REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1996

Problem 1 (Stability of contractive iteration) Let \((M, d)\) be a metric space, and suppose \(T : M \to M\) satisfies
\[
d(Tx, Ty) \leq k \cdot d(Tx, Ty) \quad \text{for all } x, y \in M
\]
where \(0 < k < 1\). Now suppose \(\varepsilon > 0\), and a sequence \(\{\tilde{x}_n\}_{n=0}^{\infty}\) in \(M\) satisfies
\[
d(\tilde{x}_n, T\tilde{x}_{n-1}) < \varepsilon \quad \text{for all } n \geq 1
\]
Prove that for \(0 \leq m < n\),
\[
d(\tilde{x}_m, \tilde{x}_n) < k^a \frac{2d(\tilde{x}_0, T\tilde{x}_0)}{1 - k} + \frac{2\varepsilon}{1 - k}
\]

Problem 2 How many zeros does the polynomial \(p(z) = z^4 - 2z + 3\) have in the unit disk \(|z| < 1\)?

Problem 3 Suppose \(f : \mathbb{R} \to \mathbb{R}\) is Lebesgue integrable and
\[
\int_{-\infty}^{\infty} \varphi(x) f(x) \, dx = 0
\]
for all continuous functions \(\varphi : \mathbb{R} \to \mathbb{R}\) which have compact support. Prove: \(f(x) = 0\) for a.e. \(x\).

Problem 4 Evaluate
\[
\int_{0}^{\pi} \frac{d\theta}{2 + \sin \theta}
\]

Problem 5 Let \((X, \Sigma, \mu)\) be a measure space with \(\mu(X) < \infty\), and let \(M\) denote the space of \(\Sigma\)-measurable extended-real-valued functions on \(X\). Define \(\rho : M \times M \to \mathbb{R}\) by
\[
\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu
\]
Show that \(\rho\) is a metric on \(M\), and that \(f_n \to f\) in the \(\rho\)-metric iff \(f_n \to f\) in measure.

Problem 6 Suppose \(f : \mathbb{C} \to \mathbb{C}\) is an entire function. Prove that there exists a point \(z_0 \in \mathbb{C}\) such that we can expand \(f(z)\) into a power series about \(z_0\),
\[
f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n
\]
for which all \(c_n \neq 0\).

Problem 7 Suppose \(f : \mathbb{R}^2 \to \mathbb{R}\) has continuous partial derivatives \(f_{xy}\) and \(f_{yx}\). Prove \(f_{xy} = f_{yx}\).

\textbf{Hint:} use Fubini's theorem to integrate \(f_{xy}\) and \(f_{yx}\) over a rectangle \([a, b] \times [c, d]\).

Problem 8 Find a conformal mapping from the unit disk \(|z| < 1\) to the region
\[
\Omega = \{x + iy : (x < 0) \text{ and } (y > 0), \text{ or } (x \geq 0) \text{ and } (y > b)\}
\]
where \(b > 0\).
REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1997

Directions: Do any **seven** of the following eight problems.

**Problem 1** Prove: if \( n \geq 2 \) is an integer, then
\[
\int_0^\infty \frac{dx}{1 + x^n} = \frac{x/n}{\sin(\pi/n)}
\]

**Problem 2** Suppose \( \Omega \) is an open connected region of the complex plane and \( f \) is a non-constant analytic function on \( \overline{\Omega} \). Prove: if \( |f(z)| = 1 \) on the boundary of \( \Omega \), then \( f(z) \) has at least one zero in \( \Omega \).

**Problem 3** Formally, we have that
\[
\frac{(-1)^n n!}{t^{n+1}} = \frac{d^n}{dt^n} \left( \frac{1}{t} \right) = \frac{d^n}{dt^n} \int_0^\infty e^{-tx} \, dx
\]
\[
= \int_0^\infty \frac{\partial^n}{\partial t^n} e^{-ix} \, dx = \int_0^\infty (-1)^n x^n e^{-ix} \, dx
\]
so that on setting \( t = 1 \) we obtain
\[
\int_0^\infty x^n e^{-x} \, dx = n!
\]

Justify the calculation.

**Problem 4** Let \( X = C([0,1]) \) be the space of all bounded continuous functions from \([0,1]\) to \( \mathbb{R} \) with the sup-norm distance,
\[
d(f,g) = \sup_{0 \leq t \leq 1} |f(t) - g(t)|
\]
You may assume that \((X,d)\) is complete. Let \( F : X \to X \) be a strict contraction, i.e., a function such that there exists \( k < 1 \) with
\[
d(Fx,Fy) \leq kd(x,y) \quad \text{for all } x, y \in X
\]
Let \( I \) denote the identity operator on \( X \), prove:
- \( I + F \) is a 1-1 mapping of \( X \) onto \( X \)
- \((I + F)^{-1}\) is continuous

**Problem 5** Let \( K : [0,1] \times [0,1] \to \mathbb{R} \) be continuous, and let \( \mathcal{F} \) be the family of all functions \( f \) on \([0,1]\) of the form
\[
f(x) = \int_0^1 g(y)K(x,y) \, dy
\]

**Problem 6** Show that for each \( \varepsilon > 0 \) the function
\[
f(z) = \sin z + \frac{1}{z}
\]
has infinitely many zeros in the strip \( |\Im z| < \varepsilon \).

**Problem 7** Determine the order of the entire function
\[
f(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n^2} \right)
\]
(Recall that the order of an entire function \( f \) is)
\[
\lim_{r \to \infty} \frac{\log \log M(r)}{r}
\]
where \( M(r) = \max_{|z| = r} |f(z)| \).

**Problem 8** Prove: if \( A \) and \( B \) are Lebesgue-measurable subsets of \( \mathbb{R} \) with positive Lebesgue measure, then the set
\[
A + B = \{ a + b : a \in A, b \in B \}
\]
has non-empty interior. (Hint: consider the convolution of the characteristic functions of \( A \) and \( B \).)
Problem 1. Let $\mu$ be a finite measure on $(X, B)$, and let $\alpha$ be a positive function on $\mathbb{R}$ such that $\alpha(x)/x \to \infty$ as $x \to \infty$. Suppose $f_n \to f$ a.e. on $X$, and $\|\alpha \circ f_n\|_1 \leq M < \infty$ for all $n \in \mathbb{N}$.

(a) Show that

$$\sup_n \|f_n \chi_{\{f_n \geq K\}}\|_1 \to 0, \quad \text{as} \quad K \to \infty.$$  

(Here $\chi_A$ denotes the characteristic function of the set $A$.)

(b) Show that for each $K > 0$, $f_n \chi_{\{f_n < K\}} \to f \chi_{\{f < K\}}$ in $L^1$ as $n \to \infty$.

(c) Show that $f_n \to f$ in $L^1$ as $n \to \infty$. 
Problem 2. Consider a nonnegative valued function $f \in L^1(\mu)$. Suppose $c \geq 0$, $A = \{x \in X : f(x) \geq c\}$, and $\mu(A) = \delta > 0$. Show that

$$\sup_{\{B: \mu(B) \leq \delta\}} \int_B f \, d\mu = \int_A f \, d\mu.$$
Problem 3. Show that for every $f, g \in L^1(\mathbb{R}, \mu)$, the following equality holds

$$
\lim_{h \to 0} \frac{1}{h} \left( \int |f + hg| \, d\mu - \int |f| \, d\mu \right) = \int_{\Sigma_0} g(x) \text{sign}(f(x)) \, d\mu + \int_{\Sigma_0} |g(x)| \, d\mu.
$$

where $\mu$ is a measure on $\mathbb{R}$ and $\Sigma_0 = \{ x : f(x) = 0 \}$. 
SPRING 1998 ANALYSIS QUALIFYING EXAM
MONDAY, MAY 4, 1998

DIRECTIONS. Do any seven of the following eight problems, using the paper and pens provided. Start each problem on a fresh sheet of paper. When you have completed the exam, be sure your name is printed on each page; sign the envelope, and return the exam papers in the envelope. You may keep this printed page.

Problem 1. Suppose $f \in L^1(\mu)$. Prove: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set $A$ with $\mu(A) < \delta$, there holds

$$\left| \int_A f \, d\mu \right| < \varepsilon.$$  

Problem 2. Let $f$ be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove: if $f$ maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

Problem 3. Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0, 1]$ which converges in the $L^1$ sense to 0, and whose derivatives $\{f'_n\}$ also converge to 0 in the $L^1$ sense. Prove: $\{f_n\}$ converges to zero uniformly.

Problem 4. Suppose $D$ is the open unit disk in $\mathbb{C}$, and $f : D \to D$ satisfies $f(1/2) = 1/2$. Show that $|f'(1/2)| \leq 3/4$.

Problem 5. Let $(X, T)$ be a topological space which has the property that every closed set $F$ is the intersection of a countable family of open sets. Prove: any finite measure $\mu$ on the Borel field of $(X, T)$ is regular: for each Borel set $E$ and each $\varepsilon > 0$, there exist an open set $G \supset E$ and a closed set $F \subset E$ such that $\mu(G \setminus F) < \varepsilon$. (Hint: consider the collection of Borel sets $F$ for which this condition is true.)

Problem 6. Let $D$ be the open unit disk in $\mathbb{C}$, and let $f : D \to D$ be analytic with $f(0) = 0$. Suppose

$$|f(z)| \geq \frac{1}{6} \quad \text{for all} \quad |z| = \frac{1}{4}.$$  

Show that $f$ assumes every value in the disk $|w| < \frac{1}{6}$.

Problem 7. Let $g : [0, 1] \to \mathbb{R}$ be Lebesgue measurable, and suppose $f(x, y) := g(x) - g(y)$ is Lebesgue integrable on $[0, 1] \times [0, 1]$. Prove: $g$ is Lebesgue integrable.

Problem 8. Evaluate: $\int_0^\infty \frac{\sqrt{x}}{1 + x^3} \, dx$.

Q4: ... $f$ analytic ... Show that $|f'(1/2)| \leq 1$

announced 11:38 am
Problem 4. Suppose the measures $\mu_n$, $n \geq 1$, on $(X, B)$ are uniformly absolutely continuous with respect to some finite measure $\nu$, that is, given $\epsilon > 0$ there is $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu_n(A) < \epsilon$ for all $n$. Suppose also that $d\mu_n/d\nu \rightarrow f$, $\nu$-a.e. Show that there exists a measure $\mu$ such that $\mu_n(A) \rightarrow \mu(A)$ for all measurable $A$. Identify the measure $\mu$.

Problem 5. Let $\mu$ be a finite measure on the Borel sets in a separable metric space $X$, and define $\text{supp}(\mu) = \{x \in X : \mu(D) > 0 \text{ for every open set } D \text{ containing } x\}$ and $G = (\text{supp}(\mu))^c$.

(a) Show that $G$ is open.

(b) Show that $\mu(G) = 0$.

(c) If $B$ is open and $\mu(B) = 0$, show that $B \subseteq G$.

Problem 6. Consider the following modes of convergence of functions $f_n \rightarrow f$:

(i) almost everywhere
(ii) in measure
(iii) in $L^1$
(iv) almost uniformly
(v) uniformly.

(a) Which implications among these are valid in all measure spaces? You need not prove these; just list all of them or make a diagram.

(b) Let $\mu$ be counting measure on the positive integers. What additional implications, if any, are valid in this special case? (Prove these implications; you may use any of the implications in (a) without proof.)
Analysis Qualifying Exam
Spring, 1999

- In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.
- Start each problem on a fresh sheet of paper.

**REAL ANALYSIS. Do only three of the following four problems.**

1. Suppose \( f_n \), where \( n = 1, 2, \ldots \), and \( f \) are nonnegative functions on a measure space \((X, \mathcal{M}, \mu)\) with \( f_n \to f \) a.e. and \( \int_X f_n \, d\mu \to \int_X f \, d\mu \). Show that \( \int_E f_n \, d\mu \to \int_E f \, d\mu \) for every measurable \( E \). (Hint: Use Fatou's Lemma.)

2. Let \((X, \mathcal{M})\) and \((Y, \mathcal{N})\) be measurable spaces and \( E \in \mathcal{M} \otimes \mathcal{N} \) (the product \( \sigma \)-algebra in \( X \times Y \)). Show that every section \( E_x = \{ y \in Y : (x, y) \in E \} \) is measurable.

3. Let \( A \) denote the set of all \( f \in C[0, 1] \) such that \( f \) is monotonic on some open subinterval of \([0, 1] \). Show that \( A \) is meager (that is, of the first category) in \( C[0, 1] \) in the topology of uniform convergence.

4. (a) Show that the class of all step functions, of form \( \sum_{j \leq n} c_j \chi_{(a_j, b_j)} \) with \( a_j, b_j \) finite, is dense in \( L^1(\mu) \) where \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). (Hint: Why is the corresponding statement true for simple functions?)

(b) Suppose \( f \in L^1(\mu) \). Show that \( \lim_{h \to 0} \int |f(x + h) - f(x)| \, dx = 0 \). (Hint: Use (a).)

**COMPLEX ANALYSIS. Do all four problems.**

5. Suppose that \( f \) is analytic on \( \mathbb{C} \) and that \( f \) is a homeomorphism of \( \mathbb{C} \) onto a set \( U \).
   (a) Show that \( f \) has a non-essential singularity at \( \infty \).
   (b) Deduce that \( f \) must be of the form \( f(z) = az + b \) for some \( a \neq 0 \) and that \( U = \mathbb{C} \).

6. (a) Suppose that \( f \) is analytic on the open unit disc \(|z| < 1\) and there is a constant \( M \) such that \( |f^{(k)}(0)| \leq k^2 M^k \) for all \( k \geq 1 \). Show that \( f \) can be extended to be analytic on \( \mathbb{C} \).

(b) Suppose that \( f \) is analytic on the open unit disc \(|z| < 1\) and there is a constant \( M > 1 \) such that \(|f(1/k)| \leq M^{-k} \) for \( k \geq 2 \). Show that \( f \) is identically zero.
In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

REAL ANALYSIS. Answer question 1 and any two of the other three questions.

1. Let \( \{f_n\} \) be a sequence of functions on \((X, A, \mu)\). Suppose \( \{f_n\} \) is Cauchy in measure, that is, for every \( \varepsilon > 0 \) there exists \( N \) such that \( m, n \geq N \) implies

\[
\mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) < \varepsilon.
\]

Show that there exists \( f \) such that \( f_n \to f \) in measure. HINT: For \( n_1 < n_2 < \ldots \) you can write \( f_{n_k} - f_{n_j} = \sum_{j=1}^{k} (f_{n_j} - f_{n_{j-1}}) \).

2. Suppose \( f : [0, 1] \to \mathbb{R} \) is a nonnegative Lebesgue measurable function satisfying

\[
\int f^n \, dm = \int f \, dm \quad \text{for all} \ n \geq 1.
\]

Show that \( f \) is the characteristic function \( \chi_E \) of some measurable \( E \subset [0, 1] \).

3. Let \((X, A, \mu)\) be a measure space and let \( m \) denote Lebesgue measure on \([0, 1]\). Suppose \( F_n \) and \( F \) map \( X \times [0, 1] \) into \( \mathbb{R} \) and satisfy

(i) \( F_n(x, \cdot) \) is absolutely continuous and nondecreasing for all \( n, x \);
(ii) \( F(x, \cdot) \) is continuously differentiable for all \( x \);
(iii) \( \frac{\partial}{\partial t} F_n(x, t) \to \frac{\partial}{\partial t} F(x, t) \) for almost every \( x, t \);
(iv) \( |F_n(x, t) - F_n(x, 0)| \leq tg(x) \) for all \( n, x, t \), where \( g \) is an integrable function on \( X \).

Show that

\[
\frac{\partial}{\partial t} \int_X F(x, t) \, d\mu(x) \bigg|_{t=0} = \int_X \frac{\partial F}{\partial t} (x, 0) \, d\mu(x).
\]

4. Let \( X \) be a metric space and \( \mu \) a regular Borel measure on \((X, B)\) with \( \mu(X) = 1 \). Let \( \mathcal{E} = \{F \in B : F \text{ closed, } \mu(F) = 1\} \) and \( H = \bigcap_{F \in \mathcal{E}} F \). (Note regular means \( \mu(E) = \sup\{\mu(K) : K \text{ compact, } K \subset E\} = \inf\{\mu(U) : U \text{ open, } E \subset U\} \) for all \( E \in B \).)

(a) Show that \( \mathcal{E} \) is closed under finite intersections.
(b) Show that \( \mu(H) = 1 \). HINT: Show that \( \mu(H^c) = 0 \).

COMPLEX ANALYSIS. See next page.
In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other.

Start each problem on a fresh sheet of paper, and write on only one side of the paper.

REAL ANALYSIS. Answer any three of the four questions.

1. Suppose that $F_n$, $n \geq 1$ are nondecreasing right-continuous functions on $\mathbb{R}$ and $F = \sum F_n$ is finite.
   (a) Show that $F$ is right continuous.
   (b) Suppose $F_n'' = 0$ a.e. for all $n$. Show that $F'' = 0$ a.e.
   [Hint: consider the corresponding measures $\mu_n$, $\mu$ with $\mu_n((a, b]) = F_n(b) - F_n(a)$].

2. Let $(X, B, \nu)$ be a finite measure space and $f_n$, $f$ non-negative bounded measurable functions on $X$. Define measures $\mu_n$, $\mu$ on $(X, B)$ by
   \[ \mu_n(A) = \int_A f_n \, d\nu \quad \text{and} \quad \mu(A) = \int_A f \, d\nu. \]
   (a) Show that if $f_n \to f$ in $L^1(\nu)$ then $\sup_{A \in B} |\mu_n(A) - \mu(A)| \to 0$.
   (b) Conversely show that if $\sup_{A \in B} |\mu_n(A) - \mu(A)| \to 0$ then $f_n \to f$ in $L^1(\nu)$.

3. Let $\mu^*$ be an outer measure on $X$ and suppose $Y$ is a $\mu^*$-measurable subset of $X$. Let $\nu^*$ be the restriction of $\mu^*$ to subsets of $Y$. Show that a set $E \subset Y$ is $\nu^*$-measurable if and only if $E$ is $\mu^*$-measurable.

4. Let $f, f_1, \ldots, f_n, \ldots$ be measurable functions from the measure space $(E, \mathcal{E}, \mu)$ to an open subset $\Omega$ of $\mathbb{R}^d$ such that for all $\varepsilon > 0$,
   \[ \mu\left(\{x \in E : \|f(x) - f_n(x)\| \geq \varepsilon\}\right) \to 0 \quad \text{as } n \to \infty. \]
   Assume that $\mu$ is finite. Show that for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $\mu(\{x : f(x) \notin K\}) \leq \varepsilon$ and $\mu(\{x : f_n(x) \notin K\}) \leq \varepsilon$ for every $n$.

COMPLEX ANALYSIS. See next page.
(1) Let $\mu$ be a finite Borel measure on $\mathbb{R}$ and let

$$f(x) = \int_{\mathbb{R}} \frac{d\mu(y)}{|x-y|^{1/2}}$$

Here $\frac{1}{|x-y|^{1/2}}$ should be interpreted as $+\infty$ when $x=y$.

(a) Prove that $f$ is finite a.e. with respect to Lebesgue measure on $\mathbb{R}$. HINT: Consider $[-M, M]$ in place of $\mathbb{R}$.

(b) Show that $f$ need not be finite a.e. with respect to $\mu$.

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions on $X$ such that $\{f_n(x)\}$ is a Cauchy sequence for almost every $x$. Show that for each $\epsilon > 0$ there is a measurable $E \subset X$ and a finite $M$ such that $\mu(X \setminus E) < \epsilon$ and $|f_n(x)| \leq M$ for all $x \in E$ and $n \geq 1$.

(3) Suppose $\{\mu_n\}$ is a sequence of finite measures on $(X, \mathcal{M})$ and $\mu_n \rightarrow \mu$ uniformly on $\mathcal{M}$, for some set function $\mu$. Show that $\mu$ is countably additive. (Note: We don’t assume $\mu$ is a measure.) HINT: For $E_1, E_2, \ldots$ disjoint and $k \geq 1$, consider $\mu(\bigcup_{i=1}^{\infty} E_i) - \sum_{i=1}^{k} \mu(E_i)$.

(4) Suppose $\mu_1, \nu_1$ are positive $\sigma$-finite measures on $(X_1, \mathcal{M}_1)$ and $\mu_2, \nu_2$ are positive $\sigma$-finite measures on $(X_2, \mathcal{M}_2)$, with $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Show that $\mu_1 \times \mu_2 \ll \nu_1 \times \nu_2$. (Here $\ll$ denotes absolute continuity.)