1. (a) Let $Z_i$ be independent $N(0, 1), i = 1, 2, \cdots, n$. Are $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2$ independent? Prove your claim.

(b) Let $X_1, X_2, \cdots, X_n$ be independent identically distributed normal with mean $\theta$ and variance $\theta^2$, where $\theta > 0$ is unknown. Let

$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$

Are $\bar{X}$ and $S^2$ independent? Prove your claim. (Hint: you can directly use the result in the first part of this problem.)

(c) Show that $(\bar{X}, S^2)$ is a sufficient statistic for $\theta$, but it is not complete.

2. (a) Let $X_1, X_2, \cdots, X_n$ be exponentially distributed with density

$f(x) = \lambda \exp(-\lambda x), \quad x > 0.$

Let $c > 0$ be a constant and if $X_i < c$, we observe $X_i$, otherwise we observe $c$.

$S_n = \sum_{i=1}^{n} X_i I(X_i < c), \quad T_n = \sum_{i=1}^{n} I(X_i > c),$

where $I(A) = 1$ if event $A$ occurs and $I(A) = 0$, otherwise. Write down the likelihood function of the observed values in terms of $T_n$ and $S_n$.

(b) Show the maximum likelihood estimator of $\lambda$ is

$\hat{\lambda}_n = \frac{n - T_n}{S_n + cT_n}.$
1. (a) Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ be an i.i.d. sample. Find the method of moments estimate $\hat{\lambda}_{\text{MOM}}$ and the maximum likelihood estimate $\hat{\lambda}_{\text{MLE}}$ of $\lambda$.

(b) Is $\hat{\lambda}_{\text{MLE}}$ unbiased? Is it efficient?

(c) Give an example of a distribution where the MOM estimate and the MLE are different.

2. (a) Prove that, for any (possibly correlated) collection of random variables $X_1, \ldots, X_k$,

$$\text{Var} \left( \sum_{i=1}^{k} X_i \right) \leq k \sum_{i=1}^{k} \text{Var}(X_i). \quad (1)$$

(b) Construct an example with $k \geq 2$ where equality holds in (1).
1. (a) Consider an independent identically distributed sequence $X_1, X_2, \cdots, X_{n+1}$ taking values 0 or 1 with probability distribution
   
   \[ P\{X_i = 1\} = 1 - P\{X_i = 0\} = p. \]

   Uniformly choose $M$ fragments $F_1, F_2, \cdots, F_M$ of length 2 starting in the interval $[1, n]$, that is, $F_i = (X_{j_i}, X_{j_i+1})$ for some $1 \leq j_i \leq n$. Let $W = (1, 1)$.

   - Let $N_W$ be the number of times the word $W$ occurs among the $M$ fragments. Calculate $E(N_W)$.
   - Calculate the probability $P(F_1 = W, F_2 = W)$.
   - Calculate $\text{Var}(N_W)$.

   (Note: Due to time constraints, you can ignore the boundary effect.)

2. Let $T$ and $C$ be independent Geometric random variables with success probability of $r$ and $s$, respectively. That is
   
   \[ P[T = j] = r(1-r)^{j-1}; j = 1, 2, \cdots, \]
   \[ P[C = j] = s(1-s)^{j-1}; j = 1, 2, \cdots, \]

   Let $X = (\min(T, C), I(T \leq C))$. Denote $X_1 = \min(T, C), X_2 = I(T \leq C)$, where $I(\cdot)$ is the indicator function.

   (a) What is the joint distribution of $X$?
   (b) Calculate $E(X) = (EX_1, EX_2)$ and the covariance matrix of $X = (X_1, X_2)$.
   (c) Let $T_1, T_2, \cdots, T_n$ be a random sample from $T$, and $C_1, C_2, \cdots, C_n$ be a random sample from $C$. Define

   \[ S_1 = \sum_{i=1}^{n} \min(T_i, C_i) \]
   \[ S_2 = \sum_{i=1}^{n} I(T_i \leq C_i). \]

   What is the maximum likelihood estimate $(\hat{r}, \hat{s})$ of $(r, s)$, in terms of $S_1$ and $S_2$?
Fall 2013 Math 541a Exam

1. For \( p \in (0, 1) \) unknown, let \( X_0, X_1, \ldots \) be independent identically distributed random variables taking values in \( \{0, 1\} \) with distribution
\[
P(X_i = 1) = 1 - P(X_i = 0) = p,
\]
and suppose that
\[
T_n = \sum_{i=0}^{n-1} I(X_i = 1, X_{i+1} = 1).
\] (1)
is observed.

(a) Calculate the mean and variance of \( T_n \).

(b) Find a consistent method of moments \( \hat{p}_n = g_n(T_n) \) estimator for the unknown \( p \) as a function \( g_n \) of \( T_n \) that may depend on \( n \), and prove that your estimate is consistent for \( p \).

(c) Show that \( T_n \) is not the sum of independent, identically distributed random variables. Nevertheless, determine the non-trivial limiting distribution of \( \hat{p}_n \), after an appropriate centering and scaling, as if (1) was the sum of i.i.d. variables and has the same mean and variance as the one computed in part (a).

(d) Explain why you would, or would not, expect \( \hat{p}_n \) to have the same limiting distribution as the one determined in part (c).

2. Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed random variables with density given by
\[
f_\beta(x) = \frac{x^{\alpha-1}}{\beta^{\alpha} \Gamma(\alpha)} \exp(-x/\beta), \text{ for } x > 0
\]
where \( \alpha > 0 \), and is known. Suppose it is desired to estimate \( \beta^3 \).

(a) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of \( \beta^3 \).

(b) Find a complete and sufficient statistic for \( \beta \). Then, compute its \( k^{th} \) moment, where \( k \) is an positive integer.

(c) If a UMVUE (uniform minimum variance unbiased estimator) exists, find its variance and compare it to the bound in part (a).
1. For known values $x_{i,1}, x_{i,2}, i = 1, \ldots, n$ let
\[ Z_i = \beta_1 x_{i,1} + \epsilon_i \]
and
\[ Y_i = \beta_1 x_{i,1} + \beta_2 x_{i,2} + \epsilon_i \quad i = 1, \ldots, n, \]
where $\epsilon_i, i = 1, 2, \ldots, n$ are independent normal random variables with mean 0 and variance 1.

(a) Given the data $Z = (Z_1, \ldots, Z_n)$ compute the maximum likelihood estimate of $\beta_1$ and show that it achieves the Cramer-Rao lower bound. Throughout this part and the following, make explicit any non-degeneracy assumptions that may need to be made.

(b) Based on $Y = (Y_1, \ldots, Y_n)$, compute the Cramer-Rao lower bound for the estimation of $(\beta_1, \beta_2)$, and in particular compute a variance lower bound for the estimation of $\beta_1$ in the presence of the unknown $\beta_2$.

(c) Compare the variance lower bound in (a), which is the same as the one for the model for $Y_i$ where $\beta_2$ is known to be equal to zero, to the one in (b), where $\beta_2$ is unknown, and show the latter one is always at least as large as the former.

2. Suppose we observe the pair $(X, Y)$ where $X$ has a Poisson($\lambda$) distribution and $Y$ has a Bernoulli($\lambda/(1 + \lambda)$) distribution, that is,
\[ P_\lambda(X = j) = \frac{\lambda^j e^{-\lambda}}{j!}, j = 0, 1, 2, \ldots \]
and
\[ P_\lambda(Y = 1) = \frac{\lambda}{1 + \lambda} = 1 - P_\lambda(Y = 0), \]
with $X$ and $Y$ independent, and $\lambda \in (0, \infty)$ unknown.

(a) Find a one-dimensional sufficient statistic for $\lambda$ based on $(X, Y)$.

(b) Is there a UMVUE (uniform minimum variance unbiased estimator) of $\lambda$? If so, find it.

(c) Is there a UMVUE of $\lambda/(1 + \lambda)$? If so, find it.
1. Let $p, q$ be values in $[0, 1]$ and $\alpha \in (0, 1]$. Assume $\alpha$ and $q$ known, and that $p$ is an unknown parameter we would like to estimate. A coin is tossed $n$ times, resulting in the sequence of zero one valued random variables $X_1, \ldots, X_n$. At each toss, independently of all other tosses, the coin has probability $p$ of success with probability $\alpha$, and probability $q$ of success with probability $1 - \alpha$.

(a) Write out the probability function of the observed sequence, and compute the maximum likelihood estimate $\hat{p}$ of $p$, when $p$ is considered a parameter over all of $\mathbb{R}$. Verify that when $\alpha = 1$ one recovers the standard estimator of the unknown probability.

(b) Show $\hat{p}$ is unbiased, and calculate its variance.

(c) Calculate the the information bound for $p$, and determine if it is achieved by $\hat{p}$.

(d) If one of the other parameters is unknown, can $p$ still be estimated consistently?

2. Let $X \in \mathbb{R}^n$ be distributed according the density or mass function $p(x; \theta)$ for $\theta \in \Theta \subset \mathbb{R}^d$.

(a) State the definition for $T(X)$ to be sufficient for $\theta$.

(b) Prove that if the (discrete) mass functions $p(x; \theta)$ can be factored as $h(x)g(T(x), \theta)$ for some functions $h$ and $g$, then $T(X)$ is sufficient for $\theta$.

(c) Let $X_1, \ldots, X_n$ be independent with the Cauchy distribution $C(\theta), \theta \in \mathbb{R}$ given by

$$p(x; \theta) = \frac{1}{\pi(1 + (x - \theta)^2)}.$$ 

Prove that the unordered sample $S = \{X_1, \ldots, X_n\}$ can be determined from any $T(X)$ sufficient for $\theta$. (Hint: Produce a polynomial from which $S$ can be determined).
1. Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be a sample from the uniform distribution on a disc \(X^2 + Y^2 \leq \theta^2\), where \(\theta > 0\) is unknown. That is, the probability density function of \((X, Y)\) is

\[
f_{(X,Y)}(x,y;\theta) = \frac{1}{\pi \theta^2} 1_{[0,\theta]}(\sqrt{x^2 + y^2})
\]

(a) Find a complete sufficient statistic of \(\theta\) and its distribution.
(b) Find the UMVU estimator of \(\theta\).
(c) Find the maximum likelihood estimator of \(\theta\).

2. Let \(Y_1, \ldots, Y_n\) be independent with \(Y_i \sim N(\alpha x_i + \beta \log x_i, \sigma^2)\), where \(x_1, \ldots, x_n\) are given positive constants, not all equal, and \(\alpha, \beta, \sigma\) are unknown parameters.

(a) Prove that the MLE of \(\beta\) is

\[
\hat{\beta} = \frac{S_{ly}S_{x2} - S_{lx}S_{xy}}{S_{x2}^2 S_{lx} - S_{lx}^2},
\]

where

\[
S_{ly} = \sum_i (\log x_i)Y_i, \quad S_{x2} = \sum_i x_i^2, \quad S_{lx} = \sum_i (\log x_i), \quad etc.
\]

(b) Find the distribution of \(\hat{\beta}\), including giving any parameter values for this distribution. Is \(\hat{\beta}\) unbiased for \(\beta\)? Justify your answers.
(c) Suppose now that you may choose the values of the \(x_i\), but each one must be either 1 or 10. How many of the \(n\) observations should you choose to make at \(x_i = 1\) in order to minimize the variance of the resulting \(\hat{\beta}\)? You can assume that \(n\) is a fixed multiple of 11.
1. Let \( \{P_{\theta}, \theta \in \Theta\} \) be a family of probability distributions. A statistic \( V \) is called \textit{ancillary} for \( \theta \) if its distribution does not depend on \( \theta \).

(a) Let \( X_1, \ldots, X_n \) have normal distribution \( N(\mu, 1) \). Show that \( V = X_1 - \bar{X} \) is an ancillary statistic for \( \mu \).

(b) Prove that if \( T \) is a complete sufficient statistic for the family \( \{P_{\theta}, \theta \in \Theta\} \), then any ancillary statistic \( V \) is independent of \( T \). (This is a theorem due to Basu).

\( \text{(Hint: show that for any (measurable) set } A, \)
\[ P_{\theta}(V \in A|T = t) = P(V \in A|T = t) = P(V \in A), \]
\( \text{and derive the conclusion).} \)

2. With \( \theta > 0 \) unknown, let a sample consist of \( X_1, \ldots, X_n \), independent observations with distribution

\[ F(y; \theta) = 1 - \sqrt{1 - \frac{y}{\theta}}, \quad 0 < y < \theta. \]

(a) Prove that the maximum likelihood estimate of \( \theta \) based on the sample is the maximum order statistic

\[ X_{(n)} = \max_{1 \leq i \leq n} X_i. \]

(b) Determine a sequence of positive numbers \( a_n \) and a non-trivial distribution for a random variable \( X \) such that

\[ a_n(\theta - X_{(n)}) \rightarrow_d X. \]

(c) Compare the rate \( a_n \) in part b) to the rate of a parametric estimation problem whose regularity would allow the application of the Cramer Rao bound. Comment.
1. For \( n \geq 2 \) let \( X_1, \ldots, X_n \) be independent samples from \( P_\theta \), the uniform distribution \( U(\theta, \theta + 1) \), \( \theta \in \mathbb{R} \). Let \( X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} \) be the order statistics of the sample.

(a) Show that \((X_{(1)}, X_{(n)})\) is a sufficient statistic for \( \theta \).

(b) Is \((X_{(1)}, X_{(n)})\) complete? Prove your claim.

(c) Find \( a_n \) and \( b(\theta) \) such that \( a_n(b(\theta) - X_{(n)}) \rightarrow Z \) in distribution, where \( Z \) has an exponential distribution with density \( f(x) = e^{-x}, x > 0 \).

(d) What is the maximum likelihood estimate of \( \theta \) given the sample?

2. Let \( X \) and \( Y \) be independent random variables with \( X \sim \text{exponential}(\lambda) \) and \( Y \sim \text{exponential}(\mu) \), where the exponential(\( \nu \)) density is given by

\[
f(x; \nu) = \frac{1}{\nu} \exp \left( -\frac{x}{\nu} \right).
\]

Let

\[ Z = \min\{X, Y\}, \]

and

\[ W = 1 \text{ if } Z = X, \text{ and } W = 0 \text{ otherwise.} \]

(a) Find the joint distribution of \( Z \) and \( W \).

(b) Prove that \( Z \) and \( W \) are independent.

(c) Suppose that \( (X, Y) \) are not observable. Instead, with \( n \geq 2 \), we observe \( (Z_1, W_1), \ldots, (Z_n, W_n) \), independent samples distributed as \( (Z, W) \). Write down the likelihood function in terms of the sample averages \( (\bar{Z}, \bar{W}) \), and find the maximum likelihood estimate \((\hat{\mu}_n, \hat{\lambda}_n)\) of \((\mu, \lambda)\).

(d) Determining whether \( 1/\hat{\lambda}_n \) is unbiased for \( 1/\lambda \), and if not, construct an estimator that is.
1. Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, 1)$. Let $u \in \mathbb{R}$ be a given threshold value, and assume that we want to estimate the probability $p = p(\mu) = P_{\mu}(X_1 \leq u)$.

(a) Find an unbiased estimator of $p$.

(b) Letting $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ denote the sample mean, show that the joint distribution of $\bar{X}$ and $X_1 - \bar{X}$ is bivariate normal and find the parameters of this distribution. Use your answer to demonstrate that $\bar{X}$ and $X_1 - \bar{X}$ are independent.

(c) Use the estimator from part (1a), along with the Rao-Blackwell theorem and part (1b), to find the uniform minimal variance unbiased estimator (UMVUE) for $p$.

2. For all $k = 0, 1, \ldots$, we have that

$$\int_{-\infty}^{\infty} x^k e^{-x^4/12} dx = 2^{\frac{k+2}{2}} 3^{\frac{k+1}{4}} ((-1)^k + 1) \Gamma\left(\frac{k+1}{4}\right).$$

(a) Determine $c_1$ such that

$$p(x) = c_1 e^{-x^4/12}$$

is a density function.

In the following we consider the location model

$$p(x; \theta) = p(x - \theta), \quad \theta \in (-\infty, \infty)$$

where $p(x)$ is as in (1). Assume a sample $X_1, \ldots, X_n$ of independent random variables with distribution $p(x; \theta)$ has been observed.

(b) Prove that the maximizer of the likelihood function is unique and can be found by setting the derivative of the log likelihood to zero.

(c) Determine the maximum likelihood estimate $\hat{\theta}$ of $\theta$ based on the sample in a form as explicit as you can in terms of the sample moments

$$m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k.$$ 

(d) Determine the information $I_X(\theta)$ for $\theta$ in the sample $X_1, \ldots, X_n$, and the non-trivial limiting distribution of $\hat{\theta}$, when properly scaled and centered. You may assume regularity conditions hold without explicitly noting them.