1^*}\Big] 1_{\{\tau_{2n+1}>\tau_{2n},\tau_1^*=\tau_2^*\}}\Big\}.
\end{aligned} \tag{3.18}$$

By Assumption A2, we get from (3.16) that

$$\lim_{n\to\infty} P\Big(X^2_{\tau_1^*} < Y^2_{\tau_1^*}, \tau_{2n+2} < \tau_{2n+1}, \tau_1^* = \tau_2^*\Big) = 0.$$

The third claim in (3.5) implies that $\{\tau_{2n} < \tau_{2n+1}\} \subset \{\tau_{2n+2} < \tau_{2n+1}\}$. Then by Assumption A3 we have

$$\lim_{n\to\infty} P\Big(X^1_{\tau_1^*} < Y^1_{\tau_1^*}, \tau_{2n} < \tau_{2n+1}, \tau_1^* = \tau_2^*\Big) = 0,$$

and therefore (3.18) leads to (3.17) immediately. ∎

We are now ready to show that (τ_1^*, τ_2^*) is an NEP.

Proof of Theorem 2.2. We recall again that Assumption A4 is used just for notational simplicity. So in the proof we may assume it.

First, by Lemmas 3.4(i), 3.5, and 3.3, we have

$$J_1(\tau, \tau_2^*) \le J_1(\tau_1^*, \tau_2^*) \quad \forall \tau \in \mathcal{T}_0. \tag{3.19}$$

Similarly, for any τ such that $P(\tau = \tau_1^* < T) = 0$, we have

$$J_2(\tau_1^*, \tau) \le J_2(\tau_1^*, \tau_2^*). \tag{3.20}$$

In the general case, for any $\tau \in \mathcal{T}_0$, set

$$\hat\tau_n \triangleq \begin{cases} [\tau + \frac{1}{n}] \wedge T & \text{if} \quad \tau = \tau_1^* < T; \\ \tau \quad \text{otherwise.} \end{cases}$$

Then $\hat\tau_n$ is a stopping time and $P(\hat\tau_n = \tau_1^* < T) = 0$. Thus (3.20) leads to

$$J_2(\tau_1^*, \hat\tau_n) \le J_2(\tau_1^*, \tau_2^*).$$

Sending $n \to \infty$, we obtain (3.20) for general τ.

Combining (3.19) and (3.20), we know (τ_1^*, τ_2^*) is an NEP. ∎

Remark 3.6. In the case when $X_2 = -Y_1$ and $Y_2 = -X_1$, then $J_1 + J_2 = 0$; i.e., we fall in the framework of the well-known zero-sum Dynkin game and then the NEP for the game is just a saddle point. Compared to the result by Lepeltier and Maingueneau [20], which is the most general paper on this subject known to date, our result provides a new construction method of the saddle point. Additionally it is obtained under fewer regularity conditions on the processes X_1 and X_2.

4. Application to game contingent claims. It is by now well known that an American contingent claim is a contract which allows its holder to exercise, i.e., to ask for the wealth, at a time she decides before or at the maturity. The only role of its issuer is to provide, if any, the pledged wealth to the buyer. In contrary, an American game contingent claim (ACC) is mainly an American contingent claim where the issuer is also allowed to recall/cancel the contract. Actually assume that a_1 (resp., a_2) is the issuer (resp., buyer) of the ACC. Both sides are allowed to exercise. Therefore it enables a_1 to terminate it and a_2 to exercise it at any time up to maturity date T when the contract expires anyway. Also if a_2 decides to exercise at σ or a_1 to terminate at τ, then a_1 pays to a_2 the amount

$$\Gamma(\tau, \sigma) = L_\sigma 1_{[\sigma \le \tau, \sigma < T]} + U_\tau 1_{[\tau < \sigma]} + \xi 1_{[\tau = \sigma = T]},$$

where the following hold:

- σ and τ are two $\mathbf{F}$-stopping times.
- L and U are $\mathbf{F}$-adapted continuous processes such that $L \leq U$. The quantity L_σ (resp., U_τ) is the amount that a_2 obtains (resp., a_1 pays) for her decision to exercise (resp., cancel) first at σ (resp., τ). The difference $U - L$ represents the compensation that a_1 pays to a_2 for the decision to terminate the contract before maturity date T.
- ξ is an $\mathcal{F}_T$-random variable which satisfies $L_T \leq \xi \leq U_T$. It stands for the money that a_1 pays to a_2 if both accept to terminate the game contingent claim (GCC) at maturity date T.

For this contingent claim, the seller a_1 (resp., buyer a_2) aims at maximizing (resp., minimizing) her cost (resp., reward) in expectation, i.e., the quantity

$$J(\tau, \sigma) := E[\Gamma(\tau, \sigma)],$$

where $E[.]$ is the expectation under the probability P on the space $(\Omega, \mathcal{F})$.

GCCs were introduced by Kifer in [17] in the framework of the Black–Scholes model. Since then, there have been several papers on the subject [3, 15, 16]. In a complete market, it is shown in those works that the nonarbitrage price V_0 of the GCC is equal to the zero-sum Dynkin game associated with L and U, i.e.,

$$V_0 = \sup_{\sigma \geq 0} \inf_{\tau \geq 0} J(\tau, \sigma) = \inf_{\tau \geq 0} \sup_{\sigma \geq 0} J(\tau, \sigma).$$

Another point of view for pricing American game options, especially in incomplete markets and in connection with the utility maximization approach, was introduced by Kuhn in [18] and is summed up as follows:

Let $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ be nondecreasing and concave functions. Those functions stand for utility functions of the seller (resp., the buyer) of the GCC. The seller a_1 (resp., the buyer a_2) chooses a stopping time τ (resp., σ) in order to maximize

$$J_1(\tau, \sigma) := E[\varphi_1(-\Gamma(\tau, \sigma))] \text{ (resp., } J_2(\tau, \sigma) := E[\varphi_2(\Gamma(\tau, \sigma))]).$$

Therefore if the nonzero-sum Dynkin game associated with J_1 and J_2 has an NEP (τ^*, σ^*), i.e.,

$$J_1(\tau^*, \sigma^*) \geq J_1(\tau, \sigma^*) \quad \text{and} \quad J_2(\tau^*, \sigma^*) \geq J_2(\tau^*, \sigma),$$

then $-\varphi_1^{-1}(J_1(\tau^*, \sigma^*))$ (resp., $\varphi_2^{-1}(J_2(\tau^*, \sigma^*))$) is the seller (resp., buyer) price of the GCC.

Note that if $\varphi_1(x) = \varphi_2(x) = x$ for all $x \in \mathbb{R}$, i.e., agents a_1 and a_2 are risk-neutral, then the nonzero-sum game is actually a zero-sum Dynkin game, (τ^*, σ^*) is a saddle point for this game, and $-J_1(\tau^*, \sigma^*) = J_2(\tau^*, \sigma^*)$. Moreover this latter quantity is the value of the game. For more details on zero-sum Dynkin games, one can see, e.g., [1, 5, 8, 15, 19, 20, 25, 26].

So pricing the GCC described above turns into the existence of an NEP for the associated nonzero-sum Dynkin game. In [18], based on the article by Morimoto [22], the author has just been able to show the existence of that NEP in the set of discrete stopping times and exponential utility functions. Also using the result of the previous section, we are able to fill in the gap between the discrete stopping times used in [18] and continuous ones which we use here and, on the other hand, to allow for arbitrary utility functions for the agents. Actually we have the following theorem.

THEOREM 4.1. *Assume that*

(i) *the utility functions* φ_1 *and* φ_2 *are, respectively, nondecreasing and increasing;*

(ii) $L_t \le U_t$ *and* $L_T \le \xi \le U_T$*, P-a.s.;*

(iii) *the processes* $\varphi_1(-L), \varphi_1(-U), \varphi_2(L), \varphi_2(U)$ *are in the space* $\mathcal{D}$*, and the random variables* $\varphi_1(-\xi)$ *and* $\varphi_2(\xi)$ *are square integrable;*

(iv) *the processes* $\varphi_1(-U)$ *and* $\varphi_2(L)$ *have only positive jumps.*

Then the nonzero-sum Dynkin game associated with the GCC has an NEP (τ^*, σ^*).

Proof. Define

$$X_t^1 \triangleq \varphi_1(-U_t)1_{\{t<T\}} + \varphi_1(-\xi)1_{\{t=T\}}, \quad X_t^2 \triangleq \varphi_2(L_t)1_{\{t<T\}} + \varphi_2(\xi)1_{\{t=T\}};$$
$$Y_t^1 \triangleq \varphi_1(-L_t)1_{\{t<T\}} + \varphi_1(-\xi)1_{\{t=T\}}, \quad Y_t^2 \triangleq \varphi_2(U_t)1_{\{t<T\}} + \varphi_2(\xi)1_{\{t=T\}}.$$

One can check straightforwardly that X^1, Y^1, X^2, Y^2 satisfy Assumptions A1–A4, and that the value functions $J_1(\tau, \sigma)$ and $J_2(\tau, \sigma)$ are the same as those defined in (2.1). Then by Theorem 2.2 we obtain the desired result. ∎

REFERENCES

[1] M. ALARIOT, J. P. LEPELTIER, AND B. MARCHAL, *Jeux de Dynkin*, in Proceedings of the 2nd Bad Honnef Workshop on Stochastic Processes, Lecture Notes in Control and Inform. Sci., Springer-Verlag, Berlin, 1982, pp. 23–32.

[2] L. H. R. ALVAREZ, *A class of solvable stopping games*, Appl. Math. Optim., 58 (2008), pp. 291–314.

[3] E. J. BAURDOUX AND A. E. KYPRIANOU, *Further calculations for Israeli options*, Stoch. Stoch. Rep., 76 (2004), pp. 549–569.

[4] A. BENSOUSSAN AND A. FRIEDMAN, *Nonzero-sum stochastic differential games with stopping times and free boundary value problem*, Trans. Amer. Math. Soc., 213 (1977), pp. 275–327.

[5] A. BENSOUSSAN AND A. FRIEDMAN, *Non-linear variational inequalities and differential games with stopping times*, J. Funct. Anal., 16 (1974), pp. 305–352.

[6] J. M. BISMUT, *Sur un problème de Dynkin*, Z. Wahrsch. Verw. Geb., 39 (1977), pp. 31–53.

[7] P. CATTIAUX AND J. P. LEPELTIER, *Existence of an quasi-Markov Nash equilibrium for nonzero sun Markov stopping games*, Stoch. Stoch. Rep., 30 (1990), pp. 85–103.

[8] J. CVITANIC AND I. KARATZAS, *Backward SDEs with reflection and Dynkin games*, Ann. Probab., 24 (1996), pp. 2024–2056.

[9] C. DELLACHERIE AND P. A. MEYER, *Probabilités et Potentiel*, Chapitres V–VIII, Hermann, Paris, 1980.

[10] E. B. DYNKIN, *A game-theoretic version of an optimal stopping problem*, Soviet Math. Dokl., 10 (1969), pp. 270–274.

[11] E. EKSTRÖM AND G. PESKIR, *Optimal stopping games for Markov processes*, SIAM J. Control Optim., 47 (2008), pp. 684–702.

[12] E. EKSTRÖM AND S. VILLENEUVE, *On the value of optimal stopping games*, Ann. Appl. Probab., 16 (2006), pp. 1576–1596.

[13] N. EL KAROUI, *Les aspects probabilistes du contrôle stochastique*, in Ecole d'été de probabilités de Saint-Flour, Lecture Notes in Math. 876, Springer-Verlag, Berlin, 1980, pp. 73–238.

[14] E. ETOURNEAU, *Résolution d'un problème de jeu de somme non nulle sur les temps d'arrêt*, Thèse de 3-ième cycle, Univ. Paris 6, 1986.

[15] S. HAMADÈNE, *Mixed zero-sum stochastic differential game and American game options*, SIAM J. Control Optim., 45 (2006), pp. 496–518.

[16] J. KALLSEN AND C. KUHN, *Pricing derivatives of American and game type in incomplete markets*, Finance Stoch., 8 (2004), pp. 261–284.

[17] Y. KIFER, *Game options*, Finance Stoch., 4 (2000), pp. 443–463.

[18] C. KUHN, *Game contingent claims in complete and incomplete markets*, J. Math. Econom., 40 (2004), pp. 889–902

[19] R. LARAKI AND E. SOLAN, *The value of zero-sum stopping games in continuous time*, SIAM J. Control Optim., 43 (2005), pp. 1913–1922.

[20] J. P. LEPELTIER, AND M. A. MAINGUENEAU, *Le Jeu de Dynkin en Théorie Générale Sans l'Hypothèse de Mokobodski*, Stochastics, 13 (1984), pp. 25–44.

[21] H. Morimoto, *Dynkin games and Martingale methods*, Stochastics, 13 (1984), pp. 213–228.
[22] H. Morimoto, *Non-zero-sum discrete parameter stochastic games with stopping times*, Probab. Theory Related Fields, 72 (1986), pp. 155–160.
[23] H. Nagai, *Non-zero-sum stopping games of symmetric Markov processes*, Probab. Theory Related Fields, 75 (1987), pp. 487–497.
[24] Y. Ohtsubo, *A nonzero-sum extension of Dynkins stopping problem*, Math. Oper. Res., 12 (1987), pp. 277–296.
[25] L. Stettner, *Zero-sum Markov games with stopping and impulsive strategies*, Appl. Math. Optim., 9 (1982/83), pp. 1–24.
[26] N. Touzi, and N. Vieille, *Continuous-time Dynkin games with mixed strategies*, SIAM J. Control Optim., 41 (2002), pp. 1073–1088.