Qualifying Exam, Spring 1996

Complex Analysis: Do 4 of the following problems.

1. Evaluate
   \[ \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx, \quad \int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2} \, dx. \]

2. If \( f \) is analytic on \( \{0 < |z| < 1\} \), can \( e^f \) have a pole at \( 0 \)?

3. Suppose \( f \) is analytic on \( \{|z| < 1\} \) and that \( f(0) = f(1/2) = 0 \) and \( |f|_\infty \leq 1 \), find the best pointwise bound for \( |f(z)| \) and discuss when the bound is achieved.

4. Find a conformal map of the region \( D \) bounded by the two circles \( C_1 : \{|z| = 1\}, \ C_2 : \{|z - (1/2)| = 1/2\} \) onto the unit disk.

5. Is the set of analytic functions satisfying the bound \( \int_{\Delta} |f(z)|^2 \, dx \, dy \leq 1 \) on the unit disc \( \Delta = \{|z| < 1\} \) a normal family?

6. Suppose \( f(z) = \sum a_n z^n \) with the property that \( \lim_{n \to \infty} a_n = a \). Does \( f(z) \) have a pole at \( z = 1 \) with residue \(-a\)? and is \( f(z) - \frac{a}{1-z} \) analytic for \( |z| < \rho, \ \rho > 1 \)? Provide a proof if yes, a counterexample if no.

7. What can you say about the location of the zeroes of the polynomials \( 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n} \) for \( n \) sufficiently large?
REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1996

Problem 1 (Stability of contractive iteration) Let $(M, d)$ be a metric space, and suppose $T : M \to M$ satisfies
\[ d(Tx, Ty) \leq k \cdot d(x, y) \quad \text{for all} \quad x, y \in M \]
where $0 < k < 1$. Now suppose $\varepsilon > 0$, and a sequence $\{\hat{x}_n\}_{n=0}^{\infty}$ in $M$ satisfies
\[ d(\hat{x}_n, T\hat{x}_{n-1}) < \varepsilon \quad \text{for all} \quad n \geq 1 \]
Prove that for $0 \leq m < n$,
\[ d(\hat{x}_m, \hat{x}_n) < k^n \frac{2d(\hat{x}_0, T\hat{x}_0)}{1 - k} + \frac{2\varepsilon}{1 - k} \]

Problem 2 How many zeros does the polynomial $p(z) = z^4 - 2z + 3$ have in the unit disk $|z| < 1$?

Problem 3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and
\[ \int_{-\infty}^{\infty} \varphi(x) f(x) \, dx = 0 \]
for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ which have compact support. Prove: $f(x) = 0$ for a.e. $x$.

Problem 4 Evaluate
\[ \int_0^{\pi} \frac{d\theta}{2 + \sin \theta} \]

Problem 5 Let $(X, \Sigma, \mu)$ be a measure space with $\mu(X) < \infty$, and let $M$ denote the space of $\Sigma$-measurable extended-real-valued functions on $X$. Define $\rho : M \times M \to \mathbb{R}$ by
\[ \rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu \]
Show that $\rho$ is a metric on $M$, and that $f_n \to f$ in the $\rho$-metric iff $f_n \to f$ in measure.

Problem 6 Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Prove that there exists a point $z_0 \in \mathbb{C}$ such that we can expand $f(z)$ into a power series about $z_0$,
\[ f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \]
for which all $c_n \neq 0$.

Problem 7 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives $f_{xy}$ and $f_{yx}$. Prove $f_{xy} = f_{yx}$.
\textbf{Hint:} use Fubini’s theorem to integrate $f_{xy}$ and $f_{yx}$ over a rectangle $[a, b] \times [c, d]$.

Problem 8 Find a conformal mapping from the unit disk $|z| < 1$ to the region
\[ \Omega = \{ x + iy : (x < 0) \text{ and } (y > 0), \text{ or } (x \geq 0) \text{ and } (y > b) \} \]
where $b > 0$. 
REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1997

Directions: Do any seven of the following eight problems.

Problem 1 Prove: if \( n \geq 2 \) is an integer, then
\[
\int_0^\infty \frac{dx}{1 + x^n} = \frac{x/n}{\sin(\pi/n)}
\]

Problem 2 Suppose \( \Omega \) is an open connected region of the complex plane and \( f \) is a non-constant analytic function on \( \bar{\Omega} \). Prove: if \( |f(z)| = 1 \) on the boundary of \( \Omega \), then \( f(z) \) has at least one zero in \( \Omega \).

Problem 3 Formally, we have that
\[
\frac{(-1)^n n!}{t^{n+1}} = \frac{d^n}{dt^n} \left( \frac{1}{t} \right) = \frac{d^n}{dt^n} \int_0^\infty e^{-tx} \, dx = \int_0^\infty e^{-tx} \frac{d^n}{dt^n} e^{-ix} \, dx = \int_0^\infty (-1)^n x^n e^{-ix} \, dx
\]
so that on setting \( t = 1 \) we obtain
\[
\int_0^\infty x^n e^{-x} \, dx = n!
\]

Justify the calculation.

Problem 4 Let \( X = C([0,1]) \) be the space of all bounded continuous functions from \([0,1]\) to \( \mathbb{R} \) with the sup-norm distance,
\[
d(f,g) = \sup_{0 \leq t \leq 1} |f(t) - g(t)|
\]
You may assume that \((X,d)\) is complete. Let \( F: X \to X \) be a strict contraction, i.e., a function such that there exists \( k < 1 \) with
\[
d(Fx, Fy) \leq kd(x, y) \text{ for all } x, y \in X
\]
Let \( I \) denote the identity operator on \( X \), prove:
- \( I + F \) is a 1-1 mapping of \( X \) onto \( X \)
- \((I + F)^{-1}\) is continuous

Problem 5 Let \( K: [0,1] \times [0,1] \to \mathbb{R} \) be continuous, and let \( \mathcal{F} \) be the family of all functions \( f \) on \([0,1]\) of the form
\[
f(x) = \int_0^1 g(y) K(x,y) \, dy
\]

Problem 6 Show that for each \( \varepsilon > 0 \) the function
\[
f(z) = \sin z + \frac{1}{z}
\]
has infinitely many zeros in the strip \( |\Im z| < \varepsilon \).

Problem 7 Determine the order of the entire function
\[
f(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n^2} \right)
\]
(Recall that the order of an entire function $f$ is
\[ \lim_{r \to \infty} \frac{\log \log M(r)}{r} \]
where $M(r) = \max_{|z|=r} |f(z)|$.)

**Problem 8** Prove: if $A$ and $B$ are Lebesgue-measurable subsets of $\mathbb{R}$ with positive Lebesgue measure, then the set
\[ A + B = \{ a + b : a \in A, b \in B \} \]
has non-empty interior. (Hint: consider the convolution of the characteristic functions of $A$ and $B$.)
Problem 4. Determine all entire functions $f(z)$ for which

$$|f(z)| \leq C|z|^{3/2}$$

for all $|z|$ sufficiently large.
Problem 5. Evaluate:
\[ \int_{0}^{+\infty} \frac{\cos 2ax - 1}{x^2} \, dx \quad (a \text{ real}). \]
Problem 6. Let $u(z)$ be harmonic in the closed unit disk $D$, with

$$u(x) = 0 \text{ for } x \in [-1, 1].$$

Show that $u(z) = -u(\bar{z})$. 
DIRECTIONS. Do any seven of the following eight problems, using the paper and pens provided. Start each problem on a fresh sheet of paper. When you have completed the exam, be sure your name is printed on each page; sign the envelope, and return the exam papers in the envelope. You may keep this printed page.

Problem 1. Suppose $f \in L^1(d\mu)$. Prove: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set $A$ with $\mu(A) < \delta$, there holds

$$\left| \int_{A} f \, d\mu \right| < \varepsilon.$$

Problem 2. Let $f$ be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove: if $f$ maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

Problem 3. Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ which converges in the $L^1$ sense to 0, and whose derivatives $\{f'_n\}$ also converge to 0 in the $L^1$ sense. Prove: $\{f_n\}$ converges to zero uniformly.

Problem 4. Suppose $D$ is the open unit disk in $\mathbb{C}$, and $f : D \to D$ satisfies $f(1/2) = 1/2$. Show that $|f'(1/2)| \leq 3/4$.

Problem 5. Let $(X,T)$ be a topological space which has the property that every closed set $F$ is the intersection of a countable family of open sets. Prove: any finite measure $\mu$ on the Borel field of $(X,T)$ is regular: for each Borel set $E$ and each $\varepsilon > 0$, there exist an open set $G \supset E$ and a closed set $F \subset E$ such that $\mu(G \setminus F) < \varepsilon$.

(Hint: consider the collection of Borel sets $E$ for which this condition is true.)

Problem 6. Let $D$ be the open unit disk in $\mathbb{C}$, and let $f : D \to D$ be analytic with $f(0) = 0$. Suppose

$$|f(z)| \geq \frac{1}{6} \quad \text{for all} \quad |z| = \frac{1}{4}.$$

Show that $f$ assumes every value in the disk $|w| < \frac{1}{6}$.

Problem 7. Let $g : [0,1] \to \mathbb{R}$ be Lebesgue measurable, and suppose $f(x,y) := g(x) - g(y)$ is Lebesgue integrable on $[0,1] \times [0,1]$. Prove: $g$ is Lebesgue integrable.

Problem 8. Evaluate:

$$\int_0^\infty \frac{\sqrt{x}}{1 + x^3} \, dx.$$
Real and Complex Analysis Exam, Fall 1998

First Name:  
Last Name:  
Social Security Number:  

You should answer all 6 questions  
Please attach these two pages to your solution pages.

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**Problem 1.** Evaluate the integrals

\[
\int_{-\infty}^{\infty} \frac{2x^2 + x + 1}{x^4 + 5x^2 + 4} \, dx \quad \text{and} \quad \int_0^{2\pi} \frac{1}{3 + \cos \theta} \, d\theta.
\]

**Problem 2.** In each case either produce a function \( f \) analytic in a neighborhood of 0 with the given property or else prove that no such function exists:

(i) \( |f^{(n)}(0)| \geq (n!)^2 \) for all \( n \geq 0 \);

(ii) \( f\left(\frac{1}{n}\right) = (n^2 - 1)^{-1} \) for all \( n \geq 2 \);

(iii) \( |f\left(\frac{1}{n}\right)| \leq e^{-n} \) for all \( n \geq 1 \) and \( f \) is not identically zero near 0.

**Problem 3.** Let \( D \) denote the open unit disc \( |z| < 1 \). For each \( a \in D \) let \( S_a \) be a fractional linear transformation of \( D \) to itself which sends \( a \) to 0. Suppose that \( f \) is analytic mapping of \( D \) to itself such that \( f(a_1) = f(a_2) = \cdots = f(a_n) = 0 \) for points \( a_1, a_2, \ldots, a_n \in D \).

(i) Write down a formula for \( S_a \).

(ii) Show that

\[
|f(z)| \leq \prod_{r=1}^{n} |S_{a_r}(z)| \quad \text{for all} \quad z \in D.
\]

(iii) Deduce that \( |f(0)| \leq |a_1 a_2 \cdots a_n| \).
Analysis Qualifying Exam
Spring, 1999

- In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.
- Start each problem on a fresh sheet of paper.

Real Analysis. Do only three of the following four problems.

1. Suppose $f_n$, where $n = 1, 2, \ldots$, and $f$ are nonnegative functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Show that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for every measurable $E$. (Hint: Use Fatou's Lemma.)

2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $E \in \mathcal{M} \otimes \mathcal{N}$ (the product $\sigma$-algebra in $X \times Y$). Show that every section $E_x = \{y \in Y : (x, y) \in E\}$ is measurable.

3. Let $A$ denote the set of all $f \in C[0, 1]$ such that $f$ is monotonic on some open subinterval of $[0, 1]$. Show that $A$ is meager (that is, of the first category) in $C[0, 1]$ in the topology of uniform convergence.

4. (a) Show that the class of all step functions, of form $\sum_{j \leq n} c_j \chi_{(a_j, b_j)}$ with $a_j, b_j$ finite, is dense in $L^1(\mu)$ where $\mu$ is the Lebesgue measure on $\mathbb{R}$. (Hint: Why is the corresponding statement true for simple functions?)

   (b) Suppose $f \in L^1(\mu)$. Show that $\lim_{h \to 0} \int |f(x + h) - f(x)| \, dx = 0$. (Hint: Use (a).)

Complex Analysis. Do all four problems.

5. Suppose that $f$ is analytic on $\mathbb{C}$ and that $f$ is a homeomorphism of $\mathbb{C}$ onto a set $U$.

   (a) Show that $f$ has a non-essential singularity at $\infty$.

   (b) Deduce that $f$ must be of the form $f(z) = az + b$ for some $a \neq 0$ and that $U = \mathbb{C}$.

6. (a) Suppose that $f$ is analytic on the open unit disc $|z| < 1$ and there is a constant $M$ such that $|f^{(k)}(0)| \leq k^2 M^k$ for all $k \geq 1$. Show that $f$ can be extended to be analytic on $\mathbb{C}$.

   (b) Suppose that $f$ is analytic on the open unit disc $|z| < 1$ and there is a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for $k \geq 2$. Show that $f$ is identically zero.
7. Suppose that $f$ is analytic on $0 < |z| < 2$ and satisfies

$$\int_{|z|=1} z^n f(z) \, dz = 0$$

for $n = 0, 1, 2, \ldots$. Show that $f$ has a removable singularity at 0.

8. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2} \, dx.$$
COMPLEX ANALYSIS. Answer any three of the four questions.

5. Starting from Cauchy’s integral formula

\[ f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \]

(with appropriate conditions on the function \( f \), the curve \( \gamma \) and the point \( z \)), explain briefly how to obtain

(a) Cauchy’s estimates on \( |f^{(n)}(a)| \) in terms of \( \sup\{|f(z)| : |z - a| = r\} \);

(b) the result that an analytic function on a connected open set has isolated zeros (unless it is identically zero).

6. (a) Suppose that \( f \) is analytic on the set \( \{z \in \mathbb{C} : |z| > R\} \), for some \( R \). Define what is means for \( f \) to have an essential singularity at \( \infty \). Prove that if \( f \) is one-to-one on the set \( \{z \in \mathbb{C} : |z| > R\} \) then \( f \) has a non-essential singularity at \( \infty \).

(b) Suppose that \( f \) is an entire function and has a non-essential singularity at \( \infty \). Prove that \( f \) is a polynomial.

(c) Suppose that \( f \) is entire and is one-to-one on \( \mathbb{C} \). Prove that \( f \) is of the form \( f(z) = az + b \) where \( a \neq 0 \).

7. Show that

\[ \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \]

[Hint: consider the function \( (1 - e^{2ix})/x^2 \).]

8. Suppose that \( u \) is harmonic on the open disc \( \{z \in \mathbb{C} : |z| < 1\} \) and \( u(z) > 0 \) for all \( z \in D \) and \( u(0) = 1 \).

(a) Show that \( u \) is the real part of some analytic function \( f \) on \( D \) with \( f(0) = 1 \). Show also that \( |f(z) - 1| < |f(z) + 1| \) for all \( z \in D \).

(b) By applying Schwarz lemma to a suitable function, show that \( 1/3 \leq u(1/2) \leq 3 \).
COMPLEX ANALYSIS. Answer any three of the four questions.

5. One of the consequences of Cauchy's integral formula is the result: "if $f$ is analytic on an open ball $B(a, r) = \{ z \in \mathbb{C} : |z - a| < r \}$ then either $f \equiv 0$ on $B(a, r)$ or else there exist $n \geq 0$ and an analytic function $g$ on $B(a, r)$ with $g(a) \neq 0$ such that $f(z) = (z - a)^n g(z)$ on $B(a, r)$.")

(a) Indicate the main steps in the proof of the result above.

(b) Deduce that if $f_1$ and $f_2$ are analytic functions on a connected open set $U$ and if $f_1(z_n) = f_2(z_n)$ for some sequence $z_n \to z_\infty$ with $z_n \in U$ for all $n$ and $z_\infty \in U$ then $f_1 \equiv f_2$.

6. Show that

$$
\int_0^\infty \frac{x^a}{1 + x^2} \, dx = \frac{\pi}{2 \cos(a\pi/2)}
$$

for $-1 < a < 1$. You should be careful to justify your calculations. In particular you should show where the assumptions on $a$ are used.

7. Suppose that $f$ is analytic on an open subset $U$ of $\mathbb{C}$ and that $f$ is a one-to-one mapping of $U$ onto a subset $V$. Show that

(a) $f^{-1} : V \to U$ is continuous;

(b) $f'(z) \neq 0$ for all $z \in U$;

(c) $V$ is open and $f^{-1} : V \to U$ is analytic.

8. Let $f$ be analytic on an open disc $\{ z \in \mathbb{C} : |z| < r \}$ for some $r > 1$.

(a) Suppose that $|f(z)| < 1$ if $|z| = 1$. How many fixed points (that is, solutions of $f(z) = z$) does $f$ have in the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$?

(b) Suppose instead that $|f(z)| > 2$ if $|z| = 1$, and $f(0) = 1$. Does $f$ have to have a zero (that is, a solution of $f(z) = 0$) in the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$?
(1) Compute
\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx \]

(2) How many roots does the equation
\[ z^6 - 5z^2 + 8z + 2 = 0 \]
have in the unit disk \(|z| \leq 1|\)?

(3) Let \( f \) be an entire function which is real on the real axis, not identically zero, and for which \( f(0) = 0 \). Prove that if \( f \) maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

(4) Prove that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the following holds:
- if \( f \) is holomorphic with \( |f| < 1 \) in \( \{ z : |\text{Im } z| < 2 \} \), and if \( |f(z)| < \delta \) for all \( x \in \mathbb{R} \) with \(-1 < x < 1\), then \( |f| < \varepsilon \) in \( \{ z : |\text{Im } z| < 1 \} \).

HINT: First do the problem for \( \{ z : |z| < 1 \} \) in place of \( \{ z : |\text{Im } z| < 1 \} \).