Geometry/Topology Qualifying, Spring 2006

Partial credit for partial solutions

1. Let \((x, y, z, w)\) be Cartesian coordinates on \(\mathbb{R}^4\). Is the set defined by the equation \(x^2 + xy^3 + yz^4 - w^5 = -1\) a smooth manifold of \(\mathbb{R}^4\)? Prove your assertion.

2. a) State the definition of the \(i\)th de Rham cohomology group \(H^i_{dR}(M)\) of a smooth manifold \(M\).
   
b) Compute the \(i\)th de Rham cohomology groups of the real line \(\mathbb{R}\) directly from the definition for all \(i \geq 0\).

3. Let \(X\) be the quotient space obtained from the \(n\)-dimensional sphere \(S^n\) by identifying three distinct points to a single common point \(p \in X\). In other words, let \(q, r, s \in S^3\) be pairwise distinct points, let \(X = S^n / \sim\) where \(x \sim y\) if \(x = y\) or if \(x, y \in \{q, r, s\}\), and let \(p \in X\) denote the equivalence class \(\{q, r, s\}\). Calculate \(\pi_1(X, p)\).

4. Let \(S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4\) and let \(\omega = w\ dx \wedge dy \wedge dz\). Compute \(\int_{S^3} \omega\).

5. Recall that the genus of a closed orientable surface \(\Sigma\) is defined to be \(\frac{1}{2} \dim_{\mathbb{R}} H^2_{dR}(\Sigma)\). Let \(S\) and \(T\) be closed orientable surfaces of respective genera \(g(S)\) and \(g(T)\). Assume \(g(S) < g(T)\). Show that the degree of any smooth map \(h : S \rightarrow T\) equals zero. [You may use the fact that on a closed orientable surface \(\Sigma\), the wedge product of one-forms induces a skew-symmetric non-degenerate bilinear pairing \(H^1_{dR}(\Sigma) \otimes H^1_{dR}(\Sigma) \rightarrow H^2_{dR}(\Sigma) \approx \mathbb{R}\), where \(H^i_{dR}(F)\) denotes the \(i\)th de Rham cohomology group of \(\Sigma\).]

6. Define the unlink to be the union of two unknotted circles in the three-dimensional sphere \(S^3\), where there are two disjoint three-dimensional balls in \(S^3\) containing the circles. Define the Hopf link to be the union of two unknotted disjoint circles in \(S^3\), where each circle meets a disk bounding the other circle in a single point. These links are illustrated in the figure below drawn in \(\mathbb{R}^3 = S^3 - \{\text{the point at infinity}\}\). Let \(U\) be the complement in \(S^3\) to the unlink and let \(H\) be the complement in \(S^3\) to the Hopf link. Calculate the homology groups of \(U\) and \(H\).

7. Let \(X\) denote a bouquet of \(n + 1\) circles, i.e., \(X\) is the quotient of the disjoint union of \(n + 1\) circles with base points obtained by identifying all the base points to a single point \(p\) in the quotient.
   
a) Prove that \(\pi_1(X, p)\) is a free group \(F_{n+1}\) on \(n + 1\) generators.
   
b) Let \(H\) be a subgroup of \(F_{n+1}\) of index \(k\). Show that \(H\) is a free group with \(kn + 1\) generators.
Geometry/Topology Qualifying Exam

September 2006

Solve all SEVEN problems. Partial credit will be given to partial solutions.

1. Let \( M, N \) be compact oriented manifolds of dimension \( n \) (without boundary), and let \( f : M \to N \) be a differentiable map. Prove that, if the induced homomorphism \( f^* : H^a_{dR}(N; \mathbb{R}) \to H^a_{dR}(M; \mathbb{R}) \) between de Rham cohomology groups is surjective, then \( f \) is surjective.

2. Let \( D^2 \) be the closed unit disk in the complex plane \( \mathbb{C} \), bounded by the unit circle \( S^1 \). Consider the 2-dimensional torus \( T^2 = S^1 \times S^1 \) and two copies \( D_1 \) and \( D_2 \) of \( D^2 \). For two integers \( p, q \), let \( X_{pq} \) be the quotient space of the disjoint union

\[ T^2 \sqcup D_1 \sqcup D_2 \]

by the equivalence relation that identifies each point \( e^{i\theta} \) in the boundary of \( D_1 \) to \((e^{ip\theta}, 1) \in S^1 \times S^1 \), and identifies each point \( e^{i\theta} \) in the boundary of \( D_2 \) to \((1, e^{iq\theta}) \in S^1 \times S^1 \). Compute the fundamental group of \( X_{pq} \).

3. Prove that any two continuous maps \( f, g : X \to S^1 \) from a simply-connected space \( X \) to the circle \( S^1 \) are homotopic.

4. Calculate the relative homology groups \( H_*(S^1 \times D^2, S^1 \times \partial D^2) \), where \( D^2 \) denotes the 2-dimensional closed disk and \( S^1 \) is the circle.

5. Let \( M \) be a compact oriented \( n \)-manifold with \( H^n_{dR}(M; \mathbb{R}) = 0 \) and let \( f : M \to T^n \) be a smooth map. Show that the degree of \( f \) is equal to 0. (Possible hint: Write \( T^n = S^1 \times \cdots \times S^1 \); if \( \theta_i \) is the angular coordinate for the \( i \)-th factor \( S^1 \), then \( d\theta_1 \wedge \cdots \wedge d\theta_n \) is a volume form for \( T^n \).)

6. Recall that the rank of a matrix is the dimension of the span of its row vectors. Show that the space of all \( 2 \times 3 \) matrices of rank 1 forms a smooth manifold.

7. Consider the group \( \text{SO}(3) \) of orientation-preserving isometries of the 2-dimensional sphere \( S^2 \). Namely, \( \text{SO}(3) \) consists of all rotations of \( \mathbb{R}^3 \) whose axis passes through the origin or, equivalently, all \( 3 \times 3 \) matrices \( A \) such that \( AA^t = \text{Id} \) and \( \det(A) = 1 \). Prove that, if \( \omega \) is a 1-form (not necessarily closed) on \( S^2 \) such that \( \phi^*(\omega) = \omega \) for every \( \phi \in \text{SO}(3) \), then \( \omega = 0 \).
1. (15 pts) Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. (This is, of course, a differentiable manifold.) For $A \in M_n(\mathbb{R})$, define a tangent vector to $M_n(\mathbb{R})$ at the identity matrix $I$ to be the class of the curve $t \mapsto tA$, $-\varepsilon < t < \varepsilon$. Denote this tangent vector by $\vec{A}$.
(a) For any $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $R_X(H) = X^T H X$. Prove that $R_X$ is differentiable.
(b) For any $\vec{A} \in T_I M_n(\mathbb{R})$, define a vector field $\xi_\vec{A}$ on $M_n(\mathbb{R})$ so that $\xi_\vec{A}(X) = (R_X)_{\ast}(I)(\vec{A})$ (Here $(R_X)_{\ast}(I)$ is the derivative of $R_X$ at $I$.) Compute the Lie bracket $[\xi_{\vec{A}}, \xi_{\vec{B}}]$.

2. (15 pts) Let $C$ be the subset of $\mathbb{C}$ with coordinates $z, w$, defined by the equation $w^2 = P(z)$, where $P(z)$ is a polynomial of degree $d$.
(a) Prove that if $P$ has no repeated roots, then $C$ is a submanifold of $\mathbb{C}^2$. (Remark: $C$ is a complex submanifold, and hence is also a real submanifold.)
(b) Suppose that $P$ has no repeated roots. Compute the fundamental group of $C - \{ (z, w) : w = 0 \}$. (Hint: Think of covering spaces.)

3. (10 pts) Prove that the tangent bundle $TM$ of a smooth manifold $M$ has the structure of a smooth orientable manifold. (Do not assume that $M$ itself is orientable.)

4. (10 pts) Consider the differential 1-form $\omega = dz - ydx$ on $\mathbb{R}^2$ with coordinates $(x, y, z)$. Prove that $\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

5. (10 pts) Define the notion of a deformation retraction of a space $X$ onto a subset $A \subseteq X$. Prove that if $A$ is the knot in the solid torus $X = S^1 \times D^3$ as drawn in the picture below, then there is no deformation retraction of $X$ onto $A$.

![Figure 1](image)

6. (10 pts) Construct a topological space $X$ such that $H_0(X; \mathbb{Z}) = \mathbb{Z}$, $H_3(X; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $H_5(X; \mathbb{Z}) = \mathbb{Z}$, and all other homology groups are zero.
Problem 1. Let \( X \) be a path connected space such that \( H_p(X, \mathbb{Z}) = 0 \) for every \( p \) with \( 0 < p \leq n \). If \( X \times S^n \) denotes the product of \( X \) with the \( n \)-dimensional sphere \( S^n \), compute the homology groups \( H_p(X \times S^n, \mathbb{Z}) \) for every \( p \) with \( 0 < p \leq n \).

Problem 2. Let \( C_1 \) and \( C_2 \) be two disjoint circles in \( \mathbb{R}^3 \), and let \( A = S^1 \times [0,1] \) denote the cylinder. Let \( X \) be the space obtained from the disjoint union \( X \sqcup A \) by gluing the boundary component \( S^1 \times \{0\} \) of \( A \) to the circle \( C_1 \) by a homeomorphism, and by gluing the other boundary component \( S^1 \times \{1\} \) to \( C_2 \) by another homeomorphism. Compute the fundamental group of the space \( X \) so obtained.

Problem 3. Let \( M_n(\mathbb{R}) \) be the vector space of \( n \times n \) matrices with coefficients in \( \mathbb{R} \), and consider the determinant function \( \text{det} : M_n(\mathbb{R}) \to \mathbb{R} \), which to a matrix \( A \) associates its determinant \( \text{det}(A) \). Compute the differential map (also called tangent map) of the function \( \text{det} \) at the identity matrix \( I_n \in M_n(\mathbb{R}) \).

Problem 4. Let \( M \) be a compact orientable \( n \)-dimensional manifold whose boundary \( \partial M \) is homeomorphism to the sphere \( S^{n-1} \subset \mathbb{R}^n \) by a homeomorphism \( f : \partial M \to S^{n-1} \). Let \( F \) be a continuous map \( F : M \to \mathbb{R}^n \) whose restriction to the boundary \( \partial M \) coincides with \( f \). Show that the image \( F(M) \) necessarily contains the center \( O \) of the sphere \( S^{n-1} \).

Problem 5. Let \( \Omega \) be the open shell in \( \mathbb{R}^2 \) consisting of those \( (x,y) \in \mathbb{R}^2 \) such that \( 1 < x^2 + y^2 < 10 \), and consider the 1-form
\[
\omega = \frac{x \, dy - y \, dx}{4x^2 + y^2}
\]

a) Show that \( \omega \) is closed in \( \Omega \).

b) Show that \( \omega \) is not closed in \( \Omega \). (Possible hint: consider an ellipse of equation \( 4x^2 + y^2 = \text{constant} \)).

Problem 6. Let \( \mathbb{RP}^2 \) denote the real projective plane of dimension 2. Consider the map \( \varphi : \mathbb{R}^2 \to \mathbb{RP}^2 \) which to \( (x,y) \in \mathbb{R}^2 \) associates the element of \( \mathbb{RP}^2 \) represented by the line passing through the point \( (x,y,1) \). (Recall that \( \mathbb{RP}^2 \) is the space of lines passing through the origin in \( \mathbb{R}^3 \).) If \( C = \{(x,y) \in \mathbb{R}^2 ; y^2 = x^3 - x \} \), show that the closure \( \overline{\varphi(C)} \) of \( \varphi(C) \) in \( \mathbb{RP}^2 \) is a differentiable submanifold of \( \mathbb{RP}^2 \).

Problem 7. Let \( M \) and \( N \) be two compact connected manifolds of the same dimension \( n \), and let \( f : M \to N \) be a continuous map. Suppose that the homomorphism \( H_n(f) : H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z}) \) induced by \( f \) is not \( 0 \). If \( f_* : \pi_1(M,x_0) \to \pi_1(N,f(x_0)) \) is the homomorphism induced by \( f \) between the fundamental groups, show that its image \( f_*(\pi(M,x_0)) \) has finite index in \( \pi(N,f(x_0)) \). (Possible hint: Consider a suitable covering of \( N \).)
1. Let $p : \tilde{X} \to X$ be a covering with path connected base $X$, and let $G$ be its automorphism group, consisting of those homeomorphisms $\varphi : \tilde{X} \to \tilde{X}$ such that $p \circ \varphi = \varphi$. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. Suppose that, for any two $\tilde{x}_0', \tilde{x}_0'' \in p^{-1}(x_0)$, there exists $\varphi \in G$ such that $\varphi(\tilde{x}_0') = \tilde{x}_0''$. Show that there is an exact sequence

$$1 \to \pi_1(\tilde{X}; \tilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \to G \to 1.$$ 

2. Consider on $\mathbb{R}^n$ the standard inner product $(\tilde{a}, \tilde{b}) = \sum_{i=1}^{n} a_i b_i$, when $\tilde{a} = (a_1, a_2, \ldots, a_n)$ and $\tilde{b} = (b_1, b_2, \ldots, b_n)$. Let $V$ be a vector subspace of $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to V$ be the orthogonal projection with respect to the above inner product. If $M$ is a submanifold of $\mathbb{R}^n$, show that the restriction $\pi|_M : M \to V$ is an immersion if and only if $T_x M \cap V^\perp = \{0\}$ for every $x \in M$.

3. Let $f : X \to X$ be a map homotopic to a constant map, and let $M_f = X \times [0, 1]/\sim$ where the equivalence relation $\sim$ identifies $(x, 0)$ to $(f(x), 1)$. Compute the homology groups of $M_f$.

4. Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree $n$ on $S^n$ such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map $f$.
   a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.
   b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of $\beta$ and $\alpha$. It may be useful to remember that the map $H^n(S^n) \to \mathbb{R}$ defined by $\gamma \mapsto \int_{S^n} \gamma$ is an isomorphism.

5. Let $\omega \in \Omega^2(S^2)$ be the restriction of the 2-form

$$xdy \wedge dz + ydz \wedge dx + ydy \wedge dx$$

to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. Compute the integral $\int_{S^2} \omega$.

6. Recall that the 1-dimensional projective space $\mathbb{R}P^1$ consists of all lines in $\mathbb{R}^2$ passing through the origin. Let $f : \mathbb{R} \to \mathbb{R}P^1$ associate to $x \in \mathbb{R}$ the line passing through $(x, 1)$ and the origin. Finally, let $P(x)$ be a polynomial function of the variable $x$.
   a) Show that there is no differential form $\omega$ on $\mathbb{R}P^1$ such that $f^*(\omega) = P(x) \, dx$.
   b) Show that there exists a vector field $V$ on $\mathbb{R}P^1$ such that $f^*(V) = P(x) \frac{\partial}{\partial x}$ if and only if the degree of $P(x)$ is $\leq 2$.

7. Let $M$ be a compact differentiable manifold, and let $C^\infty(M)$ be the algebra of all differentiable functions $M \to \mathbb{R}$. Let $\mathcal{I}$ be a maximal ideal of $C^\infty(M)$. Show that there is a point $x_0 \in M$ such that $\mathcal{I} = \{f \in C^\infty(M); f(x_0) = 0\}$. (Possible hint: Suppose that the property is not true and show that, for every $x \in M$, there exists a non-negative function $f \in \mathcal{I}$ such that $f(x) > 0.$)
Geometry/Topology Qualifying Exam

Fall 2008

Solve all SIX problems. Partial credit will be given to partial solutions.

1. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where $M$ is a smooth manifold, $\Omega^i(M)$ is the set of smooth $i$-forms on $M$, and $f$ is a smooth function on $M$.

(a) Show that $d_f$ is a cochain map, i.e., $d_f \circ d_f = 0$.

(b) Let $H^1_f(M)$ be the $i$th cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^1_f(M) \cong \mathbb{R}$ when $M$ is the real line $\mathbb{R}$.

2. Show that, when $m, n > 0$, the homomorphism $f^* : H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$ induced in de Rham cohomology by $f : S^{m+n} \to S^m \times S^n$ is trivial for all $k > 0$. Here $S^n$ is the $n$-dimensional sphere. [Possible hint: Construct a volume form for $S^m \times S^n$ from a volume form on $S^m$ and a volume form on $S^n$.]

3. Prove that the set $C = \{ (x, y) \mid y^2 - x^3 = 0 \}$ is not a smooth submanifold of the plane. [Hint: What is the space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ which are tangent to $C$?]

4. Let $T$ be the surface obtained by revolving the circle $\{(x, y, z) \mid z = 0, (x - R)^2 + y^2 = r^2 \}$ around the $y$-axis, where $R > r$. Compute the integral

$$\int_T xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

5. Let $B^3$ be the (closed) 3-dimensional ball, and let $K$ be a closed, connected 1-dimensional submanifold of $B^3$ with $\partial K = K \cap \partial B^3 = 2$ points. Compute the homology of the complement $B^3 - K$ (= an apple minus a wormhole).

6. Recall that two covering spaces $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ are isomorphic if there exists a homeomorphism $\tilde{\phi} : \tilde{X} \to \tilde{X}'$ such that $p' \circ \tilde{\phi} = p$. Consider the covering spaces $p : \tilde{X} \to X$ of the torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?
Geometry/Topology Qualifying Exam

Spring 2009

Solve all SIX problems. Partial credit will be given to partial solutions.

1. Let $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that the map
   $$f : S^2 \to \mathbb{R}^4, \quad f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1 x_2, x_1 x_3, x_2 x_3)$$
is an immersion and that $f(S^2)$ is diffeomorphic to the projective plane $\mathbb{R}P^2$.

2. Let $\omega$ be a closed $n$-form on $\mathbb{R}^{n+1} - \{0\}$. Prove that $\omega$ is exact if and only if $\int_{S^n} \omega = 0$, where $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$.

3. Find all vector fields $Z$ on $\mathbb{R}^2$ which satisfy $[X, Z] = 0$ and $[Y, Z] = 0$, where $X = e^x \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ are vector fields defined on all of $\mathbb{R}^2$.

4. Compute $\pi_n(T^p)$ for all $n \geq 1$, where $T^p = S^1 \times \cdots \times S^1$ ($p$ times) is the $p$-dimensional torus.

5. Compute $\pi_1(\mathbb{R}^3 - K)$, where $K \subset \mathbb{R}^3$ is the union of the vertical axis $\{x = 0, y = 0\}$ and the unit circle $\{x^2 + y^2 = 1, z = 0\}$.

6. Let $X$ be a compact, oriented surface of genus 2 (without boundary), and let $A$ be a simple closed curve which separates the surface $X$ into two punctured tori, as given in Figure 1 below. Then compute the relative homology groups $H_n(X, A)$ for all $n \geq 0$.

![Figure 1](image_url)
Problem 1. Let \( f : M \to N \) be a map between two compact oriented manifolds of the same dimension. Suppose that the subgroup \( f^*(\pi_1(M)) \) has finite index in \( \pi_1(N) \).

a. Show that the index \( [\pi_1(N) : f^*(\pi_1(M))] \) divides the degree of \( f \).

b. Give an example where \( [\pi_1(N) : f^*(\pi_1(M))] \) is different from the degree of \( f \).

Problem 2. Is there a differentiable map \( \mathbb{R}^2 \to \mathbb{R}^2 \) that sends the vector field \( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) to the vector field \( X = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) and sends the vector field \( \frac{\partial}{\partial x} \) to the vector field \( Y = -\frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \)?

Problem 3. Let \( f : S^n \to S^n \) be a degree 5 map from the sphere \( S^n \) to itself.

a. Show that there exists \( x_1 \in S^n \) such that \( f(x_1) = -x_1 \).

b. Show that there exists \( x_2 \in S^n \) such that \( f(x_2) = x_2 \).

Problem 4. Let \( M \) be a compact submanifold of \( \mathbb{R}^n \), of dimension at most \( n - 3 \), and let \( f : B^2 \to \mathbb{R}^n \) be a differentiable map from the 2-dimensional ball (or disk) \( B^2 \) to \( \mathbb{R}^n \). Let \( T_v : \mathbb{R}^n \to \mathbb{R}^n \) denote the translation along the vector \( v \in \mathbb{R}^n \).

a. Show that there exists arbitrarily small vectors \( v \in \mathbb{R}^n \) such that the image of \( T_v \circ f \) is disjoint from \( M \).

b. Conclude that the complement \( \mathbb{R}^n - M \) is simply connected.

Problem 5. Let \( \omega \) be a closed form of degree \( n \) on \( \mathbb{R}^{n+1} - \{0\} \). Show that, for any two differentiable maps \( f, g : S^n \to \mathbb{R}^{n+1} - \{0\} \), the ratio

\[
\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}
\]

is a rational number when the denominator is not 0.

Problem 6. Let \( S \) be the standard surface of genus 2 in \( \mathbb{R}^3 \) as in the picture below, and let \( W \) be the closure of the bounded component of \( \mathbb{R}^3 - S \). Compute the relative homology groups \( H_n(W, S) \).

![Surface of genus 2](image)

Problem 7. Let \( M \) be a compact connected submanifold of an oriented manifold \( N \), with \( \dim M = \dim N - 1 \). Show that \( M \) is orientable if and only if it admits arbitrarily small connected neighborhoods \( U \) such that \( U - M \) is disconnected. Namely, if and only if, for every open subset \( V \subset N \) containing \( M \), there is a connected open subset \( U \subset V \) such that \( U - M \) is not connected.
Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$$X_1 =$$ [Graph 1]

$$X_2 =$$ [Graph 2]

Problem 2. Let $P_1, P_2, P_3$ be three distinct points in the sphere $S^2$, and let $X$ be the topological space obtained from $S^2$ by gluing these three points together. Compute all homology groups $H_p(X; \mathbb{Z})$.

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface $\Sigma$ in $\mathbb{R}^3$ at the point $p$. Does there exist a compact immersed surface $\Sigma$ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the orthogonal group $O(n) = \{ A \in M_n(\mathbb{R}) | AA^T = \text{id} \}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $x_1, x_2, \ldots, x_n$ are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p} \omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: $S^{n-1}$.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$. If $f$ is a smooth function on $\mathbb{R}^{2n}$, find the vector field $X$ such that $i_X \omega = df$, where $i_X$ denotes the interior product. Then compute the Lie derivative $L_X \omega$.

Problem 7. Let $X$ be a topological space such that the homology group $H_p(X; \mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X; \mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X; \mathbb{Z}) = \text{Hom}(C_{p+1}(X; \mathbb{Z}), \mathbb{Z})$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X; \mathbb{Z})$ with $\partial \alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X; \mathbb{Z})$ with $k \alpha = \partial \beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k} u(\beta)$$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X; \mathbb{Z})$ with $k \alpha = \partial \beta$. Namely, show that $L_u([\alpha])$ is independent of $k$, $\beta$ and of the representative $\alpha$ of $[\alpha] \in H_p(X; \mathbb{Z})$. 

1
Geometry/Topology Qualifying Exam

Spring 2011

Solve all SIX problems. Partial credit will be given to partial solutions.

1. (10 pts) Let \( S^3 = \{ x \in \mathbb{R}^4 \mid \| x \| = 1 \} \) be the 3-dimensional sphere, oriented as the boundary of the unit ball \( B^4 \) in \( \mathbb{R}^4 \) with the standard orientation. Compute \( \int_{S^3} \omega \), where

\[
\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.
\]

(You may leave your answer in terms of volumes \( \text{vol}(S^n) \) and \( \text{vol}(B^n) \).)

2. (10 pts) Let \( M = \{ (x, y) \mid x, y \in \mathbb{R}^3, \| x \| = 1, \| y \| = 1, \langle x, y \rangle = 0 \} \), where \( \langle x, y \rangle \) is the standard inner product on \( \mathbb{R}^3 \). Show that \( M \) is a smooth compact embedded submanifold of \( \mathbb{R}^6 \) and explain how \( M \) can be identified with the unit tangent bundle of \( S^2 \).

3. (20 pts) Let \( \mathbb{RP}^n \) be the real projective space given by \( S^n / \sim \), where \( S^n = \{ \| x \| = 1 \} \subset \mathbb{R}^{n+1} \) and \( x \sim -x \) for all \( x \in S^n \).

(a) (5 pts) Use covering spaces to compute \( \pi_1(\mathbb{RP}^n) \).

(b) (5 pts) Give a cell (CW) decomposition of \( \mathbb{RP}^n \) for \( n \geq 1 \).

(c) (5 pts) Use the cell decomposition to compute the homology groups \( H_k(\mathbb{RP}^n) \), \( k \geq 0 \).

(d) (5 pts) For which values of \( n \geq 1 \) is \( \mathbb{RP}^n \) orientable? Explain.

4. (10 pts) Given a continuous map \( f : X \to Y \) between topological spaces, define

\( C_f = \left( (X \times [0, 1]) \coprod Y \right) / \sim \),

where \( (x, 1) \sim f(x) \) for all \( x \in X \) and \( (x, 0) \sim (x', 0) \) for all \( x, x' \in X \). Here \( \coprod \) is the disjoint union. Then prove that there is a long exact sequence

\[ \cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots, \]

where \( f_* \) is the map on homology induced from \( f \) and \( \tilde{H}_i \) denotes the \( i \)th reduced homology group.

5. (10 pts) Prove that the fundamental group of a connected Lie group \( G \) is abelian. (A Lie group \( G \) is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between \( fg \) and \( gf \), where \( f \) and \( g \) are loops in \( G \).]

6. (10 pts) Let \( M \subset \mathbb{R}^3 \) be an embedded compact oriented surface (without boundary) of genus \( g \geq 1 \). Show that the Gaussian curvature \( \kappa \) of \( M \) must vanish somewhere on \( M \).