ALGEBRA QUALIFYING EXAM SPRING 2012

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\mathbb{Q}$ and set of the complex numbers is $\mathbb{C}$.

1. Let $I$ be an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$. Show that $\text{dim}_\mathbb{C} R/I$ is finite $\iff I$ is contained in only finitely many maximal ideals of $R$.

2. If $G$ is a group with $|G| = 7^2 \cdot 11^2 \cdot 19$, show that $G$ must be abelian and describe the possible structures of $G$.

3. Let $F$ be a finite field and $G$ a finite group with $\text{GCD}(\text{char } F, |G|) = 1$. The group algebra $F[G]$ is an algebra over $F$ with $G$ as an $F$-basis, elements $\alpha = \sum_G a_g g$ for $a_g \in F$, and multiplication that extends $ag \cdot bh = ab \cdot gh$. Show that any $x \in F[G]$ that is not a zero left divisor (i.e. if $xy = 0$ for $y \in F[G]$ then $y = 0$) must be invertible in $F[G]$.

4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over $\mathbb{Q}$, argue that $\text{Gal}(M/\mathbb{Q})$ is solvable.

5. Let $R$ be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1y_1 + \cdots + x_ny_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \ldots, r_n) \in R^n \mid x_1r_1 + \cdots + x_nr_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that $A$ has $n$ generators, and that when $R = F[x]$ for $F$ a field then $A_k$ is free of rank $n - 1$.

6. For $p$ a prime let $F_p$ be the field of $p$ elements and $K$ an extension field of $F_p$ of dimension 72.
   i) Describe the possible structures of $\text{Gal}(K/F_p)$.
   ii) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that $K$ is a splitting field of $g(x)$ over $F_p$.
   iii) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree $d$ that split in $K$?
Algebra Graduate Exam

Fall 2012

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Use Sylow’s theorems directly to find, up to isomorphism, all possible structures of groups of order $5 \cdot 7 \cdot 23$.

2. Let $A$, $B$, and $C$ be finitely generated $F[x] = R$ modules, for $F$ a field, with $C$ torsion free. Show that $A \otimes_C B \otimes_C C$ implies that $A \otimes_C B$. Show by example that this conclusion can fail when $C$ is not torsion free.

3. Working in the polynomial ring $C[x, y]$, show that some power of $(x + y)(x^2 + y^4 - 2)$ is in $(x^3 + y^2, y^3 + xy)$.

4. For integers $n, m > 1$, let $A \subseteq M_n(Z_m)$ be a subring with the property that if $x \in A$ with $x^2 = 0$ then $x = 0$. Show that $A$ is commutative. Is the converse true?

5. Let $F$ be the splitting field of $f(x) = x^6 - 2$ over $Q$. Show that $Gal(F/Q)$ is isomorphic to the dihedral group of order 12.

6. Given that all groups of order 12 are solvable show that any group of order $2^2 \cdot 3 \cdot 7^2$ is solvable.
Algebra Graduate Exam

Spring 2013

Work all the problems. Be as explicit as possible in your solutions, and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions.

1. Let $p > 2$ be a prime. Describe, up to isomorphism, all groups of order $2p^2$.

2. Let $R$ be a commutative Noetherian ring with 1. Show that every proper ideal of $R$ is the product of finitely many (not necessarily distinct) prime ideals of $R$.  
   (Hint: Consider the set of ideals that are not products of finitely many prime ideals. Also, note that if $R$ is not a prime ring then $IJ = (0)$ for some non-zero ideals $I$ and $J$ of $R$)

3. In the polynomial ring $R = \mathbb{C}[x,y,z]$ show that there is a positive integer $m$, and polynomials $f, g, h \in R$ such that  
   
   $$(x^{16}y^{25}z^{81} - x^{7}z^{15} - y^{9} + x^{5})^m = (x - y)^3f + (y - z)^5g + (x + y + z - 3)^7h.$$  

4. Let $R \neq (0)$ be a finite ring such that for any $x \in R$ there is $y \in R$ with $xyx = x$. Show that $R$ contains an identity element and that, for $a, b \in R$, if $ab = 1$ then $ba = 1$.

5. Let $f(x) = x^{15} - 2$, and let $L$ be the splitting field of $f(x)$ over $\mathbb{Q}$.
   a) What is $[L: \mathbb{Q}]$?
   b) Show there exists a subfield $F$ of degree 8 that is Galois over $\mathbb{Q}$.
   c) What is $\text{Gal}(F/\mathbb{Q})$?
   d) Show there is a subgroup of $\text{Gal}(L/\mathbb{Q})$ that is isomorphic to $\text{Gal}(F/\mathbb{Q})$.

6. Let $F/\mathbb{Q}$ be a Galois extension of degree 60, and suppose $F$ contains a primitive ninth root of unity. Show $\text{Gal}(F/\mathbb{Q})$ is solvable.

7. Let $n$ be a positive integer. Show that $f(x,y) = x^n + y^n + 1$ is irreducible in $\mathbb{C}[x,y]$. 
Algebra Qualifying Exam - Fall 2013

1. Let \( H \) be a subgroup of the symmetric group \( S_5 \). Can the order of \( H \) be 15, 20 or 30?

2. Let \( R \) be a PID and \( M \) a finitely generated torsion module of \( R \). Show that \( M \) is a cyclic \( R \)-module if and only if for any prime \( p \) of \( R \) either \( pM = M \) or \( M/pM \) is a cyclic \( R \)-module.

3. Let \( R = \mathbb{C}[x_1, \ldots, x_n] \) and suppose \( I \) is a proper non-zero ideal of \( R \). The coefficients of a matrix \( A \in M_n(R) \) are polynomials in \( x_1, \ldots, x_n \) and can be evaluated at \( \beta \in \mathbb{C}^n \); write \( A(\beta) \in M_n(\mathbb{C}) \) for the matrix so obtained. If for some \( A \in M_n(R) \) and all \( \alpha \in \text{Var}(I) \), \( A(\alpha) = 0_{n \times n} \), show that for some integer \( m \), \( A^m \in M_n(I) \).

4. If \( R \) is a noetherian unital ring, show that the power series ring \( R[[x]] \) is also a noetherian unital ring.

5. Let \( p \) be a prime. Prove that \( f(x) = x^p - x - 1 \) is irreducible over \( \mathbb{Z}/p\mathbb{Z} \). What is the Galois group? (Hint: observe that if \( \alpha \) is a root of \( f(x) \), then so is \( \alpha + i \) for \( i \in \mathbb{Z}/p\mathbb{Z} \).)

6. Let \( R \) be a finite ring with no nilpotent elements. Show that \( R \) is a direct product of fields.

7. Let \( K \subseteq \mathbb{C} \) be the field obtained by adjoining all roots of unity in \( \mathbb{C} \) to \( \mathbb{Q} \). Suppose \( p_1 < p_2 \) are primes, \( a \in \mathbb{C} \setminus K \), and write \( L \) for a splitting field of

\[
g(x) = (x^{p_1} - a)(x^{p_2} - a)
\]

over \( K \). Assuming each factor of \( g(x) \) is irreducible, determine the order and the structure of \( Gal(L/K) \).
ALGEBRA QUALIFYING EXAM SPRING 2014

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is \( \mathbb{Q} \), and set of the complex numbers is \( \mathbb{C} \). Hand in solutions in order of the problem numbers.

1. Let \( L \) be a Galois extension of a field \( F \) with \( \text{Gal}(L/F) \cong D_{10} \), the dihedral group of order 10. How many subfields \( F \subseteq M \subseteq L \) are there, what are their dimensions over \( F \), and how many are Galois over \( F \)?

2. Up to isomorphism, using direct and semi-direct products, describe the possible structures of a group of order \( 5 \cdot 11 \cdot 61 \).

3. Let \( I \) be a nonzero ideal of \( R = \mathbb{C}[x_1, \ldots, x_n] \). Show that \( R/I \) is a finite dimensional algebra over \( \mathbb{C} \) if and only if \( I \) is contained in only finitely many maximal ideals of \( R \).

4. Let \( R \) be a commutative ring with 1, and \( M \) a noetherian \( R \) module. For \( N \) a noetherian \( R \) module show that \( M \otimes_R N \) is a noetherian \( R \) module. When \( N \) is an artinian \( R \) module show that \( M \otimes_R N \) is an artinian \( R \) module.

5. For \( n \geq 5 \) show that the symmetric group \( S_n \) cannot have a subgroup \( H \) with \( 3 \leq [S_n : H] < n \) ([\( S_n : H \)] is the index of \( H \) in \( S_n \)).

6. Let \( R \) be the group algebra \( \mathbb{C}[S_3] \). How many nonisomorphic, irreducible, left modules does \( R \) have and why?

7. Let each of \( g_1(x), g_2(x), \ldots, g_n(x) \in \mathbb{Q}[x] \) be irreducible of degree four and let \( L \) be a splitting field over \( \mathbb{Q} \) for \( \{g_1(x), \ldots, g_n(x)\} \). Show there is an extension field \( M \) of \( L \) that is a radical extension of \( \mathbb{Q} \).
Algebra Exam September 2014

Show your work. Be as clear as possible. Do all problems. Hand in solutions in numerical order.

1. Let \( G \) be a group of order 56 having at least 7 elements of order 7. Let \( S \) be a Sylow 2-subgroup of \( G \).
   
   (a) Prove that \( S \) is normal in \( G \) and \( S = C_G(S) \).
   
   (b) Describe the possible structures of \( G \) up to isomorphism. (Hint: How does an element of order 7 act on the elements of \( S \)?)

2. Show that a finite ring with no nonzero nilpotent elements is commutative.

3. If \( R = M_n(\mathbb{Z}) \), and \( A \) is an additive subgroup of \( R \), show that as additive subgroups \( [R : A] \) is finite if and only if \( R \otimes \mathbb{Q} = A \otimes \mathbb{Q} \).

4. Let \( R \) be a commutative ring with 1, \( n \) a positive integer and \( A_1, \ldots, A_k \in M_n(R) \). Show that there is a noetherian subring \( S \) of \( R \) containing 1 with all the \( A_i \in M_n(S) \).

5. Let \( R = \mathbb{C}[x, y] \). Show that there exists a positive integer \( m \) such that \(((x + y)(x^2 + y^4 - 2))^m\) is in the ideal \( (x^3 + y^2, y^3 + xy) \).

6. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial of degree \( n \geq 5 \). Let \( L \) be the splitting field of \( f \) and let \( \alpha \in L \) be a zero of \( f \). Given that \([L : \mathbb{Q}] = n!\), prove that \( \mathbb{Q}[\alpha^4] = \mathbb{Q}[\alpha] \).
ALGEBRA QUALIFYING EXAM FALL 2015

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of integers is \( \mathbb{Z} \), the set of rational numbers is \( \mathbb{Q} \), and set of the complex numbers is \( \mathbb{C} \).

Hand in the exam with problems in numerical order.

1. If \( M \) is a maximal ideal in \( \mathbb{Q}[x_1, \ldots, x_n] \) show that there are only finitely many maximal ideals in \( \mathbb{C}[x_1, \ldots, x_n] \) that contain \( M \).

2. Let \( R \) be a right Noetherian ring with 1. Prove that \( R \) has a unique maximal nilpotent ideal \( P(R) \). Argue that \( R[x] \) also has a unique maximal nilpotent ideal \( P(R[x]) \). Show that \( P(R[x]) = P(R)[x] \).

3. Up to isomorphism, describe the possible structures of any group of order 182 as a direct sum of cyclic groups, dihedral groups, other semi-direct products, symmetric groups, or matrix groups. (Note: 91 is not a prime!)

4. Let \( K = \mathbb{C}(y) \) for an indeterminate \( y \) and let \( p_1 < p_2 < \cdots < p_n \) be primes (in \( \mathbb{Z} \)). Let \( f(x) = (x^{p_1} - y) \cdots (x^{p_n} - y) \in K \) with splitting field \( L \) over \( K \)
   a) Show each \( x^{p_i} - y \) is irreducible over \( K \).
   b) Describe the structure of \( Gal(L/K) \).
   c) How many intermediate fields are between \( K \) and \( L \)?

5. In any finite ring \( R \) with 1 show that some element in \( R \) is not a sum of nilpotent elements. Note that in all \( M_n(\mathbb{Z}/n\mathbb{Z}) \) the identity matrix is a sum of nilpotent elements. (Hint: What is the trace of a nilpotent element in a matrix ring over a field?)

6. Let \( R \) be a commutative principal ideal domain.
   (1) If \( I \) and \( J \) are ideals of \( R \), show \( R/I \otimes_R R/J \cong R/(I + J) \).
   (2) If \( V \) and \( W \) are finitely generated \( R \) modules so that \( V \otimes_R W = 0 \), show that \( V \) and \( W \) are torsion modules whose annihilators in \( R \) are relatively prime.

7. Let \( g(x) = x^{12} + 5x^6 - 2x^3 + 17 \in \mathbb{Q}[x] \) and \( F \) a splitting field of \( g(x) \) over \( \mathbb{Q} \). Determine if \( Gal(F/\mathbb{Q}) \) is solvable.
1. If $R := \mathbb{C}[x, y]/(y^2 - x^3 - 1)$, then describe all the maximal ideals in $R$.

2. Suppose $F$ is a field, and $b_n(F)$ is the $F$-algebra of upper-triangular matrices, i.e., the subalgebra of $M_n(F)$ consisting of matrices $X$ such that $X_{ij} = 0$ when $i > j$. Describe the Jacobson radical of $b_n(F)$, the simple modules, and the maximal semi-simple quotient.

3. Let $F_5$ be the finite field with 5 elements, and consider the group $G = PGL_2(F_5)$ (i.e., the quotient of the group of invertible $2 \times 2$-matrices over $F_5$ by the subgroup of scalar multiples of the identity).
   (a) What is the order of $G$?
   (b) Describe $N_G(P)$ where $P$ is a Sylow-5 subgroup of $G$.
   (c) If $H \subset G$ is a subgroup, can $H$ have order 15, 20 or 30?

4. Let $A$ be an $n \times n$ matrix over $\mathbb{Z}$. Let $V$ be the $\mathbb{Z}$-module of column vectors of size $n$ over $\mathbb{Z}$.
   (a) Prove that the size of $V/AV$ is equal to the absolute value of $\det(A)$ if $\det(A) \neq 0$.
   (b) Prove that $V/AV$ is infinite if $\det(A) = 0$.
   (hint: use the theory of finitely generated modules $\mathbb{Z}$-modules)

5. Let $V$ be a finite dimensional right module over a division ring $D$. Let $W$ be a $D$-submodule of $V$.
   (a) Let $I(W) = \{ f \in \text{End}_D(V) | f(W) = 0 \}$. Prove that $I(W)$ is a left ideal of $\text{End}(V)$.
   (b) Prove that any left ideal of $\text{End}_D(V)$ is $I(W)$ for some submodule $W$.

6. Let $p$ and $q$ be distinct primes. Let $F$ be the subfield of $\mathbb{C}$ generated by the $pq$-roots of unity. Let $a, b$ be squarefree integers all greater than 1. Let $c, d \in \mathbb{C}$ with $c^p = a$ and $d^q = b$. Let $K = F(c, d)$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Describe the Galois group of $K/F$.
   (c) Show that any intermediate field $F \subset L \subset K$ satisfies $L = F(S)$ where $S$ is some subset of \{c, d\}. 

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**Algebra Qualifying Exam - Fall 2016**

1. If $R := \mathbb{C}[x, y]/(y^2 - x^3 - 1)$, then describe all the maximal ideals in $R$.

2. Suppose $F$ is a field, and $b_n(F)$ is the $F$-algebra of upper-triangular matrices, i.e., the subalgebra of $M_n(F)$ consisting of matrices $X$ such that $X_{ij} = 0$ when $i > j$. Describe the Jacobson radical of $b_n(F)$, the simple modules, and the maximal semi-simple quotient.

3. Let $F_5$ be the finite field with 5 elements, and consider the group $G = PGL_2(F_5)$ (i.e., the quotient of the group of invertible $2 \times 2$-matrices over $F_5$ by the subgroup of scalar multiples of the identity).
   (a) What is the order of $G$?
   (b) Describe $N_G(P)$ where $P$ is a Sylow-5 subgroup of $G$.
   (c) If $H \subset G$ is a subgroup, can $H$ have order 15, 20 or 30?

4. Let $A$ be an $n \times n$ matrix over $\mathbb{Z}$. Let $V$ be the $\mathbb{Z}$-module of column vectors of size $n$ over $\mathbb{Z}$.
   (a) Prove that the size of $V/AV$ is equal to the absolute value of $\det(A)$ if $\det(A) \neq 0$.
   (b) Prove that $V/AV$ is infinite if $\det(A) = 0$.
   (hint: use the theory of finitely generated modules $\mathbb{Z}$-modules)

5. Let $V$ be a finite dimensional right module over a division ring $D$. Let $W$ be a $D$-submodule of $V$.
   (a) Let $I(W) = \{ f \in \text{End}_D(V) | f(W) = 0 \}$. Prove that $I(W)$ is a left ideal of $\text{End}(V)$.
   (b) Prove that any left ideal of $\text{End}_D(V)$ is $I(W)$ for some submodule $W$.

6. Let $p$ and $q$ be distinct primes. Let $F$ be the subfield of $\mathbb{C}$ generated by the $pq$-roots of unity. Let $a, b$ be squarefree integers all greater than 1. Let $c, d \in \mathbb{C}$ with $c^p = a$ and $d^q = b$. Let $K = F(c, d)$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Describe the Galois group of $K/F$.
   (c) Show that any intermediate field $F \subset L \subset K$ satisfies $L = F(S)$ where $S$ is some subset of \{c, d\}. 

Algebra Exam February 2015

Show your work. Be as clear as possible. Do all problems.

1. Use Sylow’s theorems and other results to describe, up to isomorphism, the possible structures of a group of order 1005.

2. Let $R$ be a commutative ring with 1. Let $M, N$ and $V$ be $R$-modules.
   - (a) Show if that $M$ and $N$ are projective, then so is $M \otimes_R N$.
   - (b) Let $\text{Tr}(V) := \{ \sum_i \phi_i(v_i) | \phi \in \text{Hom}_R(V, R), v_i \in V \} \subset R$. If $1 \in \text{Tr}(V)$, show that up to isomorphism $R$ is a direct summand of $V^k$ for some $k$.

3. Let $F$ be a field and $M$ a maximal ideal of $F[x_1, \ldots, x_n]$. Let $K$ be an algebraic closure of $F$. Show that $M$ is contained in at least 1 and in only finitely many maximal ideals of $K[x_1, \ldots, x_n]$.

4. Let $F$ be a finite field.
   - (a) Show that there are irreducible polynomials over $F$ of every positive degree.
   - (b) Show that $x^4 + 1$ is irreducible over $\mathbb{Q}[x]$ but is reducible over $\mathbb{F}_p[x]$ for every prime $p$ (hint: show there is a root in $\mathbb{F}_{p^2}$).

5. Let $F$ be a field and $M$ a finitely generated $F[x]$-module. Show that $M$ is artinian if and only if $\dim_F M$ is finite.

6. Let $R$ be a right Artinian ring with with a faithful irreducible right $R$-module. If $x, y \in R$, set $[x, y] := xy - yx$. Show that if $[[x, y], z] = 0$ for all $x, y, z \in R$, then $R$ has no nilpotent elements.
Show your work. Be as clear as possible. Do all problems.

1. Let $R$ be a Noetherian commutative ring with 1 and $I \neq 0$ an ideal of $R$. Show that there exist finitely many nonzero prime ideals $P_i$ of $R$ (not necessarily distinct) so that $\cap_i P_i \subset I$ (Hint: consider the set of ideals which are not of that form).

2. Describe all groups of order 130: show that every such group is isomorphic to a direct sum of dihedral and cyclic groups of suitable orders.

3. Let $f(x) = x^{12} + 2x^6 - 2x^3 + 2 \in \mathbb{Q}[x]$. Show $f(x)$ is irreducible. Let $K$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine whether $\text{Gal}(K/\mathbb{Q})$ is solvable.

4. Determine up to isomorphism the algebra structure of $\mathbb{C}[G]$ where $G = S_3$ is the symmetric group of degree 3 (Recall that $\mathbb{C}[G]$ is the group algebra of $G$ which has basis $G$ and the multiplication comes from the multiplication on $G$).

5. If $F$ is a field and $n > 1$ show that for any nonconstant $g \in F[x_1, \ldots, x_n]$ the ideal $gF[x_1, \ldots, x_n]$ is not a maximal ideal of $F[x_1, \ldots, x_n]$.

6. Let $F$ be a field and let $P$ be a submodule of $F[x]^n$. Suppose that the quotient module $M := F[x]^n/P$ is Artinian. Show that $M$ is finite dimensional over $F$. 
Algebra Exam January 2017

Show your work. Be as clear as possible. Do all problems.

1. Let $R$ be a PID. Let $M$ be an $R$-module.
   (a) Show that if $M$ is finitely generated, then $M$ is cyclic if and only if $M/PM$ is for all prime ideals $P$ of $R$.
   (b) Show that the previous statement is false if $M$ is not finitely generated.

2. Prove that a power of the polynomial $(x + y)(x^2 + y^4 - 2)$ belongs to the ideal $(x^3 + y^2, x^3 + xy)$ in $\mathbb{C}[x, y]$.

3. Let $G$ be a finite group with a cyclic Sylow 2-subgroup $S$.
   (a) Show that $N_G(S) = C_G(S)$.
   (b) Show that if $S \neq 1$, then $G$ contains a normal subgroup of index 2 (hint: suppose that $n = [G : S]$, consider an appropriate homomorphism from $G$ to $S_n$).
   (c) Show that $G$ has a normal subgroup $N$ of odd order such that $G = NS$.

4. Show that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed in its quotient field.

5. Let $f(x) = x^{11} - 5 \in \mathbb{Q}[x]$.
   (a) Show that $f$ is irreducible in $\mathbb{Q}[x]$.
   (b) Let $K$ be the splitting field of $f$ over $\mathbb{Q}$. What is the Galois group of $K/\mathbb{Q}$.
   (c) How many subfields $L$ of $K$ are there so such that $[K : L] = 11$.

6. Suppose that $R$ is a finite ring with 1 such that every unit of $R$ has order dividing 24. Classify all such $R$. 