The Wellposedness of FBSDEs (II)*

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Abstract. This paper is a continuation of [10], in which we established the wellposedness result and a comparison theorem for a class of one dimensional Forward-Backward SDEs. In this paper we extend the wellposedness result to high dimensional FBSDEs, and weaken the key condition in [10] significantly. Compared to the existing methods in the literature, our result has the following features: (i) arbitrary time duration; (ii) random coefficients; (iii) (possibly) degenerate forward diffusion; and (iv) no monotonicity condition.

Keywords: Forward-backward SDEs, wellposedness.

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1 Introduction and Main Result

Assume $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}_0 \subset \mathcal{F}$, and W is a d-dimensional standard Brownian motion independent of $\mathcal{F}_0$. Let $\mathbf{F} \stackrel{\triangle}{=} \{\mathcal{F}_t\}_{0\le t\le T}$ be the filtration generated by W and $\mathcal{F}_0$, augmented by the null sets as usual. We study the following FBSDE:

$$\left\{\begin{array}{l} X_t = X_0 + \int_0^t b(\omega, s, \Theta_s)ds + \int_0^t \sigma^*(\omega, s, X_s, Y_s)dW_s; \\ Y_t = g(\omega, X_T) + \int_t^T f(\omega, s, \Theta_s)ds - \int_t^T Z_s dW_s. \end{array}\right. \tag{1.1}$$

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where $\Theta \stackrel{\triangle}{=} (X, Y, Z)$ and * denotes the transpose. We assume that $X_0 \in \mathcal{F}_0$, b, σ, f, g are progressively measurable, and for any $\theta \stackrel{\triangle}{=} (x, y, z)$, b, σ, f are $\mathbf{F}$-adapted and $g(\cdot, x) \in \mathcal{F}_T$. For simplicity we will always omit the variable ω in b, σ, f, g.

The wellposedness of FBSDEs has been studied by many authors (see, e.g. [1], [6], [5], [3], [4], [7], [8], and [9]). We refer the readers to [10] for a more detailed introduction on the subject. Motivated by studying numerical methods for (Markovian) FBSDEs (see [2]), in [10] we proved the following theorem.

Theorem 1.1 *Assume that all processes are one dimensional; that b, σ, f, g are differentiable with respect to x, y, z with uniformly bounded derivatives; and that*

$$\partial_y \sigma \partial_z b = 0; \quad \partial_y b + \partial_x \sigma \partial_z b + \partial_y \sigma \partial_z f = 0. \tag{1.2}$$

Denote

$$I_0^2 \stackrel{\triangle}{=} E\Big\{|X_0|^2 + |g(0)|^2 + \int_0^T [|b(t,0,0,0)|^2 + |\sigma(t,0,0)|^2 + |f(t,0,0,0)|^2]dt\Big\}. \tag{1.3}$$

If $I_0^2 < \infty$, then FBSDE (1.1) has a unique solution Θ such that

$$\|\Theta\|^2 \stackrel{\triangle}{=} E\Big\{\sup_{0\le t\le T}[|X_t|^2 + |Y_t|^2] + \int_0^T |Z_t|^2 dt\Big\} \le C I_0^2. \tag{1.4}$$

After [10] has been accepted for publication, we find that Theorem 1.1 can be improved significantly. In the sequel we assume

$$W \in \mathbb{R}^d, \quad X, b \in \mathbb{R}, \quad Y, f, g \in \mathbb{R}^n, \quad Z \in \mathbb{R}^{n\times d}, \quad \sigma \in \mathbb{R}^d. \tag{1.5}$$

Here W, Y, et al are considered as *column* vectors. Let ∂ denote partial derivatives with appropriate dimensions; and $|\cdot|$ denote the Euclidian norm. For example, $\partial_z b \in \mathbb{R}^{n\times d}, \partial_y \sigma \in \mathbb{R}^{d\times n}$ in an obvious way, and $|Y_t|^2 = Y_t^* Y_t, |Z_t|^2 = \mathrm{tr}\,(Z_t^* Z_t)$. Our main result is the following theorem.

Theorem 1.2 *Assume that b, σ, f, g are uniformly Lipschitz continuous in x, y, z; and that there exists a constant $c > 0$ such that*

$$\Lambda_t^4(y) \le -c|\Lambda_t^3(y)|, \tag{1.6}$$

for any $y \in \mathbb{R}^d$ such that $|y| = 1$, where

$$\begin{aligned}\Lambda_t^3(y) &\triangleq \sum_{i=1}^n y_i\Big[\mathrm{tr}\,([\partial_z f^i][\partial_z b]^*) - y^*[\partial_z b][\partial_z f^i]^* y + y^*[\partial_y \sigma]^*[\partial_z f^i]^* y\Big] \\ &\quad +[\partial_x\sigma]^*[\partial_z b]^* y + [\partial_y b]y \\ \Lambda_t^4(y) &\triangleq |\partial_z b|^2 - |[\partial_z b]^* y|^2 + 2y^*[\partial_z b][\partial_y \sigma]y.\end{aligned} \tag{1.7}$$

If $I_0^2 < \infty$, then FBSDE (1.1) has a unique solution Θ such that $\|\Theta\|^2 \le CI_0^2$, where C depends on c and the Lipschitz constant of the coefficients.

We note that we only assume those partial derivatives involved in (1.7) exist. Moreover, when one part of a product vanishes, we do not need to assume the other part to be differentiable. For example, if $\partial_z b = 0$, then we do not need $\partial_x \sigma$. In fact, we can even weaken (1.6) further by using approximating coefficients (see (2.13) at below).

Remark 1.3 *Following are three sufficient conditions for (1.6):*

$$[\partial_z b][\partial_z b]^* - [\partial_y \sigma]^*[\partial_z b]^* - [\partial_z b][\partial_y \sigma] \ge [|\partial_z b|^2 + c]Id_n; \tag{1.8}$$

$$\partial_y b = 0, \quad \partial_z b = 0, \quad [\partial_y \sigma]^*[\partial_z f^i]^* = 0, i = 1, \cdots, n; \tag{1.9}$$

$$n = 1, \quad -[\partial_z b][\partial_y \sigma] \ge c\Big|\partial_y b + [\partial_z f][\partial_y \sigma] + [\partial_z b][\partial_x \sigma]\Big|; \tag{1.10}$$

where $Id_n \in \mathbb{R}^{n\times n}$ is the $n \times n$ identity matrix.

Remark 1.4 *(i) A necessary condition to ensure (1.8) is $n \le d$;*

(ii) There are two typical cases for (1.9). One is that $\partial_y \sigma = 0$, then (1.1) becomes the standard decoupled FBSDE. The other one is that $\partial_z f = 0$, then (1.1) becomes

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma^*(s, X_s, Y_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{1.11}$$

We note that in this case it is allowed to have $n > d$.

Theorem 1.2 improves Theorem 1.1 in three ways. First, there is more freedom on the dimensions; second, (1.10) is obviously much weaker than (1.2); and third, we allow the coefficients to be only Lipschitz continuous (instead of differentiable). We note that the third feature is not trivial because for coefficients satisfying (1.10) (or

(1.8)), their molifiers may fail to satisfy so. We would also like to mention that, as in Theorem 1.1, our result has the following features: 1) T can be arbitrarily large; 2) the coefficients are random; 3) σ can be degenerate; 4) no monotonicity condition is required.

However, we should point out that our method does not work when X is high dimensional, mainly due to the non-commuting property of matrices multiplication. We would leave this case for future research.

2 Small Time Duration

In this section we establish some important results for FBSDEs with small time duration T. First we recall a wellposedness result due to Antonelli [1].

Lemma 2.1 *Assume b, σ, f have a uniform Lipschitz constant K, and g has a uniform Lipschitz constant K_0. There exist constants δ_0 and C_0, depending only on K and K_0, such that for $T \le \delta_0$, if $I_0^2 < \infty$, then (1.1) has a unique solution Θ and it holds that $\|\Theta\| \le C_0 I_0$.*

The following lemma, which estimates the C_0 at above in terms of (K, K_0), is the key step for the proof of Theorem 1.2.

Lemma 2.2 *Consider the following linear FBSDE:*

$$\begin{cases} X_t = 1 + \int_0^t B_s ds + \int_0^t \Gamma_s^* dW_s; \\ Y_t = G X_T + \int_t^T F_s ds - \int_t^T Z_s dW_s; \end{cases} \tag{2.1}$$

where

$$\begin{aligned} B_t &= \alpha_t^1 X_t + \beta_t^1 Y_t + \mathrm{tr}\,(\gamma_t^1 Z_t); \\ \Gamma_t &= \alpha_t^2 X_t + \beta_t^2 Y_t; \\ F_t &= \alpha_t^3 X_t + \beta_t^3 Y_t + [\mathrm{tr}\,(\gamma_t^{3,1} Z_t), \cdots, \mathrm{tr}\,(\gamma_t^{3,n} Z_t)]^*; \end{aligned}$$

and

$$\alpha_t^1 \in \mathbb{R}, \quad \beta_t^{1*}, \alpha_t^3 \in \mathbb{R}^n, \quad \alpha_t^2 \in \mathbb{R}^d, \quad \beta_t^2, \gamma_t^1, \gamma_t^{3,i} \in \mathbb{R}^{d\times n}, \quad \beta_t^3 \in \mathbb{R}^{n\times n}.$$

Assume $|\alpha^i_t|, |\beta^i_t|, |\gamma^i_t| \le K$, $|G| \le K_0$; *and*

$$\Lambda^4_t(y) \le -\frac{1}{K}|\Lambda^3_t(y)|, \tag{2.2}$$

for any $y \in \mathbb{R}$ *such that* $|y| = 1$,

Let δ_0 *be as in Lemma 2.1. There exists a constant* C_K, *depending only on* K *but independent of* K_0, *such that for any* $T \le \delta_0$, *the solution to FBSDE (2.1) satisfies*

$$|Y_0|^2 \le |\bar{K}_0|^2 \stackrel{\triangle}{=} [|K_0|^2 + 1]e^{C_K T} - 1. \tag{2.3}$$

In the sequel we use C_K to denote a generic constant which depends only on K and may vary from line to line. Recalling (1.7) one can easily check that, for linear FBSDE (2.1), we have

$$\begin{aligned} \Lambda^3_t(y) &= \sum_{i=1}^n y_i\Big[\mathrm{tr}\,(\gamma^{3,i}_t\gamma^{1*}_t) - y^*\gamma^{1*}_t\gamma^{3,i}_t y + y^*\beta^{2*}_t\gamma^{3,i}_t y\Big] + \alpha^{2*}_t\gamma^1_t y + \beta^1_t y \\ \Lambda^4_t(y) &= |\gamma^1_t|^2 - |\gamma^1_t y|^2 + 2y^*\beta^{2*}_t\gamma^1_t y. \end{aligned} \tag{2.4}$$

We also note that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ for any matrices A, B with appropriate dimensions.

Proof of Lemma 2.2. The proof is quite lengthy, we split it into two steps.

Step 1. We first assume $X_t \ne 0$ and formally derive some formulas. Note that

$$B_t \in \mathbb{R}; \quad \Gamma_t \in \mathbb{R}^d; \quad F_t \in \mathbb{R}^n.$$

Apply Ito's formula, we have

$$dX_t^{-2} = -2X_t^{-3}dX_t + 3X_t^{-4}\Gamma^*_t\Gamma_t dt = -2X_t^{-3}\Gamma^*_t dW_t - \Big[2X_t^{-3}B_t - 3X_t^{-4}\Gamma^*_t\Gamma_t\Big]dt;$$

and

$$d|Y_t|^2 = d(Y^*_tY_t) = 2Y^*_t dY_t + \mathrm{tr}\,(Z_tZ^*_t)dt = 2Y^*_tZ_t dW_t - \Big[2Y^*_tF_t - \mathrm{tr}\,(Z_tZ^*_t)\Big]dt.$$

Denote

$$\tilde{Y}_t \stackrel{\triangle}{=} Y_tX_t^{-1}; \quad \tilde{\Gamma}_t \stackrel{\triangle}{=} \Gamma_tX_t^{-1}; \quad \tilde{Z}_t \stackrel{\triangle}{=} Z_tX_t^{-1} - \tilde{Y}_t\tilde{\Gamma}^*_t; \quad d\tilde{W}_t \stackrel{\triangle}{=} dW_t - [\tilde{\Gamma}_t + \gamma^1_t\tilde{Y}_t]dt. \tag{2.5}$$

Recalling that $|Z|^2 \triangleq \mathrm{tr}\,(ZZ^*)$. Then

$$
\begin{aligned}
&d|\tilde Y_t|^2 = d(|Y_t|^2 X_t^{-2}) = X_t^{-2} d|Y_t|^2 + |Y_t|^2 dX_t^{-2} + d < |Y|^2, X^{-2} >_t \\
&= 2X_t^{-2} Y_t^* Z_t dW_t - X_t^{-2}[2Y_t^* F_t - |Z_t|^2]dt \\
&\quad -2|Y_t|^2 X_t^{-3}\Gamma_t^* dW_t - |Y_t|^2\Big[2X_t^{-3}B_t - 3X_t^{-4}|\Gamma_t|^2\Big]dt - 4X_t^{-3}Y_t^* Z_t \Gamma_t dt \\
&= \Big[2\tilde Y_t^* Z_t X_t^{-1} - 2|\tilde Y_t|^2 \tilde\Gamma_t^*\Big] dW_t \\
&\quad -2\tilde Y_t^*[\alpha_t^3 + \beta_t^3 \tilde Y_t]dt - 2\sum_{i=1}^n \tilde Y_t^i \mathrm{tr}\,(\gamma_t^{3,i} Z_t X_t^{-1})dt + |Z_t X_t^{-1}|^2 dt \\
&\quad -2|\tilde Y_t|^2\Big[\alpha_t^1 + \beta_t^1 \tilde Y_t + \mathrm{tr}\,(\gamma_t^1 Z_t X_t^{-1})\Big]dt + 3|\tilde Y_t|^2|\tilde\Gamma_t|^2 dt - 4\tilde Y_t^* Z_t X_t^{-1}\tilde\Gamma_t dt \\
&= 2\tilde Y_t^* \tilde Z_t\Big[d\tilde W_t + [\tilde\Gamma_t + \gamma_t^1 \tilde Y_t]dt\Big] + |\tilde Z_t + \tilde Y_t \tilde\Gamma_t^*|^2 dt \\
&\quad -2\mathrm{tr}\,\Big([\sum_{i=1}^n \tilde Y_t^i \gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1][\tilde Z_t + \tilde Y_t\tilde\Gamma_t^*]\Big)dt - 4\tilde Y_t^*[\tilde Z_t + \tilde Y_t \tilde\Gamma_t^*]\tilde\Gamma_t dt \\
&\quad -2\tilde Y_t^*[\alpha_t^3 + \beta_t^3\tilde Y_t]dt - 2|\tilde Y_t|^2[\alpha_t^1 + \beta_t^1 \tilde Y_t]dt + 3|\tilde Y_t|^2|\tilde\Gamma_t|^2 dt \\
&= 2\tilde Y_t^* \tilde Z_t d\tilde W_t + |\tilde Z_t|^2 dt - 2\mathrm{tr}\,\Big(\tilde Z_t[\sum_{i=1}^n \tilde Y_t^i\gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1 - \gamma_t^1 \tilde Y_t \tilde Y_t^*]\Big)dt \\
&\quad +|\tilde Y_t \tilde\Gamma_t^*|^2 dt - 2\mathrm{tr}\,\Big([\sum_{i=1}^n \tilde Y_t^i \gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1]\tilde Y_t \tilde\Gamma_t^*\Big)dt - 4\tilde Y_t^* \tilde Y_t \tilde\Gamma_t^* \tilde\Gamma_t dt \\
&\quad -2\tilde Y_t^*[\alpha_t^3 + \beta_t^3\tilde Y_t]dt - 2|\tilde Y_t|^2[\alpha_t^1 + \beta_t^1\tilde Y_t]dt + 3|\tilde Y_t|^2|\tilde\Gamma_t|^2 dt \\
&\geq 2\tilde Y_t^* \tilde Z_t d\tilde W_t - \Big|\sum_{i=1}^n \tilde Y_t^i \gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1 - \gamma_t^1\tilde Y_t \tilde Y_t^*\Big|^2 dt \\
&\quad -2\mathrm{tr}\,\Big([\sum_{i=1}^n \tilde Y_t^i\gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1]\tilde Y_t\tilde\Gamma_t^*\Big)dt - 2\tilde Y_t^*[\alpha_t^3 + \beta_t^3 \tilde Y_t]dt - 2|\tilde Y_t|^2[\alpha_t^1 + \beta_t^1\tilde Y_t]dt \\
&= 2\tilde Y_t^* \tilde Z_t d\tilde W_t - \Big[|\sum_{i=1}^n \tilde Y_t^i \gamma_t^{3,i}|^2 + |\tilde Y_t|^4|\gamma_t^1|^2 + |\tilde Y_t|^2|\gamma_t^1 \tilde Y_t|^2\Big]dt \\
&\quad +2\Big[\sum_{i=1}^n \tilde Y_t^i \tilde Y_t^* \gamma_t^{1*}\gamma_t^{3,i}\tilde Y_t + |\tilde Y_t|^2|\gamma_t^1\tilde Y_t|^2 - |\tilde Y_t|^2\sum_{i=1}^n \tilde Y_t^i \mathrm{tr}\,(\gamma_t^{3,i}\gamma_t^{1*})\Big]dt \\
&\quad -2\Big[[\alpha_t^2 + \beta_t^2\tilde Y_t]^*[\sum_{i=1}^n \tilde Y_t^i\gamma_t^{3,i} + |\tilde Y_t|^2\gamma_t^1]\tilde Y_t + \tilde Y_t^*[\alpha_t^3 + \beta_t^3\tilde Y_t] + |\tilde Y_t|^2[\alpha_t^1 + \beta_t^1 \tilde Y_t]\Big]dt \\
&\geq 2\tilde Y_t^* \tilde Z_t d\tilde W_t - C_K[1 + |\tilde Y_t|^2]dt \\
&\quad -2\Big[|\tilde Y_t|^2\sum_{i=1}^n \tilde Y_t^i \mathrm{tr}\,(\gamma_t^{3,i}\gamma_t^{1*}) - \sum_{i=1}^n \tilde Y_t^i \tilde Y_t^* \gamma_t^{1*}\gamma_t^{3,i}\tilde Y_t + |\tilde Y_t|^2\alpha_t^{2*}\gamma_t^1\tilde Y_t \\
&\qquad + \sum_{i=1}^n \tilde Y_t^i\tilde Y_t^*\beta_t^{2*}\gamma_t^{3,i}\tilde Y_t + |\tilde Y_t|^2\beta_t^1\tilde Y_t\Big]dt \\
&\quad -\Big[|\tilde Y_t|^4|\gamma_t^1|^2 - |\tilde Y_t|^2|\gamma_t^1\tilde Y_t|^2 + 2|\tilde Y_t|^2\tilde Y_t^*\beta_t^{2*}\gamma_t^1\tilde Y_t\Big]dt.
\end{aligned}
$$

Denote $\bar{Y}_t \triangleq \tilde{Y}_t|\tilde{Y}_t|^{-1}$ when $|\tilde{Y}_t| \neq 0$, and arbitrary unit vector otherwise. Then $|\bar{Y}_t| = 1$ and

$$d|\tilde{Y}_t|^2 \geq 2\tilde{Y}_t^* \tilde{Z}_t d\tilde{W}_t - C_K[1 + |\tilde{Y}_t|^2]dt - \Big[2|\tilde{Y}_t|^3\Lambda_t^3(\bar{Y}_t) + |\tilde{Y}_t|^4\Lambda_t^4(\bar{Y}_t)\Big]dt. \tag{2.6}$$

Step 2. The arguments in this step are similar to those for Lemma 3.2 in [10], so we will only sketch the main idea.

Denote

$$\tau \triangleq \inf\{t > 0 : X_t = 0\} \wedge T; \quad \tau_n \triangleq \inf\{t > 0 : X_t = \frac{1}{n}\} \wedge T.$$

Then $\tau_n \uparrow \tau$ and $X_t > 0$ for $t \in [0, \tau)$. Recall (2.5) for $t \in [0, \tau)$. By Lemma 2.1 one can easily prove that $|Y_t| \leq C_0|X_t|$, and thus

$$|\tilde{Y}_t| \leq C_0, \quad \forall t \in [0, \tau). \tag{2.7}$$

By (2.2) we have

$$2|\tilde{Y}_t|^3\Lambda_t^3(\bar{Y}_t) + |\tilde{Y}_t|^4\Lambda_t^4(\bar{Y}_t) \leq -\frac{1}{K}|\Lambda_t^3(\bar{Y}_t)||\tilde{Y}_t|^4 + 2|\tilde{Y}_t|^3|\Lambda_t^3(\bar{Y}_t)| \leq K|\Lambda_t^3(\bar{Y}_t)||\tilde{Y}_t|^2.$$

Note that $|\Lambda_t^3(\bar{Y}_t)| \leq C_K$. Thus

$$2|\tilde{Y}_t|^3\Lambda_t^3(\bar{Y}_t) + |\tilde{Y}_t|^4\Lambda_t^4(\bar{Y}_t) \leq C_K|\tilde{Y}_t|^2. \tag{2.8}$$

Then by (2.6) one gets

$$d|\tilde{Y}_t|^2 \geq 2\tilde{Y}_t^* \tilde{Z}_t d\tilde{W}_t - C_K[1 + |\tilde{Y}_t|^2]dt; \tag{2.9}$$

In light of (2.5) we define

$$M_t = 1 + \int_0^t M_s[\tilde{\Gamma}_s + \gamma_s^1 \tilde{Y}_s]^* 1_{\{\tau > s\}} dW_s; \quad L_t = e^{C_K t},$$

for the C_K in (2.9). By (2.7) M is a martingale. Moreover,

$$d(L_t M_t |\tilde{Y}_t|^2) \geq (\cdots)dW_t - C_K L_t M_t dt,$$

thanks to the obvious fact that $L_t > 0, M_t > 0$.

Now for each n, we have

$$|\tilde{Y}_0|^2 \leq L_{\tau_n} M_{\tau_n} |\tilde{Y}_{\tau_n}|^2 - \int_0^{\tau_n} (\cdots)dW_t + C_K \int_0^{\tau_n} L_t M_t dt.$$

Thus

$$|\tilde{Y}_0|^2 \le E\Big\{\Gamma_{\tau_n} M_{\tau_n} |\tilde{Y}_{\tau_n}|^2 + C_K \int_0^{\tau_n} L_t M_t dt\Big\}. \tag{2.10}$$

On the other hand, if $\tau = T$, $|Y_\tau| = |Y_T| = |GX_T| = |GX_\tau| \le K_0|X_\tau|$. If $\tau < T$, then $X_\tau = 0$, thus $|Y_\tau| \le C_0|X_\tau| = 0$. Therefore, in both cases it holds that $|Y_\tau| \le K_0|X_\tau|$. By the same arguments as in Lemma 3.2 of [10], one can prove that

$$|\tilde{Y}_{\tau_n}|^2 \le |K_0|^2 + C_K E^{\frac{1}{2}}_{\tau_n}\{|\tau - \tau_n|^2\},$$

which, combined with (2.10), implies that

$$\begin{aligned}
|\tilde{Y}_0|^2 &\le E\Big\{\Gamma_{\tau_n} M_{\tau_n}[|K_0|^2 + C_K E^{\frac{1}{2}}_{\tau_n}\{|\tau-\tau_n|^2\}] + C_K \int_0^{\tau_n} L_t M_t dt\Big\} \\
&\le E\Big\{|K_0|^2 \Gamma_{\tau_n} M_{\tau_n} + C_K \int_0^{\tau_n} L_t M_t dt\Big\} + C_K E^{\frac{1}{2}}\{|\Gamma_{\tau_n} M_{\tau_n}|^2\} E^{\frac{1}{2}}\{|\tau-\tau_n|^2\} \\
&\le E\Big\{|K_0|^2 e^{C_K T} M_{\tau_n} + C_K \int_0^T e^{C_K t} M_t dt\Big\} + C_K E^{\frac{1}{2}}\{|\tau-\tau_n|^2\} \\
&= |K_0|^2 e^{C_K T} + C_K \int_0^T e^{C_K t} dt + C_K E^{\frac{1}{2}}\{|\tau-\tau_n|^2\} \\
&= |\bar{K}_0|^2 + C_K E^{\frac{1}{2}}\{|\tau-\tau_n|^2\}.
\end{aligned}$$

Let $n \to \infty$ and note that $X_0 = 1$, we prove (2.3). ■

We note that estimate (2.8) is essential for the wellposedness of FBSDEs.

Example 1 *Consider the following one dimensional FBSDE*

$$\begin{cases} X_t = 1 - \int_0^t Y_s ds; \\ Y_t = X_T - \int_t^T Z_s dW_s. \end{cases} \tag{2.11}$$

Then

$$\Lambda^3_t(y) = -y, \quad \Lambda^4_t(y) = 0.$$

So (2.2) does not hold true. Note that $\tilde{Y}_T = Y_T X_T^{-1} = 1 > 0$. Actually one can prove in this example that $\tilde{Y}_t > 0$ for any t, then

$$2|\tilde{Y}_t|^3 \Lambda^3_t(\bar{Y}_t) + |\tilde{Y}_t|^4 \Lambda^4_t(\bar{Y}_t) = -2\tilde{Y}_t^3 < 0,$$

which implies (2.8). So we still have $|\tilde{Y}_0| \le \bar{K}_0$. Then by using the arguments in next section we can show that (2.11) is wellposeded for arbitrary T. In fact, (2.11) satisfies the monotonicity condition in [4], and thus its wellposedness is already known.

We would also like to mention that (2.8) is consistent with the four step scheme (see [5] and [3]) in the following sense. Assume an FBSDE in the four step scheme framework has two solutions Θ^1, Θ^2. Denote $\tilde{Y}_t = [Y_t^1 - Y_t^2][X_t^1 - X_t^2]^{-1}$. Note that $Y_t^i = u(t, X_t^i)$ and u is uniformly Lipschitz continuous in x, where u is the solution to the corresponding PDE. Then $\tilde{Y}_t$ is uniformly bounded and thus (2.8) holds true.

The following result connects FBSDEs (1.1) and (2.1).

Corollary 2.3 *Assume that all the conditions in Lemma 2.1 as well as (1.6) hold true with* $c = \frac{1}{K}$*. Let* $T \leq \delta_0$ *as in Lemma 2.1, and* $\Theta^i, i = 0, 1,$ *be the solution to FBSDEs:*

$$\begin{cases} X_t^i = x_i + \int_0^t b(s, \Theta_s^i)ds + \int_0^t \sigma^*(s, X_s^i, Y_s^i)dW_s; \\ Y_t^i = g(X_T^i) + \int_t^T f(s, \Theta_s^i)ds - \int_t^T Z_s^i dW_s. \end{cases}$$

Then $|Y_0^1 - Y_0^0| \leq \bar{K}_0|x_1 - x_0|$*, where* $\bar{K}_0$ *is defined in (2.3).*

Proof. We first assume that all the coefficients are differentiable. For $0 \leq \lambda \leq 1$, let $\Theta^\lambda \stackrel{\triangle}{=} (X^\lambda, Y^\lambda, Z^\lambda)$ and $\nabla\Theta^\lambda \stackrel{\triangle}{=} (\nabla X^\lambda, \nabla Y^\lambda, \nabla Z^\lambda)$ be the solutions to FBSDEs:

$$\begin{cases} X_t^\lambda = x_0 + \lambda(x_1 - x_0) + \int_0^t b(s, \Theta_s^\lambda)ds + \int_0^t \sigma^*(s, X_s^\lambda, Y_s^\lambda)dW_s; \\ Y_t^\lambda = g(X_T^\lambda) + \int_t^T f(s, \Theta_s^\lambda)ds - \int_t^T Z_s^\lambda dW_s. \end{cases}$$

and

$$\begin{cases} \nabla X_t^\lambda = 1 + \int_0^t \Big[\partial_x b(s, \Theta_s^\lambda)\nabla X_s^\lambda + \partial_y b(s, \Theta_s^\lambda)\nabla Y_s^\lambda + \mathrm{tr}\,(\partial_z b^*(s, \Theta_s^\lambda)\nabla Z_s^\lambda)\Big]ds \\ \qquad + \int_0^t [\partial_x \sigma(s, \Theta_s^\lambda)\nabla X_s^\lambda + \partial_y \sigma(s, \Theta_s^\lambda)\nabla Y_s^\lambda]^* dW_s; \\ \nabla Y_t^\lambda = \partial_x g(X_T^\lambda)\nabla X_T^\lambda - \int_t^T \nabla Z_s^\lambda dW_s \\ \qquad + \int_t^T \Big[\partial_x f(s, \Theta_s^\lambda)\nabla X_s^\lambda + \partial_y f(s, \Theta_s^\lambda)\nabla Y_s^\lambda + \sum_{j=1}^n \mathrm{tr}\,(\partial_z f^{j*}(s, \Theta_s^\lambda)\nabla Z_s^{\lambda*})\Big]ds; \end{cases} \tag{2.12}$$

respectively. One can easily prove that

$$\Theta_t^1 - \Theta_t^0 = \int_0^1 \frac{d}{d\lambda}\Theta_t^\lambda d\lambda = [x_1 - x_0]\int_0^1 \nabla\Theta_t^\lambda d\lambda.$$

In particular,

$$Y_0^1 - Y_0^0 = [x_1 - x_0]\int_0^1 \nabla Y_0^\lambda d\lambda.$$

Note that (1.6) implies (2.2) for FBSDE (2.12). Then by Lemma 2.2 we have $|\nabla Y_0^\lambda| \le \bar{K}_0$, and thus

$$|Y_0^1 - Y_0^0| \le |x_1 - x_0| \int_0^1 |\nabla Y_0^\lambda| d\lambda \le \bar{K}_0 |x_1 - x_0|.$$

In general case, for any $\varepsilon > 0$, we may find molifiers $(b^\varepsilon, \sigma^\varepsilon, f^\varepsilon, g^\varepsilon)$ such that

$$\Lambda_t^{4,\lambda,\varepsilon}(y) \le -\frac{1}{K} |\Lambda_t^{3,\lambda,\varepsilon}(y)| + \varepsilon, \tag{2.13}$$

where $\Lambda^{3,\lambda,\varepsilon}$ and $\Lambda^{4,\lambda,\varepsilon}$ are defined in an obvious way, so are other terms such as $\Theta^{\lambda,\varepsilon}$. Denote $\tilde{Y}_t^{\lambda,\varepsilon} \triangleq \nabla Y_t^{\lambda,\varepsilon} [\nabla X_t^{\lambda,\varepsilon}]^{-1}$. By Lemma 2.1 we have $|\tilde{Y}_t^{\lambda,\varepsilon}| \le C_0$ where C_0 may depend on K_0 though. Then we have

$$\begin{aligned}
&2|\tilde{Y}_t^{\lambda,\varepsilon}|^3 \Lambda_t^{3,\lambda,\varepsilon}(\bar{Y}_t^{\lambda,\varepsilon}) + |\tilde{Y}_t^{\lambda,\varepsilon}|^4 \Lambda_t^{4,\lambda,\varepsilon}(\bar{Y}_t^{\lambda,\varepsilon}) \\
&\le 2|\tilde{Y}_t^{\lambda,\varepsilon}|^3 \Lambda_t^{3,\lambda,\varepsilon}(\bar{Y}_t^{\lambda,\varepsilon}) + |\tilde{Y}_t^{\lambda,\varepsilon}|^4 [-\frac{1}{K} |\Lambda_t^{3,\lambda,\varepsilon}(y)| + \varepsilon] \\
&\le C_K |\tilde{Y}_t^{\lambda,\varepsilon}|^2 + \varepsilon |\tilde{Y}_t^{\lambda,\varepsilon}|^4 \le [C_K + \varepsilon C_0^2] |\tilde{Y}_t^{\lambda,\varepsilon}|^2.
\end{aligned}$$

Now for $\varepsilon \le C_0^{-2}$, we know (2.8) holds true for $\tilde{Y}^{\lambda,\varepsilon}$, and thus $|Y_0^{1,\varepsilon} - Y_0^{0,\varepsilon}| \le \bar{K}_0 |x_1 - x_0|$. Let $\varepsilon \to 0$, the lemma follows from the stability result for FBSDEs over small time duration (see [1]). ■

3 Proof of Theorem 1.2

We now prove Theorem 1.2 for arbitrarily large T. The arguments are exactly the same as in [10]. So again we will only sketch the main idea. In the sequel we use L_φ to denote the smallest Lipschitz constant of a function φ.

Proof. Let K and K_0 be as in Lemma 2.1. By otherwise choosing larger K, without loss of generality we assume that $c = \frac{1}{K}$ in (1.6). Define $\bar{K}_0$ as in (2.3) (for the arbitrarily large T!). Let δ_0 be a constant as in Lemma 2.1, but corresponding to $(K, \bar{K}_0)$ instead of (K, K_0). Assume $(m-1)\delta_0 < T \le m\delta_0$ for some integer m. Denote $T_i \triangleq \frac{iT}{m}, i = 0, \cdots, m$. Define a mapping $g_m : \Omega \times \mathbb{R} \to \mathbb{R}$ by $g_m(\omega, x) \triangleq g(\omega, x)$. Now for $t \in [T_{m-1}, T_m]$, consider the following FBSDE:

$$\begin{cases} X_t^m = x + \int_{T_{m-1}}^t b(s, \Theta_s^m) ds + \int_{T_{m-1}}^t \sigma^*(s, X_s^m, Y_s^m) dW_s; \\ Y_t^m = g_m(X_{T_m}^n) + \int_t^{T_m} f(s, \Theta_s^m) ds - \int_t^{T_m} Z_s^m dW_s. \end{cases}$$

Note that $L_{g_m} \le K_0 \le \bar{K}_0$, by Lemma 2.1 the above FBSDE has a unique solution for any x. Define $g_{m-1}(x) \stackrel{\triangle}{=} Y^m_{T_{m-1}}$. Then for fixed x, $g_{m-1}(x) \in \mathcal{F}_{T_{m-1}}$. Moreover, by Corollary 2.3 we have

$$|L_{g_{m-1}}|^2 \le |K_1|^2 \stackrel{\triangle}{=} [|K_0|^2 + 1]e^{C_K(T_m - T_{m-1})} - 1 \le |\bar{K}_0|^2.$$

Next we consider the following FBSDE over $[T_{m-2}, T_{m-1}]$:

$$\begin{cases} X^{m-1}_t = x + \int_{T_{m-2}}^t b(s, \Theta^{m-1}_s)ds + \int_{T_{m-1}}^t \sigma^*(s, X^{m-1}_s, Y^{m-1}_s)dW_s; \\ Y^{m-1}_t = g_{m-1}(X^{m-1}_{T_{m-1}}) + \int_t^{T_{m-1}} f(s, \Theta^{m-1}_s)ds - \int_t^{T_{m-1}} Z^{m-1}_s dW_s. \end{cases}$$

Similarly we may define $g_{m-2}(x)$ such that

$$|L_{g_{m-2}}|^2 \le |K_2|^2 \stackrel{\triangle}{=} [|K_1|^2 + 1]e^{C_K(T_{m-1} - T_{m-2})} - 1 = [|K_0|^2 + 1]e^{C_K(T_m - T_{m-2})} - 1 \le \bar{K}_0.$$

Repeat the arguments for $i = m, \cdots, 1$, we may define g_i such that

$$|L_{g_i}|^2 \le |K_{m-i}|^2 \stackrel{\triangle}{=} [|K_0|^2 + 1]e^{C_K(T_m - T_i)} - 1 \le \bar{K}_0.$$

Now for any $X_0 \in L^2(\mathcal{F}_0)$, we may construct the solution to FBSDE (1.1) piece by piece over subintervals $[T_{i-1}, T_i]$ with terminal condition g_i, $i = 1, \cdots, n$. Since on each subinterval the solution is unique, we obtain the uniqueness of the solution to FBSDE (1.1). Finally, the estimate $\|\Theta\| \le CI_0$ can also be obtained by piece by piece estimates, as done in [10]. ■

Finally we state the stability result whose proof is exactly the same as in [10] and thus is omitted.

Theorem 3.1 *Assume $(b^i, \sigma^i, f^i, g^i, X^i_0), i = 0, 1$, satisfy all the conditions in Theorem 1.2. Let Θ^i be the corresponding solutions, $\Delta\Theta \stackrel{\triangle}{=} \Theta^1 - \Theta^0$, $\Delta g \stackrel{\triangle}{=} g_1 - g_0$, and define other terms similarly. Then*

$$\|\Delta\Theta\|^2 \le CE\Big\{|\Delta X_0|^2 + |\Delta g(X^1_T)|^2 + \int_0^T \Big[|\Delta b|^2 + |\Delta\sigma|^2 + |\Delta f|^2\Big](t, \Theta^1_t)dt\Big\}.$$

Corollary 3.2 *Assume $(b^n, \sigma^n, f^n, g^n, X^n_0), n = 0, 1, \cdots$ satisfy all the conditions in Theorem 1.2 uniformly; $X^n_0 \to X^0_0$ in L^2; for $\varphi = b, \sigma, f, g$ and for any (t, θ), $\varphi^n(t, \theta) \to \varphi^0(t, \theta)$ as $n \to \infty$; and*

$$E\Big\{|X^n_0 - X_0|^2 + |g^n - g^0|^2(0) + \int_0^T [|b^n - b^0|^2 + |\sigma^n - \sigma^0|^2 + |f^n - f^0|^2](t, 0, 0, 0)dt\Big\} \to 0.$$

Let Θ^n denote the corresponding solutions. Then $\|\Theta^n - \Theta^0\| \to 0$.

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