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Stochastic Processes and their Applications 120 (2010) 403–426

stochastic processes and their applications

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Switching problem and related system of reflected backward SDEs☆

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Received 13 October 2008; received in revised form 6 January 2010; accepted 6 January 2010
Available online 14 January 2010

Abstract

This paper studies a system of backward stochastic differential equations with oblique reflections (RBSDEs for short), motivated by the switching problem under Knightian uncertainty and recursive utilities. The main feature of our system is that its components are interconnected through both the generators and the obstacles. We prove existence, uniqueness, and stability of the solution of the RBSDE, and give the expression of the price and the optimal strategy for the original switching problem via a verification theorem.

MSC: 60G40; 93E20; 62P20; 91B99

Keywords: Real options; Starting and stopping problem; Switching problem; Backward SDEs; Reflected BSDEs; Oblique reflection; Snell envelope; Optimal stopping problem

1. Introduction

This paper studies the wellposedness of a general system of Backward SDEs with oblique reflections, motivated by our study on switching problems.

☆ An earlier version of this paper was entitled *The Starting and Stopping Problem under Knightian Uncertainty and Related Systems of Reflected BSDEs*.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.01.003

The standard starting and stopping (or two modes switching) problem has attracted a lot of interest during the past decades; see a long list [1–3,6,9,7,8,12,11,17,21,20,28,31,33–35], and the references therein. Assume, for example, that a power plant produces electricity whose selling price fluctuates and depends on many factors such as consumer demand, oil prices, weather and so on. It is well known that electricity cannot be stored (or too expensive to store) and once produced it should be consumed almost immediately. Therefore electricity is produced only when there is enough profitability in the market. Otherwise the power plant is closed till the time when the profitability is coming back again. Then for this plant there are two modes: operating and closed. Accordingly, a management strategy of the plant is an increasing sequence of stopping times $\delta = (\tau_n)_{n\geq 0}$ with $\tau_0 \overset{\triangle}{=} 0$. At time τ_n, the manager switches the mode of the plant from its current one to the other. Such a switch of modes is not free and generates expenditures.

Suppose now that we have an adapted stochastic process X which stands for either the market electricity price or factors which determine the price. When the plant is run under a strategy δ, its yield is given by a quantity denoted by $J(\delta)$, which depends also on X and many other parameters such as utility functions, expenditures, etc. Therefore the main problem is to find an optimal management strategy $\delta^* = (\tau_n^*)_{n\geq 1}$ such that $J(\delta^*) = \sup_\delta J(\delta)$, and consequently, the value $J(\delta^*)$ is nothing but the fair price of the power plant in the energy market.

We note that this switching problem has also been used to model industries like copper or aluminium mines, ..., where parts of the production process are temporarily reduced or shut down when e.g. fuel, electricity or coal prices are too high to be profitable from running them. A further area of applications includes Tolling Agreements (see [4,6] for more details).

A natural extension of the two-mode problem is the multi-mode switching problem. This has been recently studied by several authors amongst we quote Carmona and Ludkovski [4], Djehiche et al. [10] and Porchet et al. [31].

The idea of using RBSDEs in starting and stopping problems was initiated by Hamadène and Jeanblanc [21]. In their model the two-dimensional RBSDE is linear and can be transformed into a one-dimensional RBSDE with double barriers. Then the wellposedness of the RBSDE is known in the literature. Moreover, via a verification theorem they express both the optimal strategy δ^* and the plant's value $J(\delta^*)$ in terms of the solution to the RBSDE. Several other papers have also used this tool (see *e.g.* [4,31]). In [4], the authors consider a multi-mode switching problem. However they left open the question of the existence of the solution of the associated RBSDEs with oblique reflection. The problem is solved by Djehiche et al. in [10].

Our first goal of the paper is to extend the work Hamadène and Jeanblanc [21] by considering Knightian uncertainty and recursive utility. All the works quoted above assume that future uncertainty of market conditions X is characterized by a certain probability measure P. The Knightian uncertainty introduced by Knight [27] assumes instead that the market evolves according to one of many possible probabilities P^u, $u \in \mathcal{U}$, but we do not know which one it is. The notion of ambiguity follows similar idea; see, e.g. [5]. The notion of recursive utilities was introduced by Duffie–Epstein [13,14]. These two new features lead to a nonlinear RBSDE with oblique reflections. There are only very few results on these kinds of RBSDEs in the literature; see e.g. [32].

We next consider a very general multi-dimensional RBSDE of which both the generators and the obstacles are interconnected. We prove the existence of solutions by using the notion of the smallest g-supermartingales introduced by Peng [29] and Peng and Xu [30]. This notion can be understood as a nonlinear version of the Snell envelope. We prove the uniqueness by a verification theorem. However, for our general case the optimal strategy does not exist, so one can only obtain approximately optimal strategies. This requires some sophisticated estimates and is in fact the main technical part of the paper. As an intermediary result we obtain some stability result for high-dimensional RBSDEs, which is interesting in its own rights.

The paper closest to ours is a recent work by Hu and Tang [26], which we learned after we finished the first version of this paper. They study RBSDEs with generator taking the form $f(t, y, z)$ and barrier $h(t, y) = y - c$ where c is a constant. Our model is more general in two aspects: (i) the generator takes the form $f(t, y_1, \ldots, y_m, z)$, that is, the utilities under different modes are interconnected in the generator; (ii) the barrier $h(t, y)$ is nonlinear and random. We think such a generalization is interesting from theoretical point of view. From applied point of view, the dependence of f on all the y-components can be interpreted as a nonzero-sum game problem, where the players' utilities affect each other and consequently the generators are interconnected. The general continuous time nonzero-sum game problem is much more difficult. We have some result on the existence of equilibrium in a different framework; see [24]. The general barrier h allows one to consider more general switching cost. In fact, even if the original cost takes the form $h(t, y) = y - c$, in some applications (especially in the case of risk-sensitive payoffs) one needs to take the standard exponential transformation and then the barrier becomes $\tilde{h}(t, \tilde{y}) \stackrel{\triangle}{=} e^{-c}\tilde{y}$; see, e.g. [23].

Hu–Tang [26] take the penalization approach to prove the existence. While their estimates are nice, their approach relies heavily on applying the Itô–Tanaka formula to $[Y_t^j - Y_t^i - c_{j,i}]^+$. It seems difficult to obtain similar estimates for $[Y_t^j - h_{j,i}(t, Y_t^i)]^+$ when general h is considered.

We also note that, in [21,26], the optimal strategy can be constructed explicitly and then the uniqueness follows immediately via the verification theorem; see Theorem 2.1. However, in our general case the optimal strategy may not exist, and thus some technical estimates have to be involved in order to prove the uniqueness of solutions.

The rest of the paper is organized as follows. In the next section we introduce the switching problem, review the works of Hamadène–Jeanblanc [21] and Hu–Tang [26], and introduce our general RBSDE. In Section 3 we prove the existence of solutions, and finally in Section 4 we prove the uniqueness. ■

2. The switching problem and reflected BSDEs

In this section we review some results in the literature and introduce our general RBSDE.

2.1. The switching problem and some existing results

We start with reviewing the work Hamadène–Jeanblanc [21]. Let $(\Omega, \mathcal{F}, P)$ denote a fixed complete probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0 \le t \le T}$, and $\mathbf{F} \stackrel{\triangle}{=} (\mathcal{F}_t)_{0 \le t \le T}$ be the filtration generated by B and augmented by all the P-null sets. Throughout this paper we assume all the processes are progressively measurable and $\mathbf{F}$-adapted. Furthermore, we let:

- $\mathcal{H}$ be the space of processes η such that $E[\int_0^T |\eta_s|^2 ds] < \infty$;
- $\mathcal{S}$ be the space ofcàdlàg processes η such that $E[\sup_{0 \le t \le T} |\eta_t|^2] < \infty$;
- $\mathcal{S}_c$ be the subspace of $\mathcal{S}$ with continuous elements;
- $\mathcal{A}$ be the space of càdlàg and non-decreasing scalar processes η with $\eta_0 = 0$ and $E[\eta_T^2] < \infty$;
- $\mathcal{A}_c$ be the subspace of $\mathcal{A}$ with continuous elements;

Let us now fix the data of the problem.

- Let $X \in \mathcal{S}_c$ with dimension k which stands for the factors determining the market electricity price.

- Let $\psi_i : [0, T] \times \mathbb{R}^k \mapsto \mathbb{R}$, $i = 1, 2$, be Borelean functions with linear growth in x which represent the rate of utilities for the power plant when it is in its operating and closed modes, respectively.

- Let c_1 (resp. c_2) be a positive constant which represents the sunk cost when the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one.

Let $\mathcal{D}$ denote the set of all admissible strategies $\delta = (\tau_n)_{n\geq 0}$ such that τ_n's are an increasing sequence of $\mathbf{F}$-stopping times with $\tau_0 = 0$ and $\lim_{n\to\infty} \tau_n = T$, P-*a.s.* Assume for convenience that the power plant is in its operating mode at the initial time $t = 0$. Then τ_{2n+1} (resp. τ_{2n}) are the times where the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one.

Under strategy $\delta \in \mathcal{D}$, the mean yield of the power plant is given by:

$$J(\delta) \triangleq E\left\{\int_0^T \psi^\delta(t, X_t)\mathrm{d}t - A_T^\delta\right\},$$

where

$$\begin{cases} \psi^\delta(t,x) \triangleq \sum_{n\geq 0}\Big[\psi_1(t,x)\mathbb{1}_{[\tau_{2n},\tau_{2n+1})}(t) + \psi_2(t,x)\mathbb{1}_{[\tau_{2n+1},\tau_{2n+2})}(t)\Big]; \\ A_t^\delta \triangleq \sum_{n\geq 0}\Big[c_1\mathbb{1}_{\{\tau_{2n+1}<t\}} + c_2\mathbb{1}_{\{\tau_{2n+2}<t\}}\Big]. \end{cases} \tag{2.1}$$

Therefore the price of the power plant in the energy market is just $\sup_{\delta\in\mathcal{D}} J(\delta)$.

As showed in [21], the above problem is closed related to the following two-dimensional RBSDEs with linear generator and oblique reflections:

$$\begin{cases} Y^1, Y^2 \in \mathcal{S}_c, \qquad Z^1, Z^2 \in \mathcal{H} \quad \text{and} \quad K^1, K^2 \in \mathcal{A}_c, \\ Y_t^i = \int_t^T \psi_i(s, X_s)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}B_s + K_T^i - K_t^i, \quad i = 1, 2; \\ Y_t^1 \geq Y_t^2 - c_1; \quad [Y_t^1 - Y_t^2 + c_1]\mathrm{d}K_t^1 = 0; \\ Y_t^2 \geq Y_t^1 - c_2; \quad [Y_t^2 - Y_t^1 + c_2]\mathrm{d}K_t^2 = 0. \end{cases} \tag{2.2}$$

Here Y_t^1 (resp. Y_t^2) stands for the optimal utility at time t if the mode at that time is operating (resp. closed).

We note that, $\Delta Y \triangleq Y^1 - Y^2$, $\Delta Z \triangleq Z^1 - Z^2$ satisfy the following RBSDE with double reflections, which has been studied by many authors (see, e.g. [19,22,30]):

$$\begin{cases} \Delta Y_t = \int_t^T \Big[\psi_1(s, X_s) - \psi_2(s, X_s)\Big]\mathrm{d}s \\ \quad - \int_t^T \Delta Z_s \mathrm{d}B_s + (K_T^1 - K_t^1) - (K_T^2 - K_t^2); \\ -c_1 \leq \Delta Y_t \leq c_2, \quad [\Delta Y_t + c_1]\mathrm{d}K_t^1 = [\Delta Y_t - c_2]\mathrm{d}K_t^2 = 0. \end{cases} \tag{2.3}$$

Then the wellposedness of (2.2) follows immediately.

Moreover, [21] obtains the following important verification theorem.

Theorem 2.1. $\sup_{\delta\in\mathcal{D}} J(\delta) = Y_0^1$ *and the optimal strategy* $\delta^* \in \mathcal{D}_2$ *is given by* $\tau_0^* \stackrel{\triangle}{=} 0$ *and,*

$$\begin{aligned} \tau^*_{2n+1} &\stackrel{\triangle}{=} \inf\{t \geq \tau^*_{2n} : Y_t^1 = Y_t^2 - c_1\} \wedge T; \\ \tau^*_{2n+2} &\stackrel{\triangle}{=} \inf\{t \geq \tau^*_{2n+1} : Y_t^2 = Y_t^1 - c_2\} \wedge T; \end{aligned} \qquad n = 0, 1, \ldots$$

The above work can be naturally extended to switching problems under Knightian uncertainty and recursive utilities. We refer to Knight [27] and Chen–Epstein [5] for Knight uncertainty and Duffie–Epstein [13,14] for recursive utilities. We note that such an extension has been carried out independently by a recent work Hu–Tang [26], which we learned after we finished the first version of this paper.

Mathematically, this amounts to solving the following BSDE with some nonlinear generator H^* and oblique reflections:

$$\begin{cases} Y^1, Y^2 \in \mathcal{S}_c, \qquad Z^1, Z^2 \in \mathcal{H} \quad \text{and} \quad K^1, K^2 \in \mathcal{A}_c, \\ Y_t^i = \displaystyle\int_t^T [\psi_i(s, X_s) + H^*(s, X., Y_s^i, Z_s^i)]\mathrm{d}s \\ \qquad - \displaystyle\int_t^T Z_s^i \mathrm{d}B_s + K_T^i - K_t^i, \quad i = 1, 2; \\ Y_t^1 \geq Y_t^2 - c_1; \quad [Y_t^1 - Y_t^2 + c_1]\mathrm{d}K_t^1 = 0; \\ Y_t^2 \geq Y_t^1 - c_2; \quad [Y_t^2 - Y_t^1 + c_2]\mathrm{d}K_t^2 = 0. \end{cases} \tag{2.4}$$

Note that one cannot transform (2.4) into a one-dimensional RBSDE with double barriers. Hu–Tang [26] establishes the wellposedness of the following RBSDE (after some obvious transformation) in higher-dimensional case:

$$\begin{cases} Y^j \in \mathcal{S}_c, \qquad Z^j \in \mathcal{H} \quad \text{and} \quad K^j \in \mathcal{A}_c, \\ Y_t^j = \xi_j + \displaystyle\int_t^T f_j(s, \omega, Y_s^j, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}B_s + K_T^j - K_t^j; \qquad j = 1, \ldots, m. \\ Y_t^j \geq \max\limits_{i\neq j}(Y_t^i - c_{j,i}); \quad [Y_t^j - \max\limits_{i\neq j}(Y_t^i - c_{j,i})]\mathrm{d}K_t^j = 0. \end{cases} \tag{2.5}$$

Here $\xi_j \in L^2(\mathcal{F}_T)$, f_j satisfies the standard measurability and Lipschitz conditions, and $c_{j,i}$ are constants satisfying

$$c_{j,i} \geq 0 \quad \text{and} \quad c_{k,j} + c_{j,i} > c_{k,i}. \tag{2.6}$$

They prove the existence of the solution by penalization approach, and the uniqueness by a verification theorem which in the meantime provides the optimal strategy δ^*, in the spirit of Theorem 2.1.

2.2. *The general BSDEs with oblique reflection*

In this paper we extend the RBSDEs (2.2) and (2.5) to the following general m-dimensional RBSDEs with oblique reflections for some $m \geq 2$: for $j = 1, \ldots, m$,

$$\begin{cases} Y^j \in \mathcal{S}_c, \qquad Z^j \in \mathcal{H} \quad \text{and} \quad K^j \in \mathcal{A}_c, \\ Y_t^j = \xi_j + \displaystyle\int_t^T f_j(s, Y_s^1, \ldots, Y_s^m, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}B_s + K_T^j - K_t^j; \\ Y_t^j \geq \max\limits_{i\in A_j} h_{j,i}(t, Y_t^i); \quad [Y_t^j - \max\limits_{i\in A_j} h_{j,i}(t, Y_t^i)]\mathrm{d}K_t^j = 0. \end{cases} \tag{2.7}$$

Here ξ_j are $\mathcal{F}_T$-measurable, the coefficients $f_j, h_{j,i}$ can depend on ω, and $A_j \subset \{1, \ldots, m\} - \{j\}$. For simplicity we denote $\overrightarrow{Y_t} \triangleq (Y_t^1, \ldots, Y_t^m)$, and similarly for other vectors. The constraint A_j means that from mode j the plant can only be switched to those modes in A_j. We emphasize that A_j can be empty and if so we take the convention that the maximum over the empty set, denoted as $\emptyset$, is $-\infty$. Then in this case Y^j has no lower barrier and then we take $K^j = 0$. Consequently, Y^j satisfies the following BSDE without reflection:

$$Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}B_s, \quad 0 \le t \le T.$$

Also, for any j we define

$$h_{j,j}(t, y) \triangleq y. \tag{2.8}$$

Then a solution of (2.7) always satisfies

$$Y_t^j \ge \max_{i \in A_j \cup \{j\}} h_{j,i}(t, Y_t^i). \tag{2.9}$$

We note that our work is done independently of Hu–Tang [26]. Our RBSDE (2.7) is not the same as the one of (2.5) in two aspects: (i) the components $Y^1, \ldots, Y^m$ are interconnected in the generators f_j; (ii) the barriers $h_{j,i}$ can be random and nonlinear. We think such a generalization is interesting from theoretical point of view. From applied point of view, the dependence of f_j on $\overrightarrow{Y}$ can be interpreted as a nonzero-sum game problem, where the players' utilities affect each other and consequently the generators are interconnected. The general barrier $h_{j,i}$ allows one to consider more general switching cost. In fact, even if the original cost takes the form $h_{j,i}(t, y) = y - c_{j,i}$, in the risk-sensitive switching problem (see [23]) one needs to take the standard exponential transformation and then the barrier becomes $\tilde{h}_{j,i}(t, \tilde{y}) \triangleq e^{-c_{j,i}} \tilde{y}$. ■

3. Existence

To prove the existence of solutions, we use the notion of the smallest g-supermartingales introduced by Peng [29] and Peng and Xu [30], which can be understood as a nonlinear version of the Snell envelope (see e.g. [15]).

Throughout this section we shall adopt the following assumptions.

Assumption 3.1. For any $j = 1, \ldots, m$, it holds that:

(i) $E\left\{\int_0^T \sup_{\overrightarrow{y}: y_j = 0} |f_j(t, \overrightarrow{y}, 0)|^2 \mathrm{d}t + |\xi_j|^2\right\} < \infty$.

(ii) $f_j(t, \overrightarrow{y}, z)$ is uniformly Lipschitz continuous in (y_j, z) and is continuous in y_i for any $i \ne j$; and $h_{j,i}(t, y)$ is continuous in (t, y) for $i \in A_j$.

(iii) $f_j(t, \overrightarrow{y}, z)$ is increasing in y_i for $i \ne j$, and $h_{j,i}(t, y)$ is increasing in y for $i \in A_j$.

(iv) For $i \in A_j$, $h_{j,i}(t, y) \le y$. Moreover, there is no sequence $j_2 \in A_{j_1}, \ldots, j_k \in A_{j_{k-1}}, j_1 \in A_{j_k}$, and $(y_1, \ldots, y_k)$ such that $y_1 \triangleq h_{j_1,j_2}(t, y_2)$, $y_2 \triangleq h_{j_2,j_3}(t, y_3)$, $\ldots$, $y_{k-1} \triangleq h_{j_{k-1},j_k}(t, y_k)$, $y_k \triangleq h_{j_k,j_1}(t, y_1)$.

(v) For any $j = 1, \ldots, m$, $\xi_j \ge \max_{i \in A_j} h_{j,i}(T, \xi_i)$. ■

We note that (i), (ii) and (v) are standard; and (iii) implies the m players are "partners". The assumption (iv) means that it is not free to make a circle of instantaneous switchings. This is satisfied, for example, in [26] under condition (2.6).

Our main result of this section is:

Theorem 3.2. *Assume Assumption 3.1 holds. Then RBSDE (2.7) has a solution.*

Proof. We shall use Picard iteration, and proceed in five steps.

Step 1. We first construct the Picard iterations. Denote:

$$\underline{f}_j(t,y,z) \triangleq \inf_{\overrightarrow{y}:y_j=y} f_j(t,\overrightarrow{y},z) \quad \text{and} \quad \bar{f}_j(t,y,z) \triangleq \sup_{\overrightarrow{y}:y_j=y} f_j(t,\overrightarrow{y},z).$$

By Assumption 3.1(i) and (ii), $\underline{f}_j$, $\bar{f}_j$ are uniformly Lipschitz continuous in (y,z) and

$$E\left\{\int_0^T [|\underline{f}_j(t,0,0)|^2 + |\bar{f}_j(t,0,0)|^2]\mathrm{d}t\right\} < \infty.$$

Let $(Y^{j,0}, Z^{j,0})$ be the solution to the following BSDE without reflection:

$$Y_t^{j,0} = \xi_j + \int_t^T \underline{f}_j(s, Y_s^{j,0}, Z_s^{j,0})\mathrm{d}s - \int_t^T Z_s^{j,0}\mathrm{d}B_s, \quad j=1,\ldots,m. \tag{3.1}$$

For $j=1,\ldots,m$ and $n=1,2,\ldots$, recursively define $Y^{j,n}$ via the following RBSDEs whose solution exists thanks to the result by El-Karoui et al. [16]:

$$\begin{cases} Y_t^{j,n} = \xi_j - \int_t^T Z_s^{j,n}\mathrm{d}B_s + K_T^{j,n} - K_t^{j,n} \\ \quad + \int_t^T f_j(s, Y_s^{1,n-1},\ldots,Y_s^{j-1,n-1}, Y_s^{j,n}, Y_s^{j+1,n-1},\ldots,Y_s^{m,n-1}, Z_s^{j,n})\mathrm{d}s; \\ Y_t^{j,n} \geq \max_{i\in A_j} h_{j,i}(t, Y_t^{i,n-1}); \quad [Y_t^{j,n} - \max_{i\in A_j} h_{j,i}(t,Y_t^{i,n-1})]\mathrm{d}K_t^{j,n} = 0. \end{cases} \tag{3.2}$$

Note that, given $Y^{i,n-1}, i=1,\ldots,m$, for each j (3.2) is a one-dimensional BSDE (when $A_j = \phi$) or RBSDE. Under Assumption 3.1, (3.2) has a unique solution. Moreover, by comparison theorem (see e.g. [16], Theorem 4.1) it is obvious that $Y^{j,1} \geq Y^{j,0}$. Then by induction one can easily show that $Y^{j,n}$ is increasing as n increases.

Step 2. We show that

$$E\left\{\sup_{0\leq t\leq T} |Y_t^{j,n}|^2 + \int_0^T |Z_t^{j,n}|^2\mathrm{d}t + |K_T^{j,n}|^2\right\} \leq C, \quad \forall j,n. \tag{3.3}$$

To this end, denote:

$$\breve{\xi} \triangleq \sum_{j=1}^m |\xi_j| \quad \text{and} \quad \breve{f}(t,y,z) \triangleq \sum_{j=1}^m |\bar{f}_j(t,y,z)|,$$

and let $(\breve{Y}, \breve{Z})$ be the solution to the following BSDE:

$$\breve{Y}_t = \breve{\xi} + \int_t^T \breve{f}(s,\breve{Y}_s,\breve{Z}_s)\mathrm{d}s - \int_t^T \breve{Z}_s\mathrm{d}B_s.$$

Denote, for $j=1,\ldots,m$,

$$\bar{Y}_t^j \triangleq \breve{Y}_t, \qquad \bar{Z}_t^j \triangleq \breve{Z}_t, \qquad \bar{K}_t^j \triangleq 0.$$

Obviously $Y_t^{j,0} \le \bar{Y}_t^j$. By Assumption 3.1(iv) we know that $(\bar{Y}^j, \bar{Z}^j, \bar{K}^j)$ satisfies

$$\begin{cases} \bar{Y}_t^j = \breve{\xi} + \int_t^T \breve{f}(s, \bar{Y}_s^j, \bar{Z}_s^j) - \int_t^T \bar{Z}_s^j \mathrm{d}B_s + \bar{K}_T^j - \bar{K}_t^j; \\ \bar{Y}_t^j \ge \max_{i\in A_j} h_{j,i}(t, \bar{Y}_t^i); \quad [\bar{Y}_t^j - \max_{i\in A_j} h_{j,i}(t, \bar{Y}_t^i)]\mathrm{d}\bar{K}_t^j = 0. \end{cases}$$

Once more apply the comparison theorem repeatedly, we get

$$Y_t^{j,n} \le \breve{Y}_t, \quad \forall n.$$

Recall that $Y_t^{j,n} \ge Y_t^{j,0}$. Then

$$\sum_{j=1}^m E\left\{\sup_{0\le t\le T} |Y_t^{j,n}|^2\right\} \le C < \infty, \quad \forall n. \tag{3.4}$$

Moreover,

$$E\left\{\sup_{0\le t\le T} |[\max_{i\in A_j} h_{j,i}(t, Y_t^{i,n-1})]^+|^2\right\} \le E\left\{\sup_{0\le t\le T} |[\max_{i\in A_j} Y_t^{i,n-1}]^+|^2\right\} \le C.$$

This, together with (3.4) and applying the results in [16], proves (3.3).

Step 3. Now let Y^j denote the limit of $Y^{j,n}$. By Peng's monotonic limit theorem [29] or [30], we know that Y^j is an càdlàg process, and following similar arguments there one can easily show that there exist $Z^j \in \mathcal{H}$ and $K^j \in \mathcal{A}$ such that

$$\begin{cases} Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}B_s + K_T^j - K_t^j; \\ Y_t^j \ge \max_{i\in A_j} h_{j,i}(t, Y_t^i). \end{cases} \tag{3.5}$$

Consider now the following RBSDEs whose solution exists thanks to the result by Hamadène [18] or Peng and Xu [30]:

$$\begin{cases} \tilde{Y}^j \in \mathcal{S}, \qquad \tilde{Z}^j \in \mathcal{H} \quad \text{and} \quad \tilde{K}^j \in \mathcal{A}; \\ \tilde{Y}_t^j = \xi_j - \int_t^T \tilde{Z}_s^j \mathrm{d}B_s + \tilde{K}_T^j - \tilde{K}_t^j \\ \quad + \int_t^T f_j(s, Y_s^1, \ldots, Y_s^{j-1}, \tilde{Y}_s^j, Y_s^{j+1}, \ldots, Y_s^m, \tilde{Z}_s^j)\mathrm{d}s; \\ \tilde{Y}_t^j \ge \max_{i\in A_j} h_{j,i}(t, Y_t^i); \qquad [\tilde{Y}_{t-}^j - \max_{i\in A_j} h_{j,i}(t, Y_{t-}^i)]\mathrm{d}\tilde{K}_t^j = 0. \end{cases} \tag{3.6}$$

We note that (3.5) and (3.6) have the same lower barrier. Since $\tilde{Y}^j$ is the smallest f_j-supermartingale with lower barrier $\max_{i\in A_j} h_{j,i}(t, Y_t^i)$, we have $\tilde{Y}_t^j \le Y_t^j$ (see [30], Theorem 2.1). On the other hand, since $Y_t^{i,n-1} \le Y_t^i$ for any $(i, n-1)$, by the monotonicity of $h_{j,i}$ we get

$$\max_{i\in A_j} h_{j,i}(t, Y_t^{i,n-1}) \le \max_{i\in A_j} h_{j,i}(t, Y_t^i).$$

Then once more by comparison theorem for RBSDEs we have $Y_t^{j,n} \le \tilde{Y}_t^j$, which implies that $Y_t^j \le \tilde{Y}_t^j$. Therefore, $\tilde{Y}_t^j = Y_t^j$. This further implies that $\tilde{Z}_t^j = Z_t^j$, $\mathrm{d}t \otimes \mathrm{d}P$-a.s., $\tilde{K}_t^j = K_t^j$ for any $0 \le t \le T$, P-a.s., and that

$$\begin{cases} Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}B_s + K_T^j - K_t^j; \\ Y_t^j \ge \max_{i \in A_j} h_{j,i}(t, Y_t^i), \qquad [Y_{t-}^j - \max_{i \in A_j} h_{j,i}(t, Y_{t-}^i)]\mathrm{d}K_t^j = 0. \end{cases} \tag{3.7}$$

Step 4. We show that Y^j is continuous. This obviously implies that K^j is also continuous and thus $(\overrightarrow{Y}, \overrightarrow{Z}, \overrightarrow{K})$ is a solution to (2.7).

We first note that, by (3.7), $\Delta Y_t^j = -\Delta K_t^j \le 0$, and if $\Delta K_t^j \neq 0$, then $Y_{t-}^j = \max_{i \in A_j} h_{j,i}(t, Y_{t-}^i)$. It is obvious that Y^j is continuous when $A_j = \emptyset$. We now assume $\Delta Y_t^{j_1} \neq 0$ for some j_1 and t. Then $A_{j_1} \neq \emptyset$ and $\Delta Y_t^{j_1} < 0$. Note that in this case $\Delta K_t^{j_1} > 0$, which further implies that

$$Y_{t-}^{j_1} = \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_{t-}^i).$$

Let $j_2 \in A_{j_1}$ be the optimal index, then

$$h_{j_1,j_2}(t, Y_{t-}^{j_2}) = Y_{t-}^{j_1} > Y_t^{j_1} \ge \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_t^i) \ge h_{j_1,j_2}(t, Y_t^{j_2}).$$

Thus $\Delta Y_t^{j_2} < 0$, and therefore $A_{j_2} \neq \emptyset$. Repeat the arguments we obtain $j_k \in A_{j_{k-1}}$ and $\Delta Y_t^{j_k} < 0$ for any k. Since each j_k can take only values $1, \ldots, m$, we may assume, without loss of generality that $j_1 = j_{k+1}$ for some $k \ge 2$ (note again that $j_1 \notin A_{j_1}$ and thus $j_2 \neq j_1$). Then we have

$$Y_{t-}^{j_1} = h_{j_1,j_2}(t, Y_{t-}^{j_2}), \ldots, Y_{t-}^{j_{k-1}} = h_{j_{k-1},j_k}(t, Y_{t-}^{j_k}), \quad Y_{t-}^{j_k} = h_{j_k,j_1}(t, Y_{t-}^{j_1}).$$

This contradicts with Assumption 3.1(iv). Therefore, all processes Y^j are continuous.

Step 5. Finally, as a by-product we show that, for $j = 1, \ldots, m$,

$$\lim_{n\to\infty} E\left\{\sup_{0\le t\le T}[|Y_t^{j,n} - Y_t^j|^2 + |K_t^{j,n} - K_t^j|^2] + \int_0^T |Z_t^{j,n} - Z_t^j|^2 \mathrm{d}t\right\} = 0. \tag{3.8}$$

In fact, since Y^j is continuous and $Y^{j,n} \uparrow Y^j$, by Dini's Theorem we know that

$$\lim_{n\to\infty} \sup_{0\le t\le T} |Y_t^{j,n} - Y_t^j| = 0, \quad a.s.$$

Applying Dominated Convergence Theorem we prove the convergence of $Y^{j,n}$ in (3.8). Now by standard arguments, see e.g. [16], one can prove (3.8). ■

By applying comparison theorem repeatedly, the following two results are direct consequences of Theorem 3.2, and their proofs are omitted.

Corollary 3.3. *The solution $\overrightarrow{Y}$ constructed in Theorem 3.2 is the minimum solution of (2.7). That is, if $\overrightarrow{\tilde{Y}}$ is another solution of (2.7), then $Y_t^j \le \tilde{Y}_t^j$, $j = 1, \ldots, m$.*

Corollary 3.4. *Assume $(\tilde{\xi}_j, \tilde{f}_j)$ also satisfy Assumption 3.1, and*

$$f_j \le \tilde{f}_j, \qquad \xi_j \le \tilde{\xi}_j.$$

Let $\overrightarrow{Y}$ *and* $\overrightarrow{\tilde{Y}}$ *denote the solution of* (2.7) *constructed in* Theorem 3.2*, with coefficients* $(\xi_j, f_j, h_{j,i})$ *and* $(\tilde{\xi}_j, \tilde{f}_j, h_{j,i})$*, respectively. Then* $Y_t^j \le \tilde{Y}_t^j$, $j = 1, \ldots, m$.

We also have the convergence of the penalized BSDEs, which is obtained by Hu and Tang [26] in their case using a different approach.

Theorem 3.5. *Assume* Assumption 3.1 *holds, and* $(\overrightarrow{Y}, \overrightarrow{Z}, \overrightarrow{K})$ *denote the solution of* (2.7) *constructed in* Theorem 3.2*. Let* $(\overrightarrow{Y^n}, \overrightarrow{Z^n})$ *denote the solutions of the following penalized BSDEs without reflection:*

$$Y_t^{n,j} = \xi_j + \int_t^T f_j(s, \overrightarrow{Y_s^n}, Z_s^{n,j})\mathrm{d}s + n\int_t^T [Y_s^{n,j} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i})]^- \mathrm{d}s - \int_t^T Z_s^{n,j}\mathrm{d}B_s. \quad (3.9)$$

Then $Y^{n,j}$ *is increasing in* n *and*

$$\lim_{n\to\infty} E\left\{\sup_{0\le t\le T}[|Y_t^{n,j} - Y_t^j|^2 + |K_t^{n,j} - K_t^j|^2] + \int_0^T |Z_t^{n,j} - Z_t^j|^2\mathrm{d}t\right\} = 0, \quad (3.10)$$

where

$$K_t^{n,j} \triangleq n\int_0^t [Y_s^{n,j} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i})]^- \mathrm{d}s.$$

Proof. The proof is similar to Theorem 3.2, we thus only introduce the main idea and leave the details to the interested readers.

First, it is obvious that the BSDEs (3.9) have a unique solution. Define $Y_t^{n,j,0} \triangleq Y_t^{j,0}$, and for $k = 0, 1, \ldots$, recursively define

$$Y_t^{n,j,k+1} = \xi_j + \int_t^T f_j(s, Y_s^{n,1,k}, \ldots, Y_s^{n,j-1,k}, Y_s^{n,j,k+1}, Y_s^{n,j+1,k}, \ldots, Y_s^{n,m,k}, Z_s^{n,j})\mathrm{d}s + n\int_t^T [Y_s^{n,j,k+1} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i,k})]^- \mathrm{d}s - \int_t^T Z_s^{n,j,k+1}\mathrm{d}B_s.$$

By standard arguments in BSDE theory one can easily see that

$$\lim_{k\to\infty} E\left\{\sup_{0\le t\le T}|Y_t^{n,j,k} - Y_t^{n,j}|^2 + \int_0^T |Z_t^{n,j,k} - Z_t^{n,j}|^2\right\} = 0.$$

Moreover, by comparison theorem we have $Y^{n,j,k}$ is increasing in n. Thus $Y^{n,j}$ is increasing in n. We note that one can also use the comparison theorem for high-dimensional BSDEs, see [25], to prove the monotonicity of $Y^{n,j}$. Let $\tilde{Y}^j$ denote the limit of $Y^{n,j}$ as $n \to \infty$. By induction one can show that $Y^{n,j,k} \le Y^{j,k}$ for any (n, j, k). Then $Y^{n,j} \le Y^j$ and thus $\tilde{Y}^j \le Y^j$. Now apply the results in [30] and the arguments in Theorem 3.2, we can prove $\tilde{Y}^j = Y^j$ and (3.10). ■

Another by-product of Theorem 3.2 is the existence of a solution of the system (2.7) considered between two stopping times. This result is in particular useful to show uniqueness of (2.7) in the next section.

To be precise, let λ_1 and λ_2 be two stopping times such that P-*a.s.*, $0 \le \lambda_1 \le \lambda_2 \le T$ and let us consider the following RBSDE over $[\lambda_1, \lambda_2]$: for $j = 1, \dots, m$, P-*a.s.*,

$$\begin{cases} (Y_t^j)_{t\in[\lambda_1,\lambda_2]} \text{ continuous, } (K_t^j)_{t\in[\lambda_1,\lambda_2]} \text{ continuous and non-decreasing,} \\ K_{\lambda_1}^j = 0, \quad \text{and} \quad E\left\{ \sup_{t\in[\lambda_1,\lambda_2]} |Y_t^j|^2 + \int_{\lambda_1}^{\lambda_2} |Z_s^j|^2 ds + (K_{\lambda_2}^j)^2 \right\} < \infty; \\ Y_t^j = \xi_{\lambda_2}^j + \int_t^{\lambda_2} f_j(s, \overrightarrow{Y}_s, Z_s^j) ds - \int_t^{\lambda_2} Z_s^j dB_s + K_{\lambda_2}^j - K_t^j, \quad \forall t \in [\lambda_1, \lambda_2]; \\ Y_t^j \ge \max_{i\in A_j} h_{j,i}(t, Y_t^i) \quad \text{and} \quad [Y_t^j - \max_{i\in A_j} h_{j,i}(t, Y_t^i)] dK_t^j = 0, \quad \forall t \in [\lambda_1, \lambda_2]. \end{cases} \tag{3.11}$$

Then we have:

Theorem 3.6. *Assume Assumption 3.1 holds and that, for $j = 1, \dots, m$, $\xi_{\lambda_2}^j \in \mathcal{F}_{\lambda_2}$ and satisfies:*

$$E\{|\xi_{\lambda_2}^j|^2\} < \infty \quad \textit{and} \quad \xi_{\lambda_2}^j \ge \max_{i\in A_j} h_{j,i}(\lambda_2, \xi_{\lambda_2}^i). \tag{3.12}$$

Then the RBSDE (3.11) has a solution. ■

4. Uniqueness

We now focus on uniqueness of the solution of RBSDE (3.11), hence that of RBSDE (2.7). To do that we need a stronger assumption.

Assumption 4.1. (i) f_j is uniformly Lipschitz continuous in all y_i.
(ii) If $i \in A_j$, $k \in A_i$, then $k \in A_j \cup \{j\}$. Moreover,

$$h_{j,i}(t, h_{i,k}(t, y)) < h_{j,k}(t, y). \tag{4.1}$$

(iii) For any $i \in A_j$,

$$|h_{j,i}(t, y_1) - h_{j,i}(t, y_2)| \le |y_1 - y_2|. \tag{4.2}$$

Note again that these assumptions are satisfied if $A_j = \{1, \dots, m\} - \{j\}$ for any $j = 1, \dots, m$ and $h_{j,i}(\omega, t, y) = y - c_{j,i}$ under condition (2.6), as in [26].

Our main result of this section is the following theorem.

Theorem 4.2 (*Uniqueness*).

(i) *Assume Assumptions 3.1 and 4.1 are in force, and $\xi_{\lambda_2}^j$ satisfies (3.12). Then the solution of RBSDE (3.11) is unique.*

(ii) *Moreover, assume for $j = 1, \dots, m$, $\tilde{f}_j$ satisfies Assumptions 3.1 and 4.1, and $\tilde{\xi}_{\lambda_2}^j$ satisfies (3.12). Let $(\tilde{Y}^j, \tilde{Z}^j)$ be the solution to RBSDE (3.11) corresponding to $(\tilde{f}_j, \tilde{\xi}_{\lambda_2}^j)$. For $j = 1, \dots, m$, denote,*

$$\begin{aligned} &\Delta Y_t^j \stackrel{\triangle}{=} Y_t^j - \tilde{Y}_t^j, \qquad \Delta\xi_{\lambda_2}^j \stackrel{\triangle}{=} \xi_{\lambda_2}^j - \tilde{\xi}_{\lambda_2}^j, \\ &\|\Delta f_t\| \stackrel{\triangle}{=} \sum_{j=1}^m \operatorname*{esssup}_{(\overrightarrow{y}, z)} |[f_j - \tilde{f}_j](t, \overrightarrow{y}, z)|. \end{aligned} \tag{4.3}$$

Then there exists a constant C, which is independent of λ_1, λ_2, such that:

$$\max_{1\le j\le m}|\Delta Y^j_{\lambda_1}|^2 \le E_{\lambda_1}\left\{e^{C(\lambda_2-\lambda_1)}\max_{1\le j\le m}|\Delta\xi^j_{\lambda_2}|^2 + C\int_{\lambda_1}^{\lambda_2}\|\Delta f_t\|^2 dt\right\}. \tag{4.4}$$

Here $E_{\lambda_1}\{\cdot\}$ denotes the conditional expectation $E\{\cdot|\mathcal{F}_{\lambda_1}\}$.

We note that the stability result (4.4) is not only interesting in its own right, we need it to prove the uniqueness in (i).

The main idea is to prove a verification theorem in the spirit of Theorem 2.1. However, the proof here is much more involved because for our general RBSDEs the optimal strategy like the δ^* in Theorem 2.1 does not exist. We can only construct some approximately optimal strategy, and then we need some precise estimates of the errors, which will be obtained by using (4.4).

The rest of this section is organized as follows. In Section 4.1 we discuss heuristically how to find the approximately optimal strategies, which will lead to the definition of admissible strategies. In Section 4.2 we define rigorously the admissible strategy δ and the corresponding value function Y^δ. In Section 4.3 we estimate the error between $Y^{\delta,j}$ and the given solution Y^j, which leads to the verification theorem. Finally in Section 4.4 we prove Theorem 4.2.

4.1. Heuristic discussion

We want to extend the arguments for Theorem 2.1 to this case. For an arbitrary solution, the idea is to express Y^1_0 as the supremum of Y^δ_0 for some appropriately defined Y^δ. The strategy δ we can use here is much more subtle and in fact the arguments are very technical. To explain the difference and to motivate our definition of admissible strategies, let us first consider the following two-dimensional RBSDEs:

$$\begin{cases} Y^j_t = \xi_j + \int_t^T f_j(s, Y^1_s, Y^2_s, Z^j_s)ds - \int_t^T Z^j_s dB_s + K^j_T - K^j_t, \quad j = 1, 2;\\ Y^1_t \ge h_1(t, Y^2_t); \quad [Y^1_t - h_1(t, Y^2_t)]dK^1_t = 0;\\ Y^2_t \ge h_2(t, Y^1_t); \quad [Y^2_t - h_2(t, Y^1_t)]dK^2_t = 0.\end{cases} \tag{4.5}$$

Assume (Y^1, Y^2) is an arbitrary solution of (4.5). As in Theorem 2.1 we want to express Y^1_0 as $Y^{\delta^*}_0$ for some δ^* and appropriately defined Y^{δ^*}. Very naturally we want to define

$$\tau^*_1 \triangleq \inf\{t \ge 0 : Y^1_t = h_1(t, Y^2_t)\} \wedge T. \tag{4.6}$$

When f_1 does not depend on Y^2_t, as in (2.2) or (2.5), we have

$$Y^1_t = \xi_1 \mathbb{1}_{\{\tau^*_1 = T\}} + h_1(\tau^*_1, Y^2_{\tau^*_1})\mathbb{1}_{\{\tau^*_1 < T\}} + \int_t^{\tau^*_1} f_1(s, Y^1_s, Z^1_s)ds - \int_t^{\tau^*_1} Z^1_s dB_s.$$

This is a BSDE without reflection and is well posed. Therefore, once we can determine $Y^2_{\tau^*_1}$, Y^1_t is unique on $[0, \tau^*_1]$. Next we can define τ^*_2 by using Y^2 and express $Y^2_{\tau^*_1}$ in terms of $Y^1_{\tau^*_2}$. Repeat the arguments we can mimic the proof of Theorem 2.1.

However, in our case, we have to consider the following RBSDE over $[0, \tau^*_1]$,

$$\begin{cases} Y^1_t = \xi_1 \mathbb{1}_{\{\tau^*_1 = T\}} + h_1(\tau^*_1, Y^2_{\tau^*_1})\mathbb{1}_{\{\tau^*_1 < T\}} + \int_t^{\tau^*_1} f_1(s, Y^1_s, Y^2_s, Z^1_s)ds - \int_t^{\tau^*_1} Z^1_s dB_s;\\ Y^2_t = Y^2_{\tau^*_1} + \int_t^{\tau^*_1} f_2(s, Y^1_s, Y^2_s, Z^2_s)ds - \int_t^{\tau^*_1} Z^2_s dB_s + K^2_T - K^2_t;\\ Y^2_t \ge h_2(t, Y^1_t); \quad [Y^2_t - h_2(t, Y^1_t)]dK^2_t = 0.\end{cases} \tag{4.7}$$

This itself takes the form of (3.11), whose wellposedness needs to be proved. We will come back to this idea later.

There is another naive approach. Define

$$\tau_1^* \stackrel{\triangle}{=} \inf\{t > 0 : Y_t^1 = h_1(t, Y_t^2) \text{ or } Y_t^2 = h_2(t, Y_t^1)\} \wedge T.$$

Then we have

$$\begin{cases} Y_t^1 = Y_{\tau_1^*}^1 + \int_t^{\tau_1^*} f_1(s, Y_s^1, Y_s^2, Z_s^1)\mathrm{d}s - \int_t^{\tau_1^*} Z_s^1 \mathrm{d}B_s; \\ Y_t^2 = Y_{\tau_1^*}^2 + \int_t^{\tau_1^*} f_2(s, Y_s^1, Y_s^2, Z_s^2)\mathrm{d}s - \int_t^{\tau_1^*} Z_s^2 \mathrm{d}B_s; \end{cases} \quad 0 \le t \le \tau_1^*.$$

This system is well posed once the terminal conditions are given. However, in this approach we will have to define

$$\tau_2^* \stackrel{\triangle}{=} \inf\{t > \tau_1^* : Y_t^1 = h_1(t, Y_t^2) \text{ or } Y_t^2 = h_2(t, Y_t^1)\} \wedge T.$$

It is very likely that $\tau_2^* = \tau_1^*$, and then we have trouble to move forward.

We now come back to the first approach. That is, we consider (4.6) and (4.7). One key observation is that, although we do not know its uniqueness yet, RBSDE (4.7) has only one reflection while the original RBSDE (4.5) has two reflections. Therefore, by doing this we reduce the number of reflections, and thus by repeating the procedure we can transform the system to BSDEs without reflection which is well posed.

There is another difficulty to prove the verification theorem for RBSDEs in the form of (4.7). To illustrate the idea let us consider the following RBSDE instead of (4.7):

$$\begin{cases} Y_t^1 = \xi_1 + \int_t^T f_1(s, Y_s^1, Y_s^2, Z_s^1)\mathrm{d}s - \int_t^T Z_s^1 \mathrm{d}B_s + K_T^1 - K_t^1; \\ Y_t^2 = \xi_2 + \int_t^T f_2(s, Y_s^1, Y_s^2, Z_s^2)\mathrm{d}s - \int_t^T Z_s^2 \mathrm{d}B_s; \\ Y_t^1 \ge h_1(t, Y_t^2); \quad [Y_t^1 - h_1(t, Y_t^2)]\mathrm{d}K_t^1 = 0. \end{cases} \tag{4.8}$$

Again we define τ_1^* by (4.6). Then over $[0, \tau_1^*]$ we have

$$\begin{cases} Y_t^1 = \xi_1 \mathbb{1}_{\{\tau_1^* = T\}} + h_1(\tau_1^*, Y_{\tau_1^*}^2)\mathbb{1}_{\{\tau_1^* < T\}} + \int_t^{\tau_1^*} f_1(s, Y_s^1, Y_s^2, Z_s^1)\mathrm{d}s - \int_t^{\tau_1^*} Z_s^1 \mathrm{d}B_s; \\ Y_t^2 = Y_{\tau_1^*}^2 + \int_t^{\tau_1^*} f_2(s, Y_s^1, Y_s^2, Z_s^2)\mathrm{d}s - \int_t^{\tau_1^*} Z_s^2 \mathrm{d}B_s. \end{cases}$$

This is well posed. However, Y^2 has no reflection, thus we cannot define τ_2^* as in Theorem 2.1. Our second key observation is that, when $\tau_1^* < T$, $Y_{\tau_1^*}^1 = h_1(\tau_1^*, Y_{\tau_1^*}^2)$. Note that Y^1, Y^2, h are all continuous. This implies that if τ_2^* is close to τ_1^*, then $Y_t^1 \approx h_1(t, Y_t^2)$ for $t \in [\tau_1^*, \tau_2^*]$, and therefore,

$$Y_t^2 \approx Y_{\tau_2^*}^2 + \int_t^{\tau_2^*} f_2(s, h_1(t, Y_t^2), Y_s^2, Z_s^2)\mathrm{d}s - \int_t^{\tau_2^*} Z_s^2 \mathrm{d}B_s. \tag{4.9}$$

Ignoring the approximation, this is a BSDE without reflection and is well posed. We should, of course, estimate the error due to this approximation.

We now summarize the above idea and discuss heuristically how to find the approximately optimal strategy for the m-dimensional RBSDE (3.11). Let μ denote the number of nonempty sets A_j in (3.11), that is, the number of reflections in (3.11). We proceed by induction on μ. First, when $\mu = 0$, (3.11) becomes an m-dimensional BSDE without reflection. By standard arguments one can easily show that Theorem 4.2 holds. Now assume Theorem 4.2 is true for $\mu = m_1 - 1$ for some $1 \le m_1 \le m$. For $\mu = m_1$, let (Y^j, Z^j, K^j) be an arbitrary solution of (3.11).

Let $\tau_0^* \stackrel{\triangle}{=} \lambda_1$, and without loss of generality assume $A_1 \neq \emptyset$. Set

$$\tau_1^* \stackrel{\triangle}{=} \inf\{t \ge \tau_0^* : Y_t^1 = \max_{i \in A_1} h_{1,i}(t, Y_t^i)\} \wedge \lambda_2.$$

When $\tau_1^* < \lambda_2$, we have

$$Y_{\tau_1^*}^1 = \max_{i \in A_1} h_{1,i}(\tau_1^*, Y_{\tau_1^*}^i).$$

That is, there exists an index, denoted as $\eta_1 \in A_1$, such that

$$Y_{\tau_1^*}^1 = h_{1,\eta_1}(\tau_1^*, Y_{\tau_1^*}^{\eta_1}).$$

So, besides the stopping time τ_1^*, we need to keep track of the *optimal index* η_1. We note that η_1 is random and is $\mathcal{F}_{\tau_1^*}$ measurable. At this point, let us denote $\eta_0 \stackrel{\triangle}{=} 1$. Note that, over $[\tau_0^*, \tau_1^*]$, it holds that:

$$\begin{cases} Y_t^j = Y_{\tau_1^*}^j + \int_t^{\tau_1^*} f_j(s, \overrightarrow{Y}_s, Z_s^j)\mathrm{d}s - \int_t^{\tau_1^*} Z_s^j \mathrm{d}B_s + K_{\tau_1^*}^j - K_t^j, \quad j \neq \eta_0; \\ Y_t^j \ge \max_{k \in A_j} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k \in A_j} h_{j,k}(t, Y_t^k)]\mathrm{d}K_t^j = 0, \; j \neq \eta_0; \\ Y_t^{\eta_0} = Y_{\tau_1^*}^{\eta_0} + \int_t^{\tau_1^*} f_{\eta_0}(s, \overrightarrow{Y}_s, Z_s^{\eta_0})\mathrm{d}s - \int_t^{\tau_1^*} Z_s^{\eta_0} \mathrm{d}B_s. \end{cases} \tag{4.10}$$

This is a system with only $m_1 - 1$ reflections, and thus is well posed by our induction assumption.

Now assume $\tau_1^* < \lambda_2$. To define (τ_2^*, η_2), we need to consider two different cases.

Case 1. $A_{\eta_1} \neq \emptyset$. Denote

$$\tau_2^* \stackrel{\triangle}{=} \inf\{t \ge \tau_1^* : Y_t^{\eta_1} = \max_{i \in A_{\eta_1}} h_{\eta_1,i}(t, Y_t^i)\} \wedge \lambda_2,$$

and, when $\tau_2^* < \lambda_2$, let $\eta_2 \in A_{\eta_1}$ such that $Y_{\tau_2^*}^{\eta_1} = h_{\eta_1,\eta_2}(\tau_2^*, Y_{\tau_2^*}^{\eta_2})$. Then $\overrightarrow{Y}$ satisfies a system with $m_1 - 1$ reflections over $[\tau_1^*, \tau_2^*]$, where the η_1th equation has no reflection.

Case 2. $A_{\eta_1} = \emptyset$. In this case, the η_1th equation has no reflection. Note that $Y_{\tau_1^*}^{\eta_0} = h_{\eta_0,\eta_1}(\tau_1^*, Y_{\tau_1^*}^{\eta_1})$. As in (4.9), choose τ_2^* "close" to τ_1^*, then for any $t \in [\tau_1^*, \tau_2^*]$, we have $Y_t^{\eta_0} \approx h_{\eta_0,\eta_1}(\tau_1^*, Y_t^{\eta_1})$. On the other hand, by (4.1) and (2.9) one can see that $Y_{\tau_1^*}^j > h_{j,\eta_0}(\tau_1^*, Y_{\tau_1^*}^{\eta_0})$ for any j such that $\eta_0 \in A_j$. Since τ_2^* is close to τ_1^*, let us assume $Y_t^j > h_{j,\eta_0}(\tau_1^*, Y_t^{\eta_0})$ for $t \in [\tau_1^*, \tau_2^*]$. So approximately, over $[\tau_1^*, \tau_2^*]$, $\{Y^j\}_{j \neq \eta_0}$ satisfy

$$\begin{cases} Y_t^j \approx Y_{\tau_2^*}^j + \int_t^{\tau_2^*} f_j(s, h_{1,\eta_1}(\tau_1^*, Y_s^{\eta_1}), Y_s^2, \ldots, Y_s^m, Z_s^j)\mathrm{d}s \\ \quad - \int_t^{\tau_2^*} Z_s^j \mathrm{d}B_s + K_{\tau_2^*}^j - K_t^j; \\ Y_t^j \ge \max_{k \in A_j - \{\eta_0\}} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k \in A_j - \{\eta_0\}} h_{j,k}(t, Y_t^k)]\mathrm{d}K_t^j = 0. \end{cases} \tag{4.11}$$

This is a system of $m-1$ equations with m_1-1 reflections, where we remove the equation for Y^{η_0} completely. In order to move forward, we need to define η_2 so that $A_{\eta_2} \neq \emptyset$. It turns out that the best way is to set $\eta_2 \triangleq \eta_0$.

Now we can continue the procedure and define a sequence of (τ_n^*, η_n).

4.2. Construction of Y^δ

The arguments in Section 4.1 is only heuristic. We now make everything rigorous. First, let us introduce the following definition:

Definition 4.3. $\delta = (\tau_0, \dots, \tau_n; \eta_0, \dots, \eta_n)$ is called an admissible strategy if

(i) $\lambda_1 = \tau_0 \le \dots \le \tau_n \le \lambda_2$ is a sequence of stopping times;
(ii) $\eta_0, \dots, \eta_n$ are random index taking values in $\{1, \dots, m\}$ such that $\eta_i \in \mathcal{F}_{\tau_i}$;
(iii) $A_{\eta_0} \neq \emptyset$;
(iv) If $A_{\eta_i} \neq \emptyset$, then $\eta_{i+1} \in A_{\eta_i}$;
(v) If $A_{\eta_i} = \emptyset$, then $\eta_{i+1} \triangleq \eta_{i-1}$.

We note that, unlike in Section 2, here δ must be a finite sequence.

Remark 4.4. By Definition 4.3(iii), $A_{\eta_i} = \emptyset$ implies that $i \ge 1$. Then the (v) above makes sense. Moreover, by induction we see in this case $A_{\eta_{i+1}} = A_{\eta_{i-1}} \neq \emptyset$.

We assume Theorem 4.2 holds for $\mu = m_1 - 1$ and for any $m \ge m_1$. Now assume $\mu = m_1$. For an admissible strategy δ, we construct $(Y^{\delta,j}, Z^{\delta,j})$ as follows.

First, for $t \in [\tau_n, \lambda_2]$ and $j = 1, \dots, m$, set

$$Y_t^{\delta,j} \triangleq Y_t^{0,j}, \qquad Z_t^{\delta,j} \triangleq Z_t^{0,j}, \tag{4.12}$$

where $(Y^{0,j}, Z^{0,j})$ is the solution to (3.11) constructed in Section 2. Then in particular we have

$$Y_{\tau_n}^{\delta,j} \ge \max_{i \in A_j} h_{j,i}(\tau_n, Y_{\tau_n}^{\delta,i}), \quad j = 1, \dots, m. \tag{4.13}$$

For $i = n-1, \dots, 0$, assume we have constructed $Y_{\tau_{i+1}-}^{\delta,j}$ for $j = 1, \dots, m$, which we will do later. Note that $Y^{\delta,j}$ may be discontinuous at τ_{i+1}. Corresponding to *Case* 1 and *Case 2* when we defined (τ_2^*, η_2) in Section 4.1, we define $(Y^{\delta,j}, Z^{\delta,j})$ over $[\tau_i, \tau_{i+1})$ in two cases.

Case 1. $A_{\eta_i} \neq \emptyset$. Assume our constructed $Y_{\tau_{i+1}-}^{\delta,j}$ satisfies

$$Y_{\tau_{i+1}-}^{\delta,j} \ge \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y_{\tau_{i+1}-}^{\delta,k}), \quad j \neq \eta_i. \tag{4.14}$$

Recall (4.10). We consider the following RBSDE by removing the constraint of the η_ith equation:

$$\begin{cases} Y_t^{\delta,j} = Y_{\tau_{i+1}-}^{\delta,j} + \int_t^{\tau_{i+1}} f_j(s, \overrightarrow{Y}_s^\delta, Z_s^{\delta,j}) \mathrm{d}s \\ \quad - \int_t^{\tau_{i+1}} Z_s^{\delta,j} \mathrm{d}B_s + K_{\tau_{i+1}}^{\delta,j} - K_t^{\delta,j}, \quad j \neq \eta_i; \\ Y_t^{\delta,j} \ge \max_{k \in A_j} h_{j,k}(t, Y_t^{\delta,k}); \quad [Y_t^{\delta,j} - \max_{k \in A_j} h_{j,k}(t, Y_t^{\delta,k})] \mathrm{d}K_t^{\delta,j} = 0, \quad j \neq \eta_i; \\ Y_t^{\delta,\eta_i} = Y_{\tau_{i+1}-}^{\delta,\eta_i} + \int_t^{\tau_{i+1}} f_{\eta_i}(s, \overrightarrow{Y}_s^\delta, Z_s^{\delta,\eta_i}) \mathrm{d}s - \int_t^{\tau_{i+1}} Z_s^{\delta,\eta_i} \mathrm{d}B_s. \end{cases} \tag{4.15}$$

It is obvious that the $f_j, h_{j,i}, A_j$ here satisfy Assumptions 3.1 and 4.1, and (4.14) implies that the terminal conditions of (4.15) satisfy (3.12). Since (4.15) has only $m_1 - 1$ reflections, by induction assumption it has the unique solution $(Y^{\delta,j}, Z^{\delta,j})$, $j = 1, \ldots, m$ over $[\tau_i, \tau_{i+1})$. ■

Case 2. $A_{\eta_i} = \emptyset$. By Remark 4.4 we have $i \geq 1$ and $A_{\eta_{i-1}} \neq \emptyset$. Assume our constructed $Y^{\delta,j}_{\tau_{i+1}-}$ satisfies

$$Y^{\delta,j}_{\tau_{i+1}-} \geq \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}), \quad j \neq \eta_{i-1}. \tag{4.16}$$

Recall (4.11). We omit the η_{i-1}th equation and consider the following $m-1$-dimensional RBSDE with at most $m_1 - 1$ reflections: for $j \neq \eta_{i-1}$,

$$\begin{cases} Y^{\delta,j}_t = Y^{\delta,j}_{\tau_{i+1}-} - \int_t^{\tau_{i+1}} Z^{\delta,j}_s \mathrm{d}B_s + K^{\delta,j}_{\tau_{i+1}} - K^{\delta,j}_t \\ \quad + \int_t^{\tau_{i+1}} \tilde{f}_j(s, Y^{\delta,1}_s, \ldots, Y^{\delta,\eta_{i-1}-1}_s, Y^{\delta,\eta_{i+1}-1}_s, \ldots, Y^{\delta,m}_s, Z^{\delta,j}_s)\mathrm{d}s; \\ Y^{\delta,j}_t \geq \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(t, Y^{\delta,k}_t), \quad [Y^{\delta,j}_t - \max_{k \in A_j - \{\eta_{i-1}\}} h_{j,k}(t, Y^{\delta,k}_t)]\mathrm{d}K^{\delta,j}_t = 0. \end{cases} \tag{4.17}$$

Here:

$$\begin{aligned} &\tilde{f}_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \ldots, y_n, z) \\ &\quad \triangleq f_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, h_{\eta_{i-1},\eta_i}(\tau_i, y_{\eta_i}), y_{\eta_{i-1}+1}, \ldots, y_n, z). \end{aligned} \tag{4.18}$$

One can easily check that $\tilde{f}_j, h_{j,i}, A_j - \{\eta_{i-1}\}$ here satisfy Assumptions 3.1 and 4.1, and (4.16) implies that the terminal conditions of (4.17) satisfy (3.12). Since RBSDE (4.17) has at most $m_1 - 1$ reflections, by induction assumption it has the unique solution $(Y^{\delta,j}, Z^{\delta,j})$, $j \neq \eta_{i-1}$, over $[\tau_i, \tau_{i+1})$. We emphasize that $Y^{\delta,\eta_{i-1}}$ is not involved in this case. ■

It remains to construct $Y^{\delta,j}_{\tau_{i+1}-}$ satisfying (4.14) or (4.16). First, set

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{0,j}_{\tau_n}, \quad \text{if } i+1 = n; \qquad Y^{\delta,j}_{\tau_{i+1}-} \triangleq \xi^j_{\lambda_2}, \quad \text{if } \tau_{i+1} = \lambda_2. \tag{4.19}$$

By (4.12) and (3.12) we know that both (4.14) and (4.16) hold. Now assume $i < n-1$ and $\tau_{i+1} < \lambda_2$. Assume we have solved either (4.15) or (4.17) over $[\tau_{i+1}, \tau_{i+2})$. Again we construct $Y^{\delta,j}_{\tau_{i+1}-}$ in two cases.

Case 2. $A_{\eta_i} = \emptyset$. In this case we need to construct $Y^{\delta,j}_{\tau_{i+1}-}$ only for $j \neq \eta_{i-1}$ and to check (4.16). By Remark 4.4 we know that $i \geq 1$, $\eta_{i+1} = \eta_{i-1}$, and $A_{\eta_{i+1}} \neq \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}$ were obtained from (4.15) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy:

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_{i+1} = \eta_{i-1}. \tag{4.20}$$

Define

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_{i-1}. \tag{4.21}$$

Then (4.16) follows immediately from (4.20). ■

Case 1. $A_{\eta_i} \neq \emptyset$. In this case we need to construct $Y^{\delta,j}_{\tau_{i+1}-}$ for all j and to check (4.14). We do it in two cases.

Case 1.1. $A_{\eta_{i+1}} = \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}$, $j \neq \eta_i$ were obtained from (4.17) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k \in A_j - \{\eta_i\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_i. \tag{4.22}$$

Define

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_i; \qquad Y^{\delta,\eta_i}_{\tau_{i+1}-} \triangleq h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}). \tag{4.23}$$

By (4.22), to prove (4.14) it suffices to show that

$$Y^{\delta,j}_{\tau_{i+1}} \geq h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})), \quad \text{if } \eta_i \in A_j. \tag{4.24}$$

Assume $\eta_i \in A_j$. By Definition 4.3(iv) and Assumption 4.1(ii), we have $\eta_{i+1} \in [A_j - \{\eta_i\}] \cup \{j\}$, and

$$h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})) < h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}). \tag{4.25}$$

If $\eta_{i+1} \in A_j - \{\eta_i\}$, then (4.24) follows from (4.22) and (4.25). If $\eta_{i+1} = j$, then (4.24) follows from (2.8) and (4.25). So in both cases (4.24) holds, then so does (4.14). ■

Case 1.2. $A_{\eta_{i+1}} \neq \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}$ were obtained from (4.15) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy:

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_{i+1}. \tag{4.26}$$

Define

$$\begin{aligned} &Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_i, \eta_{i+1}; \\ &Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} \triangleq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} \vee \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}); \\ &Y^{\delta,\eta_i}_{\tau_{i+1}-} \triangleq h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}). \end{aligned} \tag{4.27}$$

We now check (4.14) for $j \neq \eta_i$. First, for $j = \eta_{i+1}$, by (4.27),

$$Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} \geq \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}).$$

Moreover, if $\eta_i \in A_{\eta_{i+1}}$, by (4.1) and (2.8) we have

$$h_{\eta_{i+1},\eta_i}(\tau_{i+1}, Y^{\delta,\eta_i}_{\tau_{i+1}-}) = h_{\eta_{i+1},\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})) < Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}.$$

So (4.14) holds for $j = \eta_{i+1}$.

It remains to check (4.14) for $j \neq \eta_i, \eta_{i+1}$. By (4.26) and the first line in (4.27) we have

$$Y^{\delta,j}_{\tau_{i+1}-} \geq \max_{k \in A_j - \{\eta_i, \eta_{i+1}\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}). \tag{4.28}$$

If $\eta_{i+1} \in A_j$, recall the definition of $Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}$ in (4.27). First, by (4.26) we have

$$h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}) \leq Y^{\delta,j}_{\tau_{i+1}} = Y^{\delta,j}_{\tau_{i+1}-}.$$

Second, for any $k \in A_{\eta_{i+1}} - \{\eta_i\}$, similar to (4.24) one can easily prove

$$h_{j,\eta_{i+1}}(\tau_{i+1}, h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}})) \leq Y^{\delta,j}_{\tau_{i+1}} = Y^{\delta,j}_{\tau_{i+1}-}.$$

Thus, by Assumption 3.1(iii) we have

$$h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}) \le Y^{\delta,j}_{\tau_{i+1}-}. \tag{4.29}$$

Finally, if $\eta_i \in A_j$, by Definition 4.3(iv), Assumption 4.1(ii), and (4.29), we have $\eta_{i+1} \in A_j \bigcup\{j\}$ and

$$\begin{aligned} h_{j,\eta_i}(\tau_{i+1}, Y^{\delta,\eta_i}_{\tau_{i+1}-}) &= h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})) \\ &< h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}) \le Y^{\delta,j}_{\tau_{i+1}-}. \end{aligned}$$

This, together with (4.28) and (4.29), proves (4.14) for $j \neq \eta_i, \eta_{i+1}$. ■

Now repeat the arguments backward in time, we see in each $[\tau_i, \tau_{i+1})$, either (4.15) or (4.17) is well defined and is well posed. Thus we obtain $Y^{\delta,j}$ over the whole interval $[\lambda_1, \lambda_2]$, with the exception of $Y^{\delta,\eta_{i-1}}_t$ for $t \in [\tau_i, \tau_{i+1})$ when $A_{\eta_i} = \phi$. By applying Corollary 3.4 and comparison theorem repeatedly, one can easily show that:

Lemma 4.5. *Assume Assumptions 3.1 and 4.1 hold, and that Theorem 4.2 is true for $\mu = m_1 - 1$. Then for $\mu = m_1$ and for any admissible strategy δ and any j, we have $Y^{\delta,j}_t \le Y^j_t$ whenever $Y^{\delta,j}_t$ is well defined.* ■

4.3. Verification theorem

We now prove the verification theorem.

Theorem 4.6. *Assume Assumptions 3.1 and 4.1 hold, and that Theorem 4.2 is true for $\mu = m_1 - 1$. Then for $\mu = m_1$ and for any solution Y^j of RBSDE (3.11), we have $Y^j_{\lambda_1} = \operatorname{esssup}_\delta Y^{\delta,j}_{\lambda_1}$ for all j, where the* esssup *is taken over all admissible strategies δ.*

Proof. We prove the theorem in several steps.

Step 1. Fix $\varepsilon > 0$ and let $D_\varepsilon \triangleq \{i\varepsilon : i = 0, 1, \ldots\}$. We construct an approximately optimal admissible strategy as follows. First, set $\tau_0 \triangleq \lambda_1$ and choose η_0 such that $A_{\eta_0} \neq \emptyset$. For $i = 0, 1, \ldots$, we define (τ_{i+1}, η_{i+1}) in two cases.

Case 1. $A_{\eta_i} \neq \emptyset$. Set

$$\tau_{i+1} \triangleq \inf\{t \ge \tau_i : Y^{\eta_i}_t = \max_{k\in A_{\eta_i}} h_{\eta_i,k}(t, Y^k_t)\} \wedge \lambda_2.$$

If $\tau_{i+1} < \lambda_2$, set $\eta_{i+1} \in A_{\eta_i}$ be the smallest index such that

$$Y^{\eta_i}_{\tau_{i+1}} = h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\eta_{i+1}}_{\tau_{i+1}}). \tag{4.30}$$

Otherwise choose arbitrary $\eta_{i+1} \in A_{\eta_i}$.

Case 2. $A_{\eta_i} = \emptyset$. Since $A_{\eta_0} \neq \emptyset$, we have $i \ge 1$. Set $\eta_{i+1} \triangleq \eta_{i-1}$. If $\tau_i = \lambda_2$, define $\tau_{i+1} \triangleq \lambda_2$. Now assume $\tau_i < \lambda_2$. It is more involved to define τ_{i+1} in this case. By the definition of η_i, one can check that in this case we must have $A_{\eta_{i-1}} \neq \emptyset$, and thus by *Case* 1, $\eta_i \in A_{\eta_{i-1}}$ and $Y^{\eta_{i-1}}_{\tau_i} = h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_{\tau_i})$. Moreover, by Assumptions 3.1(iii) and 4.1(ii), one can easily see $Y^j_{\tau_i} > h_{j,\eta_{i-1}}(\tau_i, Y^{\eta_{i-1}}_{\tau_i})$ for any j such that $\eta_{i-1} \in A_j$. We now define

$$\tau_{i+1} \triangleq \tau^1_{i+1} \wedge \tau^2_{i+1} \wedge \lambda_2;$$

where τ^1_{i+1} is the smallest number in D_ε such that $\tau^1_{i+1} > \tau_i$; and

$$\tau^2_{i+1} \triangleq \inf\{t > \tau_i : \exists j \; s.t. \; \eta_{i-1} \in A_j, Y^j_t = h_{j,\eta_{i-1}}(t, Y^{\eta_{i-1}}_t)\}.$$

Now set $\delta \triangleq \delta^{n,\varepsilon} \triangleq (\tau_0, \ldots, \tau_n; \eta_0, \ldots, \eta_n)$. Recall Definition 4.3. One can easily check that δ is an admissible strategy.

Step 2. We estimate the errors backward in time. Recall Section 4.2 and denote

$$\Delta Y^j_t \triangleq Y^j_t - Y^{\delta,j}_t.$$

First, by (4.19) it is obvious that

$$|Y^j_{\tau_n} - Y^{\delta,j}_{\tau_n -}| = |\Delta Y^j_{\tau_n}|. \tag{4.31}$$

Now assume $i < n - 1$.

Case 1. $A_{\eta_i} \neq \phi$. We claim that

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_i}|^2 \le E_{\tau_i}\{\mathrm{e}^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|^2\}. \tag{4.32}$$

In fact, in this case (Y^j, Z^j, K^j) satisfies

$$\begin{cases} Y^j_t = Y^j_{\tau_{i+1}} + \int_t^{\tau_{i+1}} f_j(s, \overrightarrow{Y}_s, Z^j_s)\mathrm{d}s - \int_t^{\tau_{i+1}} Z^j_s \mathrm{d}B_s + K^j_{\tau_{i+1}} - K^j_t, \quad j \neq \eta_i; \\ Y^j_t \ge \max_{k\in A_j} h_{j,k}(t, Y^k_t); \quad [Y^j_t - \max_{k\in A_j} h_{j,k}(t, Y^k_t)]\mathrm{d}K^k_t = 0, \quad j \neq \eta_i; \\ Y^{\eta_i}_t = Y^{\eta_i}_{\tau_{i+1}} + \int_t^{\tau_{i+1}} f_{\eta_i}(s, \overrightarrow{Y}_s, Z^{\eta_i}_s)\mathrm{d}s - \int_t^{\tau_{i+1}} Z^{\eta_i}_s \mathrm{d}B_s. \end{cases} \tag{4.33}$$

Compare (4.33) and (4.15). They have only $m_1 - 1$ reflections, thus by induction assumption we can apply Theorem 4.2 (ii) and obtain

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_i}|^2 \le E_{\tau_i}\{\mathrm{e}^{C(\tau_{i+1}-\tau_i)} \max_{1\le j\le m} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}|^2\}.$$

So to prove (4.32) it suffices to show that

$$\max_{1\le j\le m} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| \le \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|. \tag{4.34}$$

If $\tau_{i+1} = \lambda_2$, then by (4.19), we have

$$|Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = |\xi^j_{\lambda_2} - \xi^j_{\lambda_2}| = 0, \quad \forall j.$$

Thus (4.34) holds.

Now assume $\tau_{i+1} < \lambda_2$. Note that $Y^{\delta,j}_{\tau_{i+1}-}$ is defined by either (4.23) or (4.27). In the former case, by (4.2) we have

$$\max_{j\neq\eta_i} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|;$$

$$|Y^{\eta_i}_{\tau_{i+1}} - Y^{\delta,\eta_i}_{\tau_{i+1}-}| = |h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\eta_{i+1}}_{\tau_{i+1}}) - h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})| \le |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|.$$

Since $\eta_{i+1} \in A_{\eta_i}$ and thus $\eta_{i+1} \neq \eta_i$. We prove (4.34) in this case.

In the latter case, that is, $Y^{\delta,j}_{\tau_{i+1}-}$ is defined by (4.27), we first have

$$\max_{j\neq\eta_i,\eta_{i+1}} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = \max_{j\neq\eta_i,\eta_{i+1}} |\Delta Y^j_{\tau_{i+1}}|.$$

Next, for $j = \eta_{i+1}$, by Lemma 4.5 and Assumption 3.1(iii) we have

$$Y^{\eta_{i+1}}_{\tau_{i+1}} \geq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} \quad \text{and} \quad Y^{\eta_{i+1}}_{\tau_{i+1}} \geq \max_{k\in A_{\eta_{i+1}}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^k_{\tau_{i+1}}) \geq \max_{k\in A_{\eta_{i+1}}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}).$$

Then $Y^{\eta_{i+1}}_{\tau_{i+1}} \geq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}$. Therefore,

$$|Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}| = Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} \leq Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} = |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|.$$

Finally, for $j = \eta_i$, by Assumption 4.1(iii) we have

$$\begin{aligned}|Y^{\eta_i}_{\tau_{i+1}} - Y^{\delta,\eta_i}_{\tau_{i+1}-}| &= |h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\eta_{i+1}}_{\tau_{i+1}}) - h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})| \\ &\leq |Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}| \leq |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|.\end{aligned}$$

Thus (4.34) also holds.

Case 2. $A_{\eta_i} = \phi$. In this case (Y^j, Z^j, K^j), $j \neq \eta_{i-1}$ satisfies

$$\begin{cases} Y^j_t = Y^j_{\tau_{i+1}} - \displaystyle\int_t^{\tau_{i+1}} Z^j_s \mathrm{d}B_s + K^j_{\tau_{i+1}} - K^j_t \\ \quad + \displaystyle\int_t^{\tau_{i+1}} \hat{f}_j(s, Y^1_s, \ldots, Y^{\eta_{i-1}-1}_s, Y^{\eta_{i-1}+1}_s, \ldots, Y^m_s, Z^j_s)\mathrm{d}s; \\ Y^j_t \geq \displaystyle\max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^k_t); \quad [Y^j_t - \displaystyle\max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^k_t)]\mathrm{d}K^k_t = 0; \end{cases} \tag{4.35}$$

where, recalling (4.18),

$$\begin{aligned} &\hat{f}_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \ldots, y_n, z) \\ &\quad \stackrel{\triangle}{=} \tilde{f}_j(t, y_1, \ldots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \ldots, y_n, z) + I^j_t; \end{aligned} \tag{4.36}$$

$$\begin{aligned} I^j_t \stackrel{\triangle}{=} &\ f_j(t, \overrightarrow{Y}_t, Z^j_t) - f_j(t, Y^1_t, \ldots, Y^{\eta_{i-1}-1}_t, h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_t), \\ &\ Y^{\eta_{i-1}+1}_t, \ldots, Y^n_t, Z^j_t). \end{aligned} \tag{4.37}$$

We note that here I^j_t is considered as a random coefficient. Compare (4.35) and (4.17). Recalling (4.21), by induction assumption again we get

$$\max_{j\neq\eta_{i-1}} |\Delta Y^j_{\tau_i}|^2 \leq E_{\tau_i}\left\{ \mathrm{e}^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_{i-1}} |\Delta Y^j_{\tau_{i+1}}|^2 + C \sum_{j\neq\eta_{i-1}} \int_{\tau_i}^{\tau_{i+1}} |I^j_t|\mathrm{d}t \right\}. \tag{4.38}$$

Note that $Y^{\eta_{i-1}}_{\tau_i} = h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_{\tau_i})$. Then

$$\begin{aligned} |I^j_t| &\leq C\left|Y^{\eta_{i-1}}_t - h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_t)\right|^2 \\ &\leq C\Big[|Y^{\eta_{i-1}}_t - Y^{\eta_{i-1}}_{\tau_i}|^2 + |h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_{\tau_i}) - h_{\eta_{i-1},\eta_i}(\tau_i, Y^{\eta_i}_t)|^2\Big] \\ &\leq C\Big[|Y^{\eta_{i-1}}_t - Y^{\eta_{i-1}}_{\tau_i}|^2 + |Y^{\eta_i}_{\tau_i} - Y^{\eta_i}_t|^2\Big] \leq C\sum_{k=1}^m |Y^k_t - Y^k_{\tau_i}|^2. \end{aligned}$$

Note that in this case $\tau_{i+1} - \tau_i \le \varepsilon$. Then

$$|I_t^j| \le C \sum_{k=1}^{m} \sup_{\lambda_1 \le t_1 < t_2 \le \lambda_2 : t_2 - t_1 \le \varepsilon} |Y_{t_1}^k - Y_{t_2}^k|^2 \triangleq I_\varepsilon. \tag{4.39}$$

Thus (4.38) implies

$$\max_{j \ne \eta_{i-1}} |\Delta Y_{\tau_i}^j|^2 \le E_{\tau_i}\{\mathrm{e}^{C(\tau_{i+1}-\tau_i)} \max_{1 \le j \le m} |\Delta Y_{\tau_{i+1}}^j|^2 + I_\varepsilon[\tau_{i+1} - \tau_i]\}. \tag{4.40}$$

Step 3. We claim that, for a.s. ω, $\tau_i = \lambda_2$ for i large enough. We prove it by contradiction. Assume ω is in the set that all $Y^j_\cdot(\omega)$ and $h_{j,i}(\cdot, \omega, y)$ are continuous and $\tau_i(\omega) < \lambda_2$ for all i. Denote $\tau_\infty \triangleq \lim_{i\to\infty} \tau_i$.

First, it is obvious that there can be only finitely many i such that $A_{\eta_i} = \phi$ and $\tau_{i+1} = \tau_{i+1}^1$.

Second, assume there is an infinite sequence i_k such that $A_{\eta_{i_k}} = \phi$ and $\tau_{i_k+1} = \tau_{i_k+1}^2$. Note that in this case $\eta_{i_k} \in A_{\eta_{i_k-1}}$ and there exists $\hat{\eta}_{i_k+1}$ such that $\eta_{i_k-1} \in A_{\hat{\eta}_{i_k+1}}$. Then

$$Y_{\tau_{i_k}}^{\eta_{i_k-1}} = h_{\eta_{i_k-1},\eta_{i_k}}(\tau_{i_k}, Y_{\tau_{i_k}}^{\eta_{i_k}}); \qquad Y_{\tau_{i_k+1}}^{\hat{\eta}_{i_k+1}} = h_{\hat{\eta}_{i_k+1},\eta_{i_k-1}}(\tau_{i_k+1}, Y_{\tau_{i_k+1}}^{\eta_{i_k-1}}). \tag{4.41}$$

The vector $(\hat{\eta}_{i_k+1}, \eta_{i_k-1}, \eta_{i_k})$ can take only finitely many values, then there exist (j_1, j_2, j_3) and an infinite subsequence of i_k, without loss of generality we assume it is the whole sequence i_k, such that $j_2 \in A_{j_1}$, $j_3 \in A_{j_2}$ and

$$\hat{\eta}_{i_k+1} = j_1, \qquad \eta_{i_k-1} = j_2, \qquad \eta_{i_k} = j_3, \quad \forall k.$$

By (4.41) we get

$$Y_{\tau_{i_k+1}}^{j_1} = h_{j_1,j_2}(\tau_{i_k+1}, Y_{\tau_{i_k+1}}^{j_2}), \qquad Y_{\tau_{i_k}}^{j_2} = h_{j_2,j_3}(\tau_{i_k}, Y_{\tau_{i_k}}^{j_3}), \quad \forall k.$$

Send $k \to \infty$, we have

$$Y_{\tau_\infty}^{j_1} = h_{j_1,j_2}(\tau_\infty, Y_{\tau_\infty}^{j_2}), \qquad Y_{\tau_\infty}^{j_2} = h_{j_2,j_3}(\tau_\infty, Y_{\tau_\infty}^{j_3}).$$

Then, by Assumption 4.1(ii), $j_3 \in A_{j_1} \bigcup \{j_1\}$ and

$$Y_{\tau_\infty}^{j_1} = h_{j_1,j_2}(\tau_\infty, h_{j_2,j_3}(\tau_\infty, Y_{\tau_\infty}^{j_3})) < h_{j_1,j_3}(\tau_\infty, Y_{\tau_\infty}^{j_3}).$$

This contradicts with (2.9). Therefore, there are only finitely many i such that $A_{\eta_i} = \phi$.

Finally, by the above results we must have some n_0 such that $A_{\eta_i} \ne \phi$ for all $i \ge n_0$. Then $\eta_{i+1} \in A_{\eta_i}$ and (4.30) holds for all $i \ge n_0$. We say $(\eta_i, \eta_{i+1}, \ldots, \eta_{i+l-1})$ is a *loop* if they are all different and $\eta_{i+l} = \eta_i$. Since each η_i takes only values $1, \ldots, m$, there are in total finitely many possible loops. Thus there exist $(j_1, \ldots, j_l)$ and an infinite sequence i_k such that $(\eta_{i_k}, \ldots, \eta_{i_k+l}, \eta_{i_k+l}) = (j_1, \ldots, j_l, j_1)$. Therefore, by (4.30), we have

$$Y_{\tau_{i_k+1}}^{j_1} = h_{j_1,j_2}(\tau_{i_k+1}, Y_{\tau_{i_k+1}}^{j_2}), \ldots, Y_{\tau_{i_k+l-1}}^{j_{l-1}} = h_{j_{l-1},j_l}(\tau_{i_k+l-1}, Y_{\tau_{i_k+l-1}}^{j_l}),$$

and $Y_{\tau_{i_k+l}}^{j_l} = h_{j_l,j_1}(\tau_{i_k+l}, Y_{\tau_{i_k+l}}^{j_1})$. Send $k \to \infty$, we get

$$Y_{\tau_\infty}^{j_1} = h_{j_1,j_2}(\tau_\infty, Y_{\tau_\infty}^{j_2}), \ldots, Y_{\tau_\infty}^{j_{l-1}} = h_{j_{l-1},j_l}(\tau_\infty, Y_{\tau_\infty}^{j_l}), Y_{\tau_\infty}^{j_l} = h_{j_l,j_1}(\tau_\infty, Y_{\tau_\infty}^{j_1}).$$

This contradicts with Assumption 3.1(iv). Therefore, we prove the claim.

Step 4. We are now ready to complete the proof. Given $A_{\eta_i} \neq \emptyset$, if $A_{\eta_{i+1}} = \emptyset$, by (4.32) and (4.40) we have

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_i}|^2 \le E_{\tau_i}\{e^{C(\tau_{i+2}-\tau_i)} \max_{1\le j\le m} |\Delta Y^j_{\tau_{i+2}}|^2 + I_\varepsilon[\tau_{i+2}-\tau_{i+1}]\}. \tag{4.42}$$

By Definition 4.3(v), we have $A_{\eta_{i+2}} \neq \emptyset$. Therefore, if $A_{\eta_i} \neq \emptyset$, then either $A_{\eta_{i+1}} \neq \emptyset$ and (4.32) holds, or $A_{\eta_{i+2}} \neq \emptyset$ and (4.42) holds. Since $A_{\eta_0} \neq \emptyset$, one gets immediately that

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_0}|^2 \le CE_{\tau_0}\{\max_{1\le j\le m} |\Delta Y^j_{\tau_n}|^2 + I_\varepsilon\} = CE_{\lambda_1}\{\max_{1\le j\le m} |Y^{0,j}_{\tau_n} - Y^j_{\tau_n}|^2 + I_\varepsilon\}.$$

First send $n \to \infty$. Since $\tau_n \to \lambda_2$, we get $Y^{0,j}_{\tau_n} \to \xi^j_{\lambda_2}$ and $Y^j_{\tau_n} \to \xi^j_{\lambda_2}$. By Dominating Convergence Theorem we have

$$\lim_{n\to\infty} \max_{1\le j\le m} |\Delta Y^j_{\lambda_1}|^2 \le CE_{\lambda_1}\{I_\varepsilon\}.$$

Now send $\varepsilon \to 0$. Since Y^j is continuous, by Dominating Convergence Theorem again we get

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \max_{1\le j\le m} |\Delta Y^j_{\lambda_1}|^2 = 0.$$

This, together with Lemma 4.5, proves the theorem. ∎

4.4. *Proof of* Theorem 4.2

As mentioned before, we prove the theorem by induction on μ. When $\mu = 0$, (3.11) is an m-dimensional BSDE without reflections. Then (i) holds, and by standard arguments one can easily prove (ii).

Assume Theorem 4.2 holds for $\mu = m_1 - 1$. Now assume $\mu = m_1$.

(i) By Theorem 4.6, $Y^j_{\lambda_1}$ is unique. Similarly Y^j_t is unique for any $t \in [\lambda_1, \lambda_2]$. By the uniqueness of the Doob–Meyer decomposition we get Z^j is unique, which further implies the uniqueness of K^j immediately.
(ii) For any admissible strategy δ, define $\tilde{Y}^{\delta,j}$ similarly and denote

$$\Delta Y^{\delta,j}_t \triangleq Y^{\delta,j}_t - \tilde{Y}^{\delta,j}_t.$$

If $A_{\eta_i} \neq \emptyset$, recalling (4.15), (4.23) and (4.27), by induction we have:

$$\max_{1\le j\le m} |\Delta Y^{\delta,j}_{\tau_i}|^2 \le E_{\tau_i}\left\{e^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_i} |\Delta Y^{\delta,j}_{\tau_{i+1}}|^2 + C\int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt\right\}.$$

If $A_{\eta_i} = \emptyset$, recalling (4.17) and (4.21), by induction we have:

$$\max_{j\neq\eta_{i-1}} |\Delta Y^{\delta,j}_{\tau_i}|^2 \le E_{\tau_i}\left\{e^{C(\tau_{i+1}-\tau_i)} \max_{1\le j\le m} |\Delta Y^{\delta,j}_{\tau_{i+1}}|^2 + C\int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt\right\}.$$

Note that $A_{\eta_0} \neq \emptyset$. Applying the above estimates repeatedly we get:

$$\max_{1\le j\le m} |\Delta Y^{\delta,j}_{\lambda_1}|^2 \le E_{\lambda_1}\left\{e^{C(\lambda_2-\lambda_1)} \max_{1\le j\le m} |\Delta\xi^j_{\lambda_2}|^2 + C\int_{\lambda_1}^{\lambda_2} \|\Delta f_t\|^2 dt\right\}, \quad \forall\delta.$$

Then (ii) follows from Theorem 4.6 immediately. ∎

Acknowledgements

The second author's research was supported in part by NSF grants DMS 04-03575 and DMS 06-31366. Part of the work was done while this author was visiting Université du Maine, whose hospitality is greatly appreciated.

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