Topics for the Graduate Exam in Differential Equations

Most of the following topics are normally covered in the courses Math 555a and 565a.

This is a two hour exam.

Existence, uniqueness and dependence of initial data and continuation of solutions.

Linear systems, periodic linear systems, Floquet’s Theorem, stability of critical points and periodic orbits. Dichotomies, perturbations of linear systems, Lyapunov functions.

2-dimensional systems, classification of elementary critical points, Poincare-Bendixson Theorem, flows, invariant sets. Anosov diffeomorphisms, Stable Manifold Theorem, invariant manifolds, bifurcations and skew-product dynamical systems.

First order equations (the Cauchy problem, method of characteristics)
Sobolev spaces (imbedding theorem, Rellich compactness theorem)
Laplace equation (mean value property, Harnack principle, maximum principle, Liouville’s theorem)
Heat equation (Cauchy problem, energy equality, maximum principle, nonhomogeneous heat equation)
Wave equation (D’Alamert’s formula, energy equality, Duhamel’s principle)

References:

P. Hsieh and Y. Sibuya, Basic Theory of ODE, Springer
J. Hale, ODE, Wiley
E.A. Coddington and N. Levinson, Theory of ODE
L.C. Evans, Partial Differential Equations
ODE/PDE Qualifying Exam, May 1997

1) Consider the equation
   \[ x'' + (a + b \cos t)x = 0 \]
   and let \( u, v \) be solutions such that
   \[ u(0) = 1, \quad u'(0) = 0; \quad v(0) = 0, \quad v'(0) = 1. \]
   Set \( F(a, b) = u(2\pi) + v'(2\pi) \) and show that if \( |F(a, b)| > 2 \) then no solutions remain bounded for all (real) \( t \).

2) Consider the system
   \[
   \begin{align*}
   \dot{x} &= -x + y \\
   \dot{y} &= kx - y - xz \\
   \dot{z} &= -z + xy
   \end{align*}
   \]
   where \( k > 0 \). Show that all solutions exist and remain bounded for \( t \geq 0 \).

3) Show that \( x' + x = x^2 \) has a solution \( u(t) \) with \( u(0) = x_0 \) that satisfies for all \( t \geq 0 \)
   \[ |u(t)| < 2|x_0| \]
   provided \( |x_0| \) is small enough.

4) Consider \( \dot{x} = F(t, x), \quad x(t_0) = x_0, \) where \( F \) is continuous and satisfies a Lipschitz condition in some open neighborhood of \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n\). Show that the solution \( x = \varphi(t, t_0, x_0) \) satisfies a Lipschitz condition in \( x_0 \) near \((t_0, x_0)\).

5) Show that a \( C^1 \) solution of the following equation can not exist in a large time interval
   \[
   \begin{align*}
   \{ & u_t + uu_x = 0, \quad t \geq 0, x \in (-\infty, +\infty), \\
   & u(x, 0) = h(x), \quad x \in (-\infty, +\infty),
   \end{align*}
   \]
   where \( h \) is a smooth function with compact support.

6) Show that the solution \( U \in C^2(\overline{\Omega}) \) of
   \[ \Delta U = 0 \text{ in } \Omega, \quad U = f \text{ on } \partial \Omega \]
   minimizes the integral
   \[ \int_{\Omega} |\nabla U|^2 \, dx \]
   among all functions in \( C^1(\overline{\Omega}) \) with boundary values \( f \).

7a) Let \( f(x) \) be bounded and continuous for \( x \in \mathbb{R}^n \) and satisfy
   \[ \int |f(y)| \, dy < \infty. \]
Show that there exists a solution $u(x, t)$ of

$$
\begin{align*}
  &u_t - \Delta u = 0 \text{ for } x \in \mathbb{R}^n, t > 0, \\
  &u(x, 0) = f(x),
\end{align*}
$$

such that $\lim_{t \to \infty} u(x, t) = 0$.

7b) Let $n = 1$. Show that the same conclusion holds for $f \in C^2(\mathbb{R})$ that have period $2\pi$ and satisfy

$$
\int_0^{2\pi} f(y) dy = 0.
$$

8) Let $u = u(x_1, x_2)$ be a solution of class $C^2$ on the semi-strip

$$
x_1 \geq 0, \ a \leq x_2 \leq b,
$$

of the equation

$$
u_{x_1} - u_{x_2x_2} = 0.
$$

Show that $u$ is determined uniquely by its Cauchy data on

$$
x_1 = 0, a < x_2 < b.
$$
1. Consider $\dot{x} = [A + \varepsilon B(t)]x$ where $A$ is an $n \times n$ real matrix whose eigenvalues $\lambda$ satisfy $\text{Re}\lambda < 0$ and $B(t)$ is continuous and periodic with period 1. 

Show that for $|\varepsilon|$ sufficiently small, all solutions satisfy $\lim_{t \to \infty} |x(t)| = 0$.

2. Show that solutions of 

$$\dot{x} = \frac{t^2 x^5}{1 + x^2 + x^4}$$

can be continued to the whole real line.

3. Let $f$ and $g$ be $C^1$ vector fields in $\mathbb{R}^2$ such that the inner product $(f(x), g(x)) = 0$ for all $x$. If $f$ has a closed orbit prove $g$ has a zero.

4. Solve 

$$(y + u) \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = x - y$$

$u(x, 1) = 1 + x.$

Show that this solutions is unique in the class $C^1(\Omega)$ where $\Omega = \{(x, y) \in \mathbb{R}^2, y > 0\}$.

5. Let $u$ be a harmonic function in a bounded connected open set $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. Show that there exists $x_0 \in \partial \Omega$ such that

$$|\nabla u(x_0)|^2 = \max_{x \in \Omega} |\nabla u(x)|^2.$$ 

6. Let $u \in C^2(\mathbb{R}^n \times \mathbb{R})$ be a solution of

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$u(x, 0) = f(x)$

$$\frac{\partial u}{\partial t}(x, 0) = g(x)$$

where $f, g \in C^\infty(\mathbb{R}^n)$ have compact support. Compute $\int_{\mathbb{R}^n} u(x, t)dx$ for every $t \in \mathbb{R}$. 

1. Assume that \( u \in C^2(\mathbb{R}^3) \) is a harmonic function such that

\[
|u(x)| \leq C(|x|^{1/2} + 1), \quad x \in \mathbb{R}^3
\]

for some \( C > 0 \). Prove that \( u \) is a polynomial.

2. Assume that \( u(x, t) \) is such that

\[
u \in C^\infty(\mathbb{R} \times (0, \infty)) \cap C(\mathbb{R} \times [0, \infty)) \cap L^\infty(\mathbb{R} \times (0, \infty))
\]

and

\[
u_t = \nu_{xx},
\]

\[u(x, 0) = u_0(x)
\]

where

\[u_0(x + 1) = u_0(x), \quad x \in \mathbb{R}.
\]

Prove that

\[u(x + 1, t) = u(x, t)\]

for \((x, t) \in \mathbb{R} \times [0, \infty)\).

3. Let \( u \in C^\infty(\mathbb{R}^3 \times \mathbb{R})\) be a solution of the equation

\[
u_{tt} = \Delta u,
\]

\[u(x, 0) = 0, \quad x \in \mathbb{R}^3
\]

\[u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3.
\]

If the support of \( g \) is compact, prove that

\[
limit_{t \to +\infty} ||u(\cdot, t)||_{L^\infty} = 0.
\]

4. Consider the equation

\[\varphi'' + f \varphi' + g \varphi = 0\]

where \( f(z), g(z) \) are complex functions. What conditions must \( f \) and \( g \) satisfy if \( \infty \) is a regular point.

Assume \( f, g \) are not constant, and \( \infty \) regular. Show that \( 0 \) is a singular point.

5. Consider the following system of differential equations

\[x' = y + \alpha \sin x
\]

\[y' = -\beta x.
\]

Linearize at the critical points and classify the critical points.
6. Let $A(t)$ be a real continuous matrix for $t \in \mathbb{R}^+$ such that

$$\lim_{t \to \infty} A(t) = C$$

where $C$ is a constant matrix. Show that every solution $\bar{x}(t)$ of $\ddot{x} = A\bar{x}$ has the property that

$$\lim_{t \to \infty} \frac{1}{t} \log \|\bar{x}(t)\| = \mu$$

where $\mu$ is the real part of an eigenvalue of $C$. 
Show your work and explain to obtain full credit

1. Show that for all \( \epsilon \) every solution of
\[
\begin{align*}
x' &= y \\ y' &= -x - \epsilon y
\end{align*}
\]
goes to zero as \( t \to \infty \).

2. Consider the second order differential equation \( x'' + (a + b \sin t)x = 0 \) and let \( u, v \) be two solutions with the initial data \( u(0) = 1, u'(0) = 0, v(0) = 0, v'(0) = 1 \). Show that if \( u(2\pi) + v'(2\pi) = 2 \) then there exists at least one periodic solution with period \( 2\pi \).

3. Show that the trivial solution of
\[
u'' + (u^2 - 2u + 1)u' + u^3 - u^5 = 0
\]
is asymptotically stable.

4. a) Find the solution \( u(x, t), t \geq 0, x \geq 0 \) of
\[
\begin{align*}
u_{tt} - c^2 u_{xx} &= 0, & t \geq 0, x \geq 0, \\
u(0, t) &= h(t), & t \geq 0, \\
u(x, 0) &= f(x), u_t(x, 0) &= g(x), & x \geq 0,
\end{align*}
\]
where \( f, g, h \in C^2([0, \infty)) \).
b) Let \( f = g = 0 \) and \( h(t) = \sin \pi t \). Is the solution a continuous and differentiable function? For each \( t \), determine the set \( \{ x : u(x, t) \neq 0 \} \).

5. (Dirichlet principle). Show that a solution \( v \in C^2(\bar{D}) \) of
\[
\Delta v = 0 \text{ in } \bar{D}, \quad v = f \text{ on } \partial D
\]
minimizes the Dirichlet integral \( \int_D |\nabla u|^2 \, dx \) among all functions \( u \in C^1(\bar{D}) \) with boundary value \( f \).

6. A. Consider the following problem for \( u(x, y), y \geq 0, x \in (-\infty, \infty), \)
\[
\begin{align*}
u_y &= u_{xx}, \\
u(x, 0) &= f(x)
\end{align*}
\]
where \( f \in C^\infty \).

a) Is the line \( y = 0 \) non characteristic?
b) Are the derivatives of \( u \) determined uniquely on \( y = 0 \) provided they exist?

B. Given a \( 2\pi \)-periodic function \( f \in C^2(\mathbb{R}) \), solve the Cauchy problem
\[
\begin{align*}
u_t(x, t) &= u_{xx}(x, t) & t > 0, x \in (-\infty, \infty), \\
u(x, 0) &= f(x), & x \in (-\infty, \infty);
\end{align*}
\]
Find \( \lim_{t \to \infty} u(x, t) \).