Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems
   (a) What is the LU-decomposition of an n by n matrix $A$, and how is it related to Gaussian elimination? Does it always exist? If not, give sufficient condition for its existence.
   (b) What is the relation of Cholesky factorization to Gaussian elimination? Give an example of a symmetric matrix for which Cholesky factorization does not exist.
   (c) Let $C = A + iB$ where $A$ and $B$ are real n by n matrices. Give necessary and sufficient conditions on $A$ and $B$ for $C$ to be Hermitian, and give a nontrivial example of a 3 by 3 Hermitian matrix.

2. Least squares
   (a) Give a simple example which shows that loss of information can occur in forming the normal equations. Discuss how the accuracy can be improved by using iterative improvement.
   (b) Compute the pseudoinverse, $x^\dagger$, of a nonzero row or column vector, $x$, of length n. Let $a = [1, 0]$ and let $b = [1, 1]^T$. Show that $(ab)^\dagger \neq b^\dagger a^\dagger$.

3. Iterative Methods
   Consider the stationary vector-matrix iteration given by
   $$x_{k+1} = Mx_k + c$$
   where $M \in \mathbb{C}^{n \times n}$, $c \in \mathbb{C}^n$, and $x_0 \in \mathbb{C}^n$ are given.
   (a) If $x^* \in \mathbb{C}^n$ is a fixed point of (1) and $\|M\| < 1$ where $\| \cdot \|$ is any compatible matrix norm induced by a vector norm, show that $x^*$ is unique and that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$.
   (b) Let $r(M)$ denote the spectral radius of the matrix $M$ and use the fact that $r(M) = \inf\|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k \to \infty} x_k = x^*$ for any $x_0 \in \mathbb{C}^n$ if and only if $r(M) < 1$.

   Now consider the linear system
   $$Ax = b$$
   where $A \in \mathbb{C}^{n \times n}$ nonsingular and $b \in \mathbb{C}^n$ are given.
   (c) What are the matrix $M \in \mathbb{C}^{n \times n}$ and the vector $c \in \mathbb{C}^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2).
   (d) Use part (a) to show that if $A \in \mathbb{C}^{n \times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).
   (e) Use part (b) together with the Gershgorin Circle Theorem to show that if $A \in \mathbb{C}^{n \times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).
4. Computation of Eigenvalues and Eigenvectors

Consider an \( n \times n \) Hermitian matrix \( A \) and a unit vector \( q_1 \). For \( k = 2, \cdots, n \), let \( p_k = Aq_{k-1} \) and set

\[
q_k = \frac{h_k}{\|h_k\|_2}, \quad h_k = p_k - \sum_{j=1}^{k-1} (q_j^H \cdot p_k) q_j.
\]

where \( \|\cdot\|_2 \) is the Euclidian norm in \( \mathbb{C}^n \).

(a) Show that the vectors \( q_k \), for \( k = 1, \cdots, n \), form an orthogonal set if none of the vectors \( h_k \) is the zero vector.

(b) Consider the matrix \( Q^H A Q \). Use part (a) to show that it is a tridiagonal matrix (Hint: \( [Q^H A Q]_{i,j} = q_i^H A q_j \)).

(c) Suggest a possible approach that uses the result of part (b) to reduce the number of operations in the QR-algorithm for the computation of the eigenvalues of the matrix \( A \).
Direct Methods for Linear Equations.

a. Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite (SPD) matrix. There exists a nonsingular lower triangle matrix \( L \) satisfying \( A = L \cdot L^t \). Is this factorization unique? If not, propose a condition on \( L \) to make the factorization unique.

b. Compute the above factorization for
\[
A = \begin{pmatrix}
1 & 2 & 1 \\
2 & 13 & 8 \\
1 & 8 & 14
\end{pmatrix}.
\]

Iterative Methods for Linear Equations.

Consider the iterative method:
\[
Nx_{k+1} = Px_k + b, \quad k = 0, 1, \ldots
\]
where \( N, P \) are \( n \times n \) matrices with \( \det N \neq 0 \); and \( x_0, b \) are arbitrary \( n \)-dim vectors. Then the above iterates satisfy the system of equations
\[
x_{k+1} = Mx_k + N^{-1}b, \quad k = 0, 1, \ldots
\]
where \( M = N^{-1}P \). Now define \( N_\alpha = (1 + \alpha)N, P_\alpha = P + \alpha N \) for some real \( \alpha \neq -1 \) and consider the related iterative method
\[
x_{k+1} = M_\alpha x_k + N^{-1}_\alpha b, \quad k = 0, 1, \ldots
\]
where \( M_\alpha = N^{-1}_\alpha P_\alpha \).

a. Let the eigenvalues of \( M \) be denoted by: \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Show that the eigenvalues \( \mu_{\alpha,k} \) of \( M_\alpha \) are given by:
\[
\mu_{\alpha,k} = \frac{\lambda_k + \alpha}{1 + \alpha}, \quad k = 1, 2, \ldots, n
\]
b. Assume the eigenvalues of \( M \) are real and satisfy: \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 1 \). Show that the iterations in eq. (2) converge as \( k \to \infty \) for any \( \alpha \) such that \( \alpha > \frac{1 + \lambda_1}{2} > -1 \).

Eigenvalue Problem.

a. Let \( \lambda \) be an eigenvalue of a \( n \times n \) matrix \( A \). Show that \( f(\lambda) \) is an eigenvalue of \( f(A) \) for any polynomial \( f(x) = \sum_{k=0}^n a_k x^k \).

b. Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a symmetric matrix satisfying:
\[
a_{1i} \neq 0, \quad \sum_{j=1}^n a_{ij} = 0, \quad a_{ii} = \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n
\]
Show all eigenvalues of \( A \) are non-negative and determine the dimension of eigenspace corresponding to the smallest eigenvalue of \( A \).
Least Square Problem.

a. Let $A$ be an $m \times n$ real matrix with the following singular value decomposition:

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix},$$

where $U = (U_1 \ U_2)$ and $V = (V_1 \ V_2)$ are orthogonal matrices, $U_1$ and $V_1$ have $r = \text{rank}(A)$ columns, and $\Sigma$ is invertible.

For any vector $b \in \mathbb{R}^n$, show that the minimum norm, least squares problem:

$$\min_{x \in S} \|x\|_2, \quad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = \text{min}\}$$

always has a unique solution, which can be written as $x = V_1 \Sigma^{-1} U_1^T b$.

b. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using part a) above, find the minimum norm, least squares solution to the problem:

$$\min_{x \in S} \|x\|_2, \quad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = \text{min}\}$$

Hint: You can assume that the $U$ in the SVD of $A$ must be of the form $U = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ for some real $a > 0$. 
Numerical Analysis Screening Exam, Spring 2013

Problem 1. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite (SPD) matrix. At the end of the first step of Gaussian Elimination without partial pivoting, we have:

$$A_1 = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    0 & & & \\
    \vdots & & & \hat{A} \\
    0 & & & \\
  \end{pmatrix}$$

a. Show that $\hat{A}$ is also a SPD.

b. Use the first conclusion to show the existence of the LU factorization and Cholesky factorization of any SPD.

Problem 2.

A matrix $A$ with all non-zero diagonal elements can be written as $A = D_A(I - L - U)$ where $D_A$ is a diagonal matrix with identical diagonal as $A$ and matrices $L$ and $U$ are lower and upper triangular matrices with zero diagonal elements. The matrix $A$ is said to be consistently ordered if the eigenvalues of matrix $\rho L + \rho^{-1} U$ are independent of $\rho \neq 0$. Consider a tri-diagonal matrix $A$ of the form

$$A = \begin{pmatrix}
    \alpha & \beta & 0 & \cdots & 0 \\
    \beta & \ddots & \ddots & \ddots & \vdots \\
    0 & \ddots & \ddots & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \beta \\
    0 & \cdots & 0 & \beta & \alpha \\
  \end{pmatrix}$$

with $|\alpha| \geq 2\beta > 0$.

a. Show that the matrix $A$ is consistently ordered.

b. Show that if $\lambda \neq 0$ is an eigenvalue of the iteration matrix $B_\omega$ of the Successive Over Relaxation (SOR) method for matrix $A$

$$B_\omega = (I - \omega L)^{-1}((1 - \omega)I + \omega U),$$

then $\mu = (\lambda + \omega - 1)(\omega \sqrt{\lambda})^{-1}$ is an eigenvalue of $L + U$. 

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Problem 3.

a. Assume that $v_1 = (1, 1, 1)^T$ is an eigenvector of a $3 \times 3$ matrix $B$. Find a real unitary matrix $V$ such that the first column of the matrix $V^T BV$ contains all zeros except on the first row.

b. Consider a matrix $A$ defined by

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

verify that $v_1 = (1, 1, 1)^T$ is an eigenvector of $A$ and the first column of the matrix $V^T AV$ contains all zeros except on the first row where $V$ is the matrix you obtained in (a).

c. Assume that $V^T AV$ has the form

$$V^T AV = \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

Find a Schur decomposition of the matrix $A$. That is, find a unitary matrix $U$ such that $U^H AU = R$ where $R$ is an upper triangular matrix and $U^H$ is the conjugate transpose of $U$.

Problem 4.

Consider an $n$ by $m$ matrix $A$ and a vector $b \in \mathbb{R}^n$. A minimum norm solution of the least squares problem is a vector $x \in \mathbb{R}^m$ with minimum Euclidian norm that minimizes $\|Ax - b\|$. Consider a vector $x^*$ such that $\|Ax^* - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^n$. Show that $x^*$ is a minimum norm solution if and only if $x^*$ is in the range of $A^*$. 
Preliminary Exam in Numerical Analysis Fall 2013

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems
   For \( x \in \mathbb{R}^n \) and \( M \in \mathbb{R}^{n \times n} \) let \( \| x \| \) denote the norm of \( x \), and let \( \| M \| \) denote the corresponding induced matrix norm of \( M \). Let \( S \in \mathbb{R}^{n \times n} \) be nonsingular and define a new norm on \( \mathbb{R}^n \) by \( \| x \|_S = \| Sx \| \).
   (a) Show that \( \| \cdot \|_S \) is in fact a norm on \( \mathbb{R}^n \).
   (b) Show that \( \| \cdot \|_S \) and \( \| \cdot \| \) are equivalent norms on \( \mathbb{R}^n \).
   (c) Show that the induced norm of \( M \in \mathbb{R}^{n \times n} \) with respect to the \( \| \cdot \|_S \) norm is given by \( \| M \|_S = \| SMS^{-1} \| \).
   (d) Let \( \kappa(M) \) denote the condition number of \( M \in \mathbb{R}^{n \times n} \) with respect to the \( \| \cdot \| \) norm, let \( \kappa_S(M) \) denote the condition number of \( M \in \mathbb{R}^{n \times n} \) with respect to the \( \| \cdot \|_S \) norm and show that \( \kappa_S(M) \leq \kappa(S)^2 \kappa(M) \).

2. Least squares
   (a) Assume you observe four \((x, y)\) data points: \((0,1), (1, 1), (-1, -1), (2,0)\). You want to fit a parabola of the form \( y = a + bx^2 \) to these data points that is best in the least squares sense. Derive the normal equations for this problem and put them in matrix vector form (you do not need to solve the equations).
   (b) Let \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \) and consider the linear system \( Ax = b \), for \( b \in \mathbb{R}^3 \). Find the QR or SVD decomposition of \( A \) and the rank of \( A \).
   (c) For a given \( b \in \mathbb{R}^3 \), state the condition such that the equation in part (b) has a solution, and the condition such that the solution is unique.
   (d) Find the pseudoinverse of the matrix \( A \) given in part (b).
   (e) For \( b = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} \) find the solution \( x \) to the system given in part (b).

3. Iterative Methods
   Consider the stationary vector-matrix iteration given by
   \[ x_{k+1} = Mx_k + c \] (1)
   where \( M \in \mathbb{C}^{n \times n}, \ c \in \mathbb{C}^n, \) and, \( x_0 \in \mathbb{C}^n \) are given.
   (a) Let \( r(M) \) denote the spectral radius of the matrix \( M \) and show that if \( \lim_{k \to \infty} x_k = x^* \) for any \( x_0 \in \mathbb{C}^n \), then \( r(M) < 1 \).
Now consider the linear system
\[ Ax = b \]  \hspace{1cm} (2)
where \( A \in \mathbb{C}^{n \times n} \) nonsingular and \( b \in \mathbb{C}^{n} \) are given.

(b) Derive the matrix \( M \in \mathbb{C}^{n \times n} \) and the vector \( c \in \mathbb{C}^{n} \) in (1) in the case of the Gauss-Seidel iteration for solving the linear system given in (2).

c) Derive the matrix \( M \in \mathbb{C}^{n \times n} \) and the vector \( c \in \mathbb{C}^{n} \) in (1) in the case of the Successive Over Relaxation Method (SOR) with parameter \( \theta \) for solving the linear system given in (2). (Hint: Use your answer in part (b) and write \( D + \frac{1}{\theta} D = \frac{1}{\theta} D + (1 - \frac{1}{\theta}) D \).)

(d) Show that if for the SOR method, \( \lim_{k \to \infty} x_k = x^* \) for any \( x_0 \in \mathbb{C}^{n} \), then it is necessary that \( \theta \in (0, 2) \).

4. Computation of Eigenvalues and Eigenvectors

Let \( A \) be a nondefective \( n \times n \) matrix with eigenvalues, \( \lambda_1, \lambda_2, \ldots, \lambda_n \), with \( |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \), and corresponding eigenvectors \( u_1, u_2, \ldots, u_n \). Let \( x_0 \in \mathbb{C}^{n} \) be such that \( x_0 = \sum_{i=1}^{n} \alpha_i u_i \), with \( \alpha_1 \neq 0 \). Define the sequence of vectors \( \{x_k\}_{k=1}^{\infty} \subseteq \mathbb{C}^{n} \) recursively by \( x_{k+1} = Ax_k, k = 0, 1, 2, \ldots \)

(a) Let \( v \in \mathbb{C}^{n} \) be any fixed vector that is not orthogonal to \( u_1 \). Show that \( q_k = \frac{v^T x_{k+1}}{v^T x_k} \) converges to \( \lambda_1 \) as \( k \to \infty \).

(b) Now suppose that \( |\lambda_2| > |\lambda_3| \), \( v \in \mathbb{C}^{n} \) is orthogonal to \( u_1 \) but is not orthogonal to \( u_2 \) and \( \alpha_2 \neq 0 \). Show that \( \lim_{k \to \infty} q_k = \lambda_2 \).

(c) Now suppose \( |\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n| \), \( v \in \mathbb{C}^{n} \) is such that \( \alpha_1 v^T u_1 \neq 0 \). Show that for \( k \) sufficiently large, \( q_k \approx \lambda_1 + C(\lambda_2/\lambda_1)^k \) for some constant \( C \). (Hint: Show that \( \lim_{k \to \infty} (q_k - \lambda_1)(\lambda_1/\lambda_2)^k = C \), for some constant \( C \)).
1. (a) Let \( \{ f_k \}_{k=1}^{n} \) be \( n \) linearly independent real valued functions in \( L_2(a, b) \), and let \( Q \) be the \( n \times n \) matrix with entries \( Q_{i,j} = \int_{a}^{b} f_i(x) f_j(x) \, dx \). Show that \( Q \) is positive definite symmetric and therefore invertible.

(b) Let \( g \) be a real valued functions in \( L_2(a, b) \) and find the best (in \( L_2(a, b) \)) approximation to \( g \) in \( \text{span}\{ f_k \}_{k=1}^{n} \).

2. Let \( A \) be a \( 3 \times 3 \) nonsingular matrix which can be reduced to the matrix

\[
U = \begin{bmatrix}
1 & u_1 & u_2 \\
0 & 1 & u_3 \\
0 & 0 & 1
\end{bmatrix}
\]

using the following sequence of elementary row operations:

(i) \( \alpha_1 \) times Row 1 is added to Row 2.

(ii) \( \alpha_2 \) times Row 1 is added to Row 3.

(iii) Row 2 is multiplied by \( \frac{1}{\alpha_3} \).

(iv) \( \alpha_4 \) times Row 2 is added to Row 3.

(a) Find an \( LU \) decomposition for the matrix \( A \).

(b) Let \( b = [b_1 \ b_2 \ b_3]^T \) be an arbitrary vector in \( R^3 \) and let the vector \( x = [x_1 \ x_2 \ x_3]^T \) in \( R^3 \) be the unique solution to the linear system \( Ax = b \). Find an expression for \( x_3 \) in terms of the \( \alpha_i \)'s, the \( h_i \)'s, and the \( u_i \)'s, \( i = 1, 2, 3 \).

3. In this problem we consider the iterative solution of the linear system of equations \( Ax = b \) with the following \( (n-1) \times (n-1) \) matrices

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
\]
(a) Show that the vectors \( x^k = \left( \sin \frac{\pi k}{n}, \sin \frac{2\pi k}{n}, \ldots, \sin \frac{(n-1)\pi k}{n} \right) \), for \( k = 1, \ldots, n - 1 \) are eigenvectors of \( B_j \), the Jacobi iteration matrix corresponding to the matrix \( A \) given above.

(b) Determine whether or not the Jacobi’s method would converge for all initial conditions \( x^0 \).

(c) Let \( L \) and \( U \) be, respectively, the lower and upper triangular matrices with zero diagonal elements such that \( B_j = L + U \), and show that the matrix \( \alpha L + \alpha^{-1} U \) has the same eigenvalues as \( B_j \) for all \( \alpha \neq 0 \).

(d) Show that an arbitrary nonzero eigenvalue, \( \lambda \), of the iteration matrix

\[
H(\omega) = (I - \omega L)^{-1} ((1 - \omega) I + \omega U)
\]

for the Successive Over Relaxation (SOR) method satisfies the following equation

\[
\lambda^2 - 2(1 - \omega) \lambda - \mu^2 \omega^2 \lambda + (1 - \omega)^2 = 0,
\]

where \( \mu \) is an eigenvalue of \( B_j \) (Hint: use the result of (c)).

(e) For \( n = 4 \), find the spectral radius of \( H(1) \).

4. (a) Find the singular value decomposition (SVD) of the matrix

\[
A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.
\]

(b) Let \( \{\lambda_k\} \) and \( \{\sigma_k\} \) be the sets of eigenvalues and singular values of \( n \times n \) matrix \( A \). Show that:

\[
\min_k \sigma_k \leq \min_k |\lambda_k| \quad \text{and} \quad \max_k \sigma_k \geq \max_k |\lambda_k|.
\]

(c) Let \( A \) be a full column rank \( m \times n \) matrix with singular value decomposition \( A = U\Sigma V^* \), where \( V^* \) indicates the conjugate transpose of \( V \).

(1) Compute the SVD of \( A(A^*A)^{-1}A^* \) in terms of \( U \), \( \Sigma \), and \( V \).

(2) Let \( || \cdot || = \sup_{x \neq 0} \frac{||Ax||^2}{||x||^2} \) be the matrix norm induced by the vector 2-norm, and let \( \sigma_{\text{max}} \) be the largest singular value of \( A \). Show that \( ||A|| = \sigma_{\text{max}} \).
Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems
   (a) Let $A$ be an $n \times n$ matrix, $B$ be an $n \times m$ matrix, $C$ be an $m \times n$ matrix, and let $D$ be an $m \times m$ matrix with the matrices $A$ and $D - CA^{-1}B$ nonsingular. Show that the partitioned $(n + m) \times (n + m)$ matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is nonsingular and find an explicit formula for its inverse.
   (b) How many multiplications would be needed to compute $M^{-1}$ if you do not take advantage of its special structure.
   (c) How many multiplications would be needed to compute $M^{-1}$ if you do take advantage of its special structure (i.e. using the formulas you derived in part (a)).

2. Least squares
   Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, with $m \leq n$ and rank $A = m$. Let $x_0 \in \mathbb{R}^n$ and consider the constrained optimization problem given by
   $$\min \|x - x_0\|,$$
   subject to $Ax = b$, where the norm in the above expression is the Euclidean norm on $\mathbb{R}^n$. Show that this problem has a unique solution given by
   $$x^* = A^T (AA^T)^{-1} b + (I_n - A^T (AA^T)^{-1} A) x_0,$$
   where $I_n$ is the $n \times n$ identity matrix.

3. Iterative Methods
   Consider the following matrix
   $$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}.$$
   (a) Find a range for the real parameter $\omega$ such that Richardson’s method for the solution of the linear system $Ax = b$, 
   $$x^{k+1} = x^k - \omega (Ax^k - b)$$
   converges for any initial guess $x^0$.
   (b) Find an optimal value for $\omega$ and justify your answer.
4. Computation of Eigenvalues and Eigenvectors

Consider the following matrix

\[ A = \begin{bmatrix}
1 & 1 \\
0 & 1 + \epsilon
\end{bmatrix}. \]

(a) Find a non-singular matrix \( T(\epsilon) \) such that the matrix \( T^{-1}(\epsilon)AT(\epsilon) \) is in Jordan canonical form.

(b) Find the limit of \( \|T(\epsilon)\| \) as \( \epsilon \) tends toward zero.

(c) Explain what this implies with regard to using the LR or QR methods to compute the eigenvalues of a matrix whose Jordan canonical form contains a Jordan block of size larger than 1.
Problem 1 Consider the following $3 \times 3$ matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \epsilon & \epsilon \\ 0 & \epsilon & \epsilon + \epsilon^2 \end{pmatrix}$$

(a) Use the Gram-Schmidt orthogonalization method on the columns of matrix $A$ to derive a QR-decomposition of $A$.

(b) Use the Householder QR-factorization method to perform the same task as (a).

(c) Based on your calculations in (a) and (b), determine which method would lead to a more accurate factorization on an actual computer and justify your reasoning.

(d) A graduate student programmed both techniques in Matlab and tested them for the case $\epsilon = 10^{-10}$. He evaluated the norm of $A - Q \cdot R$ in Matlab and found that the norm was equal to 0 for the Gram-Schmidt factorization and $4.6032 \times 10^{-26}$ for the Householder factorization. Is this consistent with your conclusion in (c)? What other quantities would you suggest him to examine that may support your conclusion in (c)?
Name: ________________________________
Name:
PROBLEM 2. Let $A$ be an $n \times n$ real-valued, symmetric positive definite matrix and $b \in \mathbb{R}^n$. Consider the following two-stage iterative procedure for solving the system of equations $Ax = b$:

$$
\begin{align*}
    x_{n+\frac{1}{2}} &= x_n + \omega_1(b - Ax_n) \\
    x_{n+1} &= x_{n+\frac{1}{2}} + \omega_2(b - Ax_{n+\frac{1}{2}})
\end{align*}
$$

(a) Let $e_n = x - x_n$ be the error between the $n$-th iterate $x_n$ and the exact solution $x$. Find the matrix $K$ such that $e_{n+1} = Ke_n$.

(b) Find the eigenvalues of $K$ in terms of the eigenvalues of $A$, $\omega_1$, and $\omega_2$.

(c) Show that $\omega_1$ and $\omega_2$ can be chosen so that the method converges with any initial condition. Express the rate of convergence in terms of $\lambda_M$ and $\lambda_m$ which correspond to the largest and smallest eigenvalues of $A$, respectively.
Name: _________________________________
Name:
Problem 3. Consider a least square minimization problem:

\[
\text{minimize} \|Ax - b\|_2^2, \quad x \in \mathbb{R}^n, \; b \in \mathbb{R}^m, \quad (1)
\]

and a regularized form of the (1):

\[
\text{minimize} \|Ax - b\|_2^2 + \alpha \|x\|_2^2, \quad \alpha > 0. \quad (2)
\]

(a) State and justify a necessary and sufficient condition for a vector \(x_0\) to be a solution of (1) and determine whether or not this problem always has a unique solution.

(b) State and justify a necessary and sufficient condition for a vector \(x_0\) to be a solution of (2) and determine whether or not this problem always has a unique solution.

(c) Let \(\mathcal{R}(A^T)\) be the range of \(A^T\) and let \(\mathcal{N}(A)\) be the null space of \(A\). Explain why a solution of (2) must be in \(\mathcal{R}(A^T)\).

(d) Suggest a method for approximating a minimal norm solution of (1) using a sequence of solutions of (2) and justify your answer.
Name:
Name:
Problem 4. Let $A$ be an $n \times n$ skew-Hermitian (i.e. $A^H = -A$) matrix.

(a) Show that $I + A$ is invertible.

(b) Show that $U = (I + A)^{-1}(I - A)$ is unitary.

(c) Show that if $U$ is unitary with $-1 \notin \sigma(U)$, then there exists a skew-Hermitian matrix $A$ such that $U = (I + A)^{-1}(I - A)$.

(d) Show that if $B$ is an $n \times n$ normal matrix (i.e. $B^H B = BB^H$) then it is unitarily similar to a diagonal matrix.

(e) Let $C$ be an $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Show that if $\lambda$ is an eigenvalue of $C$, then $|\lambda| \leq \sigma_1$ and that $|\det(C)| = \prod_{i=1}^{n} \sigma_i$. 


Name: ________________________________
Name: ________________________________
Problem 1. (Iterative Methods) Consider the block matrix

\[
A = \begin{bmatrix}
I_n & 0 & -M_1 \\
-M_2 & I_n & 0 \\
0 & -M_3 & I_n \\
\end{bmatrix}
\]

where \(I_n\) is the \(n \times n\) identity matrix and \(M_1, M_2,\) and \(M_3,\) are \(n \times n\) matrices.

a) Find the Jacobi and Gauss-Seidel iteration matrices for solving the system of equations \(Ax = b\) (for some fixed vector \(b\)).

b) Show the methods either both converge or both diverge and, in the case of the former, the Gauss-Seidel method converges three times as fast as the Jacobi method.

c) Now consider the Jacobi and Gauss-Seidel methods for the system of equations \(A^T x = b\). Show they either both converge or both diverge and, in the case of the former, the Gauss-Seidel method converges one-and-a-half times as fast as the Jacobi method.
Problem 2. (Least Squares) Let \( A \in \mathbb{C}^{m \times n} \) and consider the following set of equations in the unknown matrix \( X \in \mathbb{C}^{n \times m} \) known as the Moore-Penrose equations (MP):

\[
AXA = A \\
XAX = X \\
(AX)^* = AX \\
(XA)^* = XA
\]

(a) Show that the system (MP) has at most one solution (Hint: Show that if both \( X \) and \( Y \) are solutions to (MP) then \( X = XAY \) and \( X = YAX \).

(b) When \( A = \text{zeros}(m, n) \), show that there exists a solution to the system (MP). (Hint: Find one!)

(c) Assume that \( A \) has full column rank and find the solution to the least squares problem given by \( \min \| AX - I \|_F^2 \) where \( I \) denotes the \( m \times m \) identity matrix and \( \| \cdot \|_F \) denotes the Frobenius norm on \( \mathbb{C}^{m \times m} \) \( \| B \|_F = (\sum_{i=1}^m \sum_{j=1}^m |b_{i,j}|^2)^{1/2} \) \( = (\sum_{i=1}^m \sum_{j=1}^m b_{i,j}^2)^{1/2} \). (Hint: Note that the given least squares problem decouples into \( m \) separate standard least squares problems!)

(d) Assume that \( A \) has full column rank and use part (c) to show that the system (MP) has a solution. (Hint: Find one!).

(e) Assume \( \text{rank}(A) = r, 0 < r < n \), and show that there exists a permutation matrix \( P \in \mathbb{C}^{n \times n} \) such that \( AP = [\hat{A}|\hat{A}R] \), or \( A = [\hat{A}|\hat{A}R]P^T \) where \( \hat{A} \in \mathbb{C}^{m \times r} \) has full column rank and \( R \in \mathbb{C}^{r \times (n-r)} \). (A permutation matrix is a square matrix that has precisely a single 1 in every row and column and zeros everywhere else.)

(f) Assume \( \text{rank}(A) = r, 0 < r < n \), assume that a solution to the system (MP) has the form \( X = P \begin{bmatrix} S \\ T \end{bmatrix} \), where \( S \in \mathbb{C}^{r \times m} \) and \( T \in \mathbb{C}^{(n-r) \times m} \) and use parts (c), (d), and (e) and the first equation in the system (MP) to determine the matrices \( S \in \mathbb{C}^{r \times m} \) and \( T \in \mathbb{C}^{(n-r) \times m} \) in terms of the matrix \( \hat{A} \), and therefore show that the system (MP) has a solution.
Problem 3. (Direct Methods) Consider a vector norm $\|\cdot\|_V$ for $\mathbb{R}^n$. Define another norm $\|\cdot\|_{V'}$ for $\mathbb{R}^n$ by $\|x\|_{V'} = \max_{y \in \mathbb{R}^n, \|y\|_V \leq 1} |x^T y|$. It is known that for every vector $x \neq 0 \in \mathbb{R}^n$, there exists a vector $y \in \mathbb{R}^n$ such that $y^T x = \|y\|_{V'} \|x\|_V = 1$. A vector $y$ with this property is called a dual element of $x$.

a. Consider a nonsingular $n \times n$ matrix $A$. We define the distance between $A$ and the set of singular matrices by

$$\text{dist}(A) = \min \{ \|\delta A\|_V : \delta A \in \mathbb{R}^{n \times n}, \text{where } A + \delta A \text{ is singular} \}$$

where $\|A\|_V$ is the operator norm induced by $\|\cdot\|_V$. Show that

$$\text{dist}(A) \geq \frac{1}{\|A^{-1}\|_V}.$$  

**Hint:** Suppose the matrix $A + \delta A$ is singular. Then there exists $x \neq 0$ such that $(A + \delta A)x = 0$.

b. Let $x$ be a unit vector such that $\|A^{-1}x\|_V = \|A^{-1}\|_V$ and $y = A^{-1}x/\|A^{-1}\|_V$. Consider a dual element $z$ of $y$ and the matrix

$$\delta A = -\frac{xz^T}{\|A^{-1}\|_V}.$$  

Show that $A + \delta A$ is singular.

c. Show that $\|\delta A\|_V = \|A^{-1}\|_V^{-1}$ and

$$\text{dist}(A) = \frac{1}{\|A^{-1}\|_V}.$$
Problem 4. (Eigenvalue/Eigenvector Problems) Let $A$ be a real symmetric matrix and $q_1$ a unit vector. Let

$$\mathcal{K}(A, q_1, j) = \text{Span}\{q_1, Aq_1, A^2q_1, ..., A^{j-1}q_1\}$$

be the corresponding Krylov subspaces. Suppose $\{q_1, q_2, ..., q_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and $\{q_1, q_2, ..., q_j\}$ is an orthonormal basis for $\mathcal{K}(A, q_1, j), 1 \leq j \leq n$. Let $Q_j$ be the $n \times j$ matrix whose columns are $q_1, q_2, ..., q_j$, i.e. $Q_j = [q_1, q_2, ..., q_j]$. It is easy to see that $Q_j^T A Q_j = T_j$ where $T_j$ has the form

$$T_j = \begin{pmatrix}
\alpha_1 & \beta_1 & 0 & \cdots & 0 \\
\beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\
0 & \beta_2 & \alpha_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \alpha_{j-1} & \beta_{j-1} & 0 \\
\beta_{j-1} & \alpha_j & \cdots & \beta_2 & \alpha_1
\end{pmatrix}$$

a) Derive an algorithm to compute the vectors $q_j$ and the numbers $\alpha_j$ and $\beta_j$ iteratively from the fact that $A Q_n = Q_n T_n$. This algorithm is known as the Lanczos algorithm.

b) Let $M_j$ be the largest eigenvalue of $T_j$. Show that $M_j$ increases as $j$ increases and that $M_n$ is equal to the largest eigenvalue of $A$, $\lambda_1$.

Hint: Recall that $M_j = \max_{x \in \mathbb{R}^j} x^T T_j x$ and $\lambda_1 = \max_{x \in \mathbb{R}^n} x^T A x$.

c) A standard approach for finding $\lambda_1$ is to use Householder transformations to tridiagonalize $A$ and then use the QR algorithm. Suppose $A$ is large and sparse. How could you use the Lanczos algorithm to improve on this method? What are the advantages of this alternative approach?
Exam in Numerical Analysis Spring 2015

Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems

Consider the symmetric positive definite (spd) $n \times n$ matrix $A$ and its LU-decomposition $A = LU$ with $l_{ii} = 1, i = 1, 2, \ldots, n$.

a. Show that $u_{11} = a_{11}$, and $u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}$, $k = 2, \ldots, n$, where for each $k = 1, 2, \ldots, n$, $A_k$ is the $k \times k$ matrix formed by taking the intersection of the first $k$ rows and $k$ columns of $A$.

b. Show that $A$ can be written as $A = R^T R$ with $R$ upper triangular with positive diagonal entries. (Hint: Let $D = \text{diag}(u_{11}, u_{22}, \ldots, u_{nn})$ and consider the identity $A = LDD^{-1}U$.)

c. Show how the decomposition found in part (b) suggests a scheme for solving the system $A x = b$ with $A$ spd, that like Choleski's method requires only $U$, but that unlike Choleski's method, does not require the computation of square roots.

2. Least squares

Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, be given.

(a) Show that the $x$-component of any solution of linear system

$$
\begin{pmatrix}
I_m & A^T \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
r
\end{pmatrix}
= 
\begin{pmatrix}
b \\
0
\end{pmatrix},
$$

is a solution of the minimization problem

$$
\min_x ||b - Ax||_2.
$$

(b) Show that the solution of the linear system (1) is unique if and only if the solution of the least squares problem (2) is unique.

3. Iterative Methods

Let $A$ be an $n \times n$ nonsingular matrix and consider the matrix iteration

$$
X_{k+1} = X_k + X_k(I - AX_k),
$$

with $X_0$ given. Find and justify necessary and sufficient conditions on $A$ and $X_0$ for this iteration to converge to $A^{-1}$. 
4. **Computation of Eigenvalues and Eigenvectors**

Consider the matrices $A$ and $T$ given by

$$A = \begin{bmatrix} 3 & \alpha & \beta \\ -1 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix},$$

where $|\alpha|, |\beta| \leq 1$.

(a) Use the similarity transform $T^{-1}AT$ to show that the matrix $A$ has at least two distinct eigenvalues. (Hint: Gershgorin’s Theorem)

(b) In the case $\alpha = \beta = 1$ verify that $x = [1 \quad 1 \quad 1]^T$ is an eigenvector of $A$ and find a unitary matrix $U$ such that the matrix $U^TAU$ has the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

(c) Explain, without actually calculating, how one could find a Schur decomposition of $A$. 

1. Suppose

$$A = \begin{pmatrix}
D_1 & F_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
E_2 & D_2 & F_2 & 0 & \ldots & 0 & 0 & 0 \\
0 & E_3 & D_3 & F_3 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & E_{n-1} & D_{n-1} & F_{n-1} \\
0 & 0 & 0 & 0 & \ldots & 0 & E_n & D_n
\end{pmatrix}$$

is a real block tridiagonal matrix where the blocks are all size $q \times q$ and the diagonal blocks $D_i$ are all invertible, $1 \leq i \leq n$. Suppose, moreover, that $A^t$ is block diagonally dominant, in other words

$$||D_i^{-1}||_1 (||F_i^{-1}||_1 + ||E_{i+1}||_1) < 1$$

for $1 \leq i \leq n$ where $F_0 = E_{n+1} = 0$.

(a) Show $A$ is invertible.

(b) Show $A$ has a block $LU$ decomposition of the form

$$A = \begin{pmatrix}
I & 0 & 0 & \ldots & 0 & 0 \\
L_2 & I & 0 & \ldots & 0 & 0 \\
0 & L_3 & I & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I & 0 \\
0 & 0 & 0 & \ldots & L_n & I
\end{pmatrix} \begin{pmatrix}
U_1 & F_1 & 0 & \ldots & 0 & 0 \\
0 & U_2 & F_2 & \ldots & 0 & 0 \\
0 & 0 & U_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & U_{n-1} & F_{n-1} \\
0 & 0 & 0 & \ldots & 0 & U_n
\end{pmatrix}$$

where

* $||L_i||_1 \leq 1$, $2 \leq i \leq n - 1$ and
* each matrix $U_i$ is invertible with $||U_i||_1 \leq ||A||_1$, $i \leq i \leq n$.

Hint: Recall, if a square matrix $M$ has $||M||_1 < 1$ then $I - M$ is invertible and

$$(I - M)^{-1} = I + M + M^2 + M^3 + \ldots$$

(c) Show how you can find this block $LU$ decomposition numerically and how you can use it to solve the system of equations $Ax = b$ (for a given vector $b$). Explain the significance of the bounds in (b) and why this approach might be preferable to employing Gaussian elimination with pivoting on the whole of $A$. 
2. Consider the following $2 \times 3$ matrix $A$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$ 

(a) Find a two dimensional subspace $S^*$ such that

$$\min_{x \in S^*, \|x\|_2 = 1} \|Ax\|_2 = \max_{\dim S = 2} \min_{x \in S, \|x\|_2 = 1} \|Ax\|_2.$$ 

Justify your answer.

(b) Find a rank one $2 \times 3$ matrix $B$ such that $\|A - B\|_2$ is minimized and justify your answer.
3. Let $X$ be a linear vector space over $C$ and let $P$ be the $n \times n$ matrix defined by the linear transformation on $X^n$ given by

$$P\vec{x} = P[x_1, x_2, ..., x_n]^T = [x_2, x_3, ..., x_n, x_1]^T.$$ 

(a) What are the matrices $P$, $P^0$, $P^2$, $P^{n-1}$ and $P^n$? (Hint: Although you could do this with matrix multiplication, it’s easier to base your answer on the underlying transformation.)

Let $F$ be the $n \times n$ matrix given by

$$[F]_{j,k} = \frac{1}{\sqrt{n}} \omega^{(j-1)(k-1)}, j, k = 1, 2, ..., n,$$

where $\omega = e^{\frac{2\pi i}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$.

(b) Show that for $k = 1, 2, ..., n$, $P\vec{F}_k = \omega^{(k-1)}\vec{F}_k$, where $\vec{F}_k$ is the $k^{th}$ column of the matrix $F$.

(c) Show that the matrix $F$ is unitary.

Let $a = \{a_i\}_{i=1}^n \subseteq C$, set $p_a(z) = \sum_{i=1}^n a_i z^{i-1}$ and let the $n \times n$ matrix $A_a$ be given by

$$A_a = \begin{bmatrix}
    a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\
    a_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\
    a_{n-1} & a_n & a_1 & a_2 & \cdots & a_{n-2} \\
    a_{n-2} & a_{n-1} & a_n & a_1 & \cdots & a_{n-3} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_2 & a_3 & a_4 & a_5 & \cdots & a_1
\end{bmatrix}$$

(d) Show that $A_a$ is diagonalizable with eigenvalue/eigenvector pairs given by $\{p_a(\omega^{(k-1)}), \vec{F}_k\}$, $k = 1, 2, ..., n$. (Hint: Parts (a), (b) and (c).)
4. Let \( \{z_k^m\}_{k=1}^n \) be \( n \) points in the complex plane and consider the following iteration:

\[
z_k^{m+1} = \text{the average of } \left\{ \begin{array}{l}
z_k^m \text{ and } z_{k+1}^m \quad k = 1, 2, \ldots, n - 1 \\
z_n^m \text{ and } z_1^m \quad k = n
\end{array} \right.
\]

(a) Let \( Z^m = [z_1^m, z_2^m, \ldots, z_n^m]^T \) and rewrite the transformation from \( Z^m \) to \( Z^{m+1} \) given above in the form of a matrix iteration.

(b) Show that \( \lim_{m \to \infty} z_k^m = \hat{z}, k = 1, 2, \ldots, n, \) where \( \hat{z} = \frac{1}{n} \sum_{j=1}^n z_j^0. \) (Hint: The RESULT of Problem 3(d) might be of some help to you here. Note that you may use the result of problem 3(d) even if you were not able to prove it yourself.)

(c) What happens if, in parts (a) and (b), the phrase “the average” in the definition of the iteration is replaced with “an arbitrary convex combination”; that is:

\[
z_k^{m+1} = \left\{ \begin{array}{ll}
\alpha z_k^m + (1 - \alpha) z_{k+1}^m & k = 1, 2, \ldots, n - 1 \\
\alpha z_n^m + (1 - \alpha) z_1^m & k = n
\end{array} \right.
\] for some \( \alpha \in (0,1) \)?
1. Let $A \in \mathbb{C}^{n \times n}$ and let $A_j \in \mathbb{C}^n$ $j = 1,2,\ldots,n$ be the $j^{th}$ column of $A$. Show that
\[ |\det A| \leq \prod_{j=1}^{n} \| A_j \|_1. \]
Hint: Let $D = \text{diag}(\| A_1 \|_1, \| A_2 \|_1, \ldots, \| A_n \|_1)$, and consider $\det B$, where $B = AD^{-1}$.

2. a) Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and let $A_j \in \mathbb{R}^n$ $j = 1,2,\ldots,n$ be the $j^{th}$ column of $A$. Use Gram-Schmidt to show that $A = QR$, where $Q$ is orthogonal and $R$ is upper triangular with $\| A_j \|_2^2 = \sum_{i=1}^{j} R_{i,j}^2$ $j = 1,2,\ldots,n$.

b) Given $A, Q, R \in \mathbb{R}^{n \times n}$ as in part (a) above with $A = QR$, and given $b \in \mathbb{R}^n$, perform an operation count (of multiplications only) for solving the linear system $Ax = b$.

3. Consider the constrained least squares problem:
\[ * \left\{ \begin{array}{l}
\min_x ||Ax - b||_2 \\
\text{subject to } Cx = d
\end{array} \right. \]
where the $m \times n$ matrix $A$, the $p \times n$ matrix $C$, and the vectors $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$ are given.

a) Show that the unconstrained least squares problem
\[ \min_x ||Ax - b||_2 \]
is a special case of the constrained least squares problem $*$. 

b) Show that the minimum norm problem
\[ \left\{ \begin{array}{l}
\min_x ||x||_2 \\
\text{subject to } Cx = d
\end{array} \right. \]
is a special case of the constrained least squares problem $*$. 

c) By writing $x = x_0 + Nz$, show that solving the constrained least squares problem $*$ is equivalent to solving an unconstrained least squares problem
\[ ** \min_z ||Az - \tilde{b}||_2. \]
What are the matrices $N$ and $A$ and vectors $x_0$ and $\tilde{b}$?

d) Use part c) to solve the constrained least squares problem $*$ where
Consider a stationary iteration method for solving a system of linear equations $Ax = b$ given by

$$\begin{align*}
y^k &= x^k + \omega_0 (b - Ax^k), \\
x^{k+1} &= y^k + \omega_1 (b - Ay^k).
\end{align*}$$

a) Show that the matrix $B$ defined by $x^{k+1} = Bx^k + c$ has the form $B = \mu p(A)$ where $p(\lambda)$ is a second order polynomial in $\lambda$ with leading coefficient equal to 1.

b) Show that the scaled Chebyshev polynomial $T_2(\lambda) = \lambda^2 - 1/2$ has the property that

$$\frac{1}{2} = \max_{-1 \leq \lambda \leq 1} |T_2(\lambda)| \leq \max_{-1 \leq \lambda \leq 1} |q(\lambda)|$$

for all second order polynomial $q$ with leading coefficient 1.

c) If we know that matrix $A$ is Hermitian with eigenvalues in $(-1, 1)$, find coefficients $\omega_0$ and $\omega_1$ such that the proposed iterative scheme converges for any initial vector $x^0$.

d) What could you do if the eigenvalues of the matrix $A$ is in $(\alpha, \beta)$ to make the scheme convergent?