ALGEBRA QUALIFYING EXAM SPRING 2006

Work all the problems. Be as explicit as possible in your solutions and justify your statements with specific reference to the results that you use. Partial credit will be given for partial solutions. Let \( \mathbb{Q} \) denote the field of rational numbers, \( \mathbb{C} \) the field of complex numbers, and \( \mathbb{F}_q \) the finite field of \( q \) elements.

1. Up to isomorphism, describe the groups of order 3·17·19.

2. i) For \( p \) a prime, \( q = p^k \), and \( n \) a positive integer, describe a condition that guarantees that the multiplicative group \( \mathbb{F}_q^* = (\mathbb{F}_q, \cdot) \) contains an element of order \( n \).
   ii) Determine the cardinality of a splitting field \( L \) over \( \mathbb{F}_3 \) of \( x^{13} - 1 \in \mathbb{F}_3[x] \), and the structure of \( \text{Gal}(L/\mathbb{F}_3) \).

3. Let \( f(x) = (x^3 - 2)(x^3 - 3) \in \mathbb{Q}[x], M \subseteq \mathbb{C} \) a splitting field over \( \mathbb{Q} \) for \( f(x) \), \( G = \text{Gal}(M/\mathbb{Q}) \), and \( \omega \in \mathbb{C} \) a primitive cubic root of unity.
   i) Show that \( \omega \in M \).
   ii) Assume that \( 3^{1/3} \notin \mathbb{Q}(\omega, 2^{1/3}) \subseteq M \), and use this to find the order of \( G \).
   iii) Describe how the elements of \( G \) act on \( M \).
   iv) Determine the structure of \( G \).

4. In \( \mathbb{C}[x, y] \) show that for some integer \( m \geq 1 \), \( (3x^2 + 10xy + 3y^2)^m \in (x + y - 2, x^2 + y^2 - 10) \), the ideal of \( \mathbb{C}[x, y] \) generated by \( x + y - 2 \) and \( x^2 + y^2 - 10 \).

5. Let \( g_1, g_2, \ldots, g_n, \ldots \in R \), a commutative Noetherian ring with 1, and let \( I \) be an ideal of \( R \). Assume that for each \( i \) there is \( k_i \geq 1 \) so that \( g_i^{k_i} \in I \). Show that there is a positive integer \( K \) so that \( g_1 g_2 \cdots g_n \in I \) for any choices of \( g_{ij} \in \{ g_i \} \).

6. Let \( S \) be a finite ring so that for each \( x \in S \), \( x^3 = x \).
   i) Show that \( S \) contains no nonzero nilpotent element.
   ii) Show that, up to isomorphism, \( S \) is a direct sum of copies of \( \mathbb{F}_2, \mathbb{F}_3 \), and \( \mathbb{F}_4 \).
1. (a) Find the number of Sylow $p$-subgroups of the symmetric group $S_p$. Here $p$ is a prime.
(b) Use (a) to prove that
\[(p - 1)! + 1 = 0 \mod p.\]

2. Let $G$ be a finite solvable group and $H$ a minimal (non-trivial) normal subgroup. Show $H$ is isomorphic to a direct sum of cyclic groups of order $p_i$, for some prime $p_i$. (Hint: First show that the commutator subgroup $H'$ of $H$ is $\{e\}$.)

3. Let $m_1, m_2, \ldots, m_n$ be positive integers which are pairwise relatively prime (that is, gcd$(m_i, m_j) = 1$ for all $i \not= j$).
(a) Show that $F := \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n})$. (Hint: induction.)
(b) Show that $F = \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n})$ is Galois over $\mathbb{Q}$. What is its Galois group?

4. Let $k$ be a field, let $R$ be a commutative $k$-algebra, and let $S = M_n(R)$ be all $n \times n$-matrices with entries in $R$. Choose $A_1, \ldots, A_m \in S$. Show that there exists a (left) Noetherian $k$-subalgebra $S_0$ of $S$ which contains all of the matrices $A_i$. (Hint: Consider the subalgebra $R_0 \subset R$ generated by all entries of $A_1, \ldots, A_m \in S$.)

5. Let $R = \mathbb{C}[x, y, z]$, let $I = (x^2z^3 - y^2z + xyz - x^2y)$ be an ideal of $R$, and define $S = R/I$.
(a) Prove that the polynomial $x^2z^3 - y^2z + xyz - x^2y$ is irreducible in $R$. (Hint: consider it as an element of $\mathbb{C}[x, y][z]$.)
(b) Show that $S$ is a Noetherian integral domain.
(c) Prove that in $S$ the intersection of all maximal ideals is $\{0\}$.

6. Let $k$ be a finite field and let $R$ be a finite dimensional semi-simple $k$-algebra such that for all $r \in R$, there exists a positive integer $n = n(r) > 0$ such that $r^n(r)$ is in the center of $R$. Prove that $R$ is commutative.

7. Let $M$ be a finitely generated $\mathbb{Z}$-module with torsion submodule $T(M)$.
(a) Justify: $M/T(M)$ is a free $\mathbb{Z}$-module.
For parts (b) and (c), set $r(M) := \text{rank}(M/T(M))$.
(b) Show that $r(M) = \text{dim}_\mathbb{Q}(M \otimes \mathbb{Q})$.
(c) Assume that
\[0 \to M \to N \to P \to 0\]
is an exact sequence of $\mathbb{Z}$-modules. Show that
\[r(N) = r(M) + r(P).\]
1. For $G$ a finite group with $|G| > 1$ and $p$ a prime dividing the order of $G$, let $O_p(G) = \cap \{P \in \text{Syl}_p(G)\}$.
   a) Show that $O_p(G)$ is a normal subgroup of $G$.
   b) Show that if $N$ is a normal subgroup of $G$ with $|N| = p^k$, then $N \subseteq O_p(G)$.
   c) Prove that if $G$ is solvable then for some $p$, $|O_p(G)| \neq 1$.

2. Let $F = GF(p^n)$ be a field of (exactly) $p^n$ elements. Suppose that $k$ is a positive integer dividing $n$, and set $B = \{a^{p^k} + a^{p^{2k}} + \cdots + a^{p^{nk}} \mid a \in F\}$.
   i) Show that $B \subseteq E$, a subfield of $F$ with $p^k$ elements.
   ii) Show that $B = E$.

3. Let $A \in M_n(\mathbb{Q})$ with $A^k = I_n$. If $j$ is a positive integer with $(j, k) = 1$, show that $\text{tr}(A) = \text{tr}(A^j)$.
   (Hint: Consider $A \in M_n(\mathbb{Q}(\varepsilon))$ for $\varepsilon = e^{2\pi i/k}$, where $e^{2\pi i} = -1$.)

4. Let $R$ be a commutative ring with 1 and let $M$ be a Noetherian $R$-module. If $f \in \text{Hom}_R(M_R, M_R)$ is surjective, show that $f$ is an automorphism of $M_R$.

5. Let $f, g \in \mathbb{C}[x, y]$ so that $(0, 0) \in \mathbb{C}^2$ is the only common zero of $f$ and $g$. Prove that there is a positive integer $m$ so that whenever $h \in \mathbb{C}[x, y]$ has no monomial of degree less than $m$, then $h \in f\mathbb{C}[x, y] + g\mathbb{C}[x, y]$.

6. For a fixed positive integer $n > 1$, describe all finite rings $R$ so that $x^n = x$ for all $x \in R$. 

1. Let $G$ be a group of order 105.
   (a) Show $G$ has a normal subgroup of index 3.
   (b) Show $Z(G) \neq 1$.
   (c) Determine all possibilities for $G$.

2. Let $p$ be a prime. A group $G$ is called $p$-divisible if the map $x \to x^p$ is surjective. Suppose that $G$ is a finitely generated abelian group. Show that $G$ is $p$-divisible if and only if $G$ is finite and $p$ does not divide the order of $G$.

3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Suppose that $f \in R$ is irreducible. If $g(a) = h(a)$ whenever $f(a) = 0$, show that $g + (f) = h + (f)$ in $R/(f)$.

4. Let $F$ be a field. Suppose that $A$ is an $F$-subalgebra of $M_n(F)$ containing the identity of $M_n(F)$.
   (a) If $A$ is a domain, show that $A$ is a division algebra and $\dim A \leq n$.
   (b) If $A$ is simple, show that $(\dim A) | n^2$ (hint: Let $V$ be the space of column vectors of size $n$ over $F$ — this is a left $M_n(F)$-module of dimension $n$; show that $V$ is a direct sum of say $s$ isomorphic copies of a simple $A$-module $U$. Relate the dimension of $A$ and the dimension of $U$).

5. Let $p$ be a prime. Let $F := F_{p^n}$ be the field of size of $p^n$. Let $f(x) \in F[x]$ be irreducible of degree $t$.
   (a) Show that the splitting field for $f$ has size $p^{nt}$.
   (b) If $n = 1$, show that $f(x)|(x^{p^m} - x)$ if and only if $t | n$.
   (c) How many irreducible polynomials of degree 6 are there over $F_2$?

6. Let $R$ be a commutative ring with 1. Assume that $R = a_1 R + \ldots + a_n R$ for some $a_i \in R$. Let $M = \{(r_1, \ldots, r_n) \in R^n | \sum a_i r_i = 0\}$. Show that $M$ is a projective $R$-module and can be generated by $n$ elements as an $R$-module.
Algebra Qualifying Examination, Spring 2008

Directions

This exam consists of 7 problems. Please do 6 of them and show your work. If you are using a well-known result in your proof, please refer to it by name. Good Luck!

1. Let $G$ be a finite group with $A$ a normal subgroup of $G$. Let $P$ be a Sylow $p$-subgroup of $A$. (a) If $g \in G$, show that $gPg^{-1} = xPx^{-1}$ for some $x \in A$. (b) Prove that $G = AN_G(P)$. (c) Prove that if $[G : A] = p$, then the number of Sylow $p$-subgroups of $G$ that contain $A$ is equal to $[N_G(P) : N_G(Q)]$ where $Q$ is a Sylow $p$-subgroup of $G$ containing $P$.

2. Let $G$ be a finite group of order $n$. Let $f : G \to S_n$ be the regular representation of $G$. (a) Show that the image of $f$ is contained in the alternating group if and only if the Sylow 2-subgroup of $G$ is not cyclic. (b) Use (a) to show that if $G$ has a cyclic nontrivial Sylow 2-subgroup, then $G$ contains a normal subgroup $N$ of odd order (hint: show first that $G$ has a normal subgroup of index 2).

3. Let $A$ be a (commutative) integral domain that is not a field and let $K$ be the quotient field of $A$.

(a) Prove that $K$ is not finitely generated as an $A$-module. Hint: use Nakayama’s lemma.

(b) Can $K$ ever be finitely generated as an $A$-algebra?

4. Let $K$ be a field and $p$ be a prime number that is not equal to the characteristic of $K$. Assume that $K$ does not contain a primitive $p$-th root of unity. Let $a \in K^*$ that is not a $p$-th power and let $b$ be an
element of an separable closure of \( K \) with \( b^p = a \). Consider the field \( L = K(\zeta, b) \), where \( \zeta \) is a primitive \( p \)-th root of unity. Prove that \( L/K \) is a Galois extension and that \( \text{Gal}(L/K) \) is isomorphic to the group of 2 by 2 matrices
\[
\begin{pmatrix}
1 & r \\
0 & s
\end{pmatrix}
\]
where \( r \in \mathbb{Z}/p\mathbb{Z} \) and \( s \in (\mathbb{Z}/p\mathbb{Z})^* \). Is this group abelian?

5. Let \( R = \mathbb{C}[x, y] \) and consider the two ideals \( I = (2x + y) \) and \( J = (x^2 - y) \). (a) show that \( I \) and \( J \) are both prime ideals of \( R \), and that each of them is the intersection of all of the maximal ideals containing it. (b) Consider the ideal \( I + J \). Is it a prime ideal? (c) Same question for \( I \cap J \).

(hint: you can get a lot of intuition for this problem by thinking about the analogous varieties in \( \mathbb{R}^2 \)).

6. (i) Let \( A \) be a finitely generated abelian group. If \( \ell \) is a prime number, let \( A[\ell] \) denote the set of elements of \( A \) that are killed by \( \ell \). Then \( A[\ell] \) and \( A/\ell A \) are finite groups, say of orders \( n_1 \) and \( n_2 \), respectively. Express the difference \( n_2 - n_1 \) in terms of other invariants of \( A \).

(ii) If \( A \) is an abelian group such that the groups \( A[\ell] \) and \( A/\ell A \) are finite, is \( A \) necessarily finitely generated? Give a proof or a counterexample.

7. Let \( R \) be a (left) Artinian ring which is an algebra over the field \( k \). Assume that every \( r \in R \) is algebraic over \( k \), with a minimal polynomial of the form
\[
x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.
\]
such that either \( a_0 \) or \( a_1 \) is non-zero. Show that \( R \) is a direct sum of division rings.
ALGEBRA QUALIFYING EXAM, Fall 2008

1. Let $p, q$ be odd primes with $p > 7$ and $q > 8p$. Let $G$ be a group of order $8pq$.
   (a) Show that $G$ has a normal subgroup of order $pq$.
   (b) Show that $G$ has a normal subgroup of index 2.
   (c) Show that $G$ has a nontrivial center.

2. Let $G = L_1 \times \ldots \times L_t$, for $t > 1$, where all of the $L_i$ are simple groups.
   (a) Assuming that all of the $L_i$ are nonabelian, prove that the only normal subgroups of $G$ are direct products of some subset of the $L_i$. (Hint: Let $N$ be a normal subgroup of $G$ and show that if the $i$th projection of $N$ into $L_i$ is nontrivial, then $N$ contains $L_i$).
   (b) Now suppose that all $L_i \cong L$, with $L$ simple (possibly abelian). Show that there is no nontrivial proper subgroup of $G$ which is invariant under all automorphisms of $G$. (Hint: Consider the abelian and nonabelian cases separately.)
   (c) Suppose that $G = L \times L$ with $L$ a nonabelian simple group. Let $D = \{(x, x) | x \in L\}$ be the diagonal subgroup. Show that $D$ is a maximal subgroup of $G$.

3. Consider $f(x) = x^4 + x^3 + 9 \in \mathbb{Q}[x]$.
   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$. (Hint: first show that the only possible factors are quadratic, and then see what happens when $x$ is replaced by $-x$.)
   (b) Find the Galois group of $f(x)$ over $\mathbb{Q}$.
   (c) Describe the splitting field of $f$ over $\mathbb{Q}$ and the intermediate fields.

4. Let $R$ be a commutative Noetherian ring. Show that any surjective ring endomorphism $\phi: R \to R$ is an automorphism.
   (Hint: consider the iterations $\phi, \phi^2, \phi^3, \ldots$)

5. Let $I$ be the ideal
   
   $I = (x^{37}, y^{31}, z^{29}, t^{23}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subset \mathbb{C}[x, y, z, t].$

   If $f(x, y, z, t)$ is any polynomial without constant term show that some power of $f$ is in $I$.

6. Let $A$ be a finite-dimensional algebra over $\mathbb{C}$. Show that if $x, y \in A$ such that $xy = 1$, then also $yx = 1$.

7. Let $A, B, C$ be finitely generated modules over a PID $R$. Show that $B$ is isomorphic to $C$ if and only if $A \oplus B$ is isomorphic to $A \oplus C$. 
Throughout, \( \mathbb{Z} \) denotes the integers, \( \mathbb{Q} \) the rational numbers, \( \mathbb{R} \) the real numbers, and \( \mathbb{C} \) the complex numbers.

1. Let \( G \) be a finite group. Define the Frattini subgroup of \( G \) to be \( \Phi(G) \), the intersection of all maximal subgroups of \( G \).
   (1) Show that \( \Phi(G) \) is characteristic in \( G \) (i.e. invariant under any automorphism of \( G \)).
   (2) Show that if \( G = \langle \phi(G), S \rangle \) for some subset \( S \) of \( G \), then \( G = \langle S \rangle \).
   (3) Let \( P \) be a Sylow \( p \)-subgroup of \( \phi(G) \). Show that \( P \) is normal in \( G \) (hint: first show that \( G = \Phi(G)N_G(P) \) by using Sylow’s theorems and then use (2)).
   (4) Show that \( \Phi(G) \) is nilpotent.

2. Let \( G \) be a finite group acting on the finite set \( X \) with \( |X| = n > 1 \), and suppose that \( G \) has \( N \) orbits on \( X \). If \( g \in G \), let \( F(g) \) be the number of \( x \in X \) fixed by \( g \).
   (1) Prove that \( \sum_{g \in G} F(g) = |G|N \) (this is known as Burnside’s Lemma).
   (2) Prove that if \( G \) is transitive on \( X \), then \( F(g) = 0 \) for some \( g \in G \) (either use (1) or prove directly).
   (3) Show that this is not always true if \( G \) is not transitive on \( X \).

3. Let \( f(x) = x^4 - x^3 + x^2 - x + 1 \in \mathbb{Q}[x] \). Find the splitting field (over \( \mathbb{Q} \)) of \( f(x) \), and compute \( Gal(K/\mathbb{Q}) \).

4. Construct an example of each of the following (with reasons):
   (1) A field extension \( F \subseteq K \) which is normal but not separable.
   (2) A field extension \( F \subseteq K \) which is separable but not normal.
   (3) A field extension \( F \subseteq K \) which is neither separable nor normal.

5. Let \( F \) be the field of \( p \) elements. Let \( A \in G := GL(n, F) \).
   (1) Show that \( A \) has order a power of \( p \) if and only if \( (A - I)^n = 0 \).
   (2) Show that if this is the case then the order of \( A \) is less than \( np \).
   (3) Show that any such \( A \) is similar to an upper triangular matrix.

6. Let \( M \) be a finitely generated abelian group, and \( N \) a subgroup. If \( M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q} \), show that \( M/N \) is torsion.

CONTINUED →
7. Consider the polynomial ring $\mathbb{C}[x, y]$ and let $I$ be the ideal
$I = (x + y - 2, x^2 + y^2 - 10)$.

(1) Show that there exists some $m > 0$ such that $(3x^2 + 10xy + 3y^2)^m \in I$.
(2) Show that the two ideals $I_1 = (x + y - 2)$ and $I_2 = (x^2 + y^2 - 10)$ are
prime ideals. Are they maximal?
(3) Can $I$ be written as an intersection of maximal ideals? Why or why not?

8. Let $A$ be a finite-dimensional algebra over $\mathbb{R}$, with center $Z = Z(A)$ and
Jacobson radical $J = J(A)$. Assume that for any $a \in A$, there is some
$n = n(a) \geq 1$ such that $a^{2^n} - a \in Z$.

(1) Show that $J \subseteq Z$.
(2) Show that $A/J$ is commutative.

In fact $A$ itself is commutative, although you do not have to show this.
ALGEBRA QUALIFYING EXAM, Fall 2009

Notation: \( \mathbb{Q} \) denotes the rational numbers, \( \mathbb{R} \) the real numbers, \( \mathbb{C} \) the complex numbers, and \( \mathbb{F}_p \) the field with \( p \) elements, for \( p \) a prime.

1. Determine up to isomorphism all groups of order \( 1005 = 3 \cdot 5 \cdot 67 \).

2. (a) Let \( G \) be a group of order \( 2^m k \), where \( k \) is odd. Prove that if \( G \) contains an element of order \( 2^m \), then the set of all elements of odd order in \( G \) is a (normal) subgroup of \( G \).
   (Hint: consider the action of \( G \) on itself by left multiplication \( \Phi_L \), and then consider the structure of the permutations \( \Phi_L(x) \), for \( x \in G \).)
   (b) Conclude from (a) that a finite simple group of even order must have order divisible by 4.

3. Give a brief argument or a counterexample for each statement:
   (a) \( x^{2n} + 1 \in \mathbb{Q}[x] \) is irreducible for all positive integers \( n \);  
   (b) Any splitting field for \( x^{13} - 1 \in \mathbb{F}_3[x] \) has \( 3^{12} \) elements. (c) \( \text{Gal}(L/\mathbb{Q}) \) for \( L \) a splitting field over \( \mathbb{Q} \) of \( x^5 - 2 \in \mathbb{Q}[x] \) has a normal 5-Sylow subgroup.

4. Let \( A \) denote the commutative ring \( \mathbb{R}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 + 1) \).
   (a) Prove that \( A \) is a Noetherian domain.
   (b) Give an infinite family of prime ideals of \( A \) that are not maximal.

5. Let \( R = \mathbb{C}[x_1, \ldots, x_n] \), let \( A = [a_{ij}] \in M_n(\mathbb{C}) \), and choose \( b_1, \ldots, b_n \in \mathbb{C} \).
   For each \( i = 1, \ldots, n \), set \( L_i = a_{1i} x_1 + a_{2i} x_2 + \cdots + a_{ni} x_n - b_i \in R \), and consider the ideal \( I = (L_1, \ldots, L_n) \subseteq R \).
   Prove that \( R/I \) is finite-dimensional \( \iff \) the matrix \( A \) is invertible in \( M_n(\mathbb{C}) \).

6. Let \( R = K[x] \), for \( K \) a field, and let \( M \) be a finitely-generated torsion module over \( R \). Prove that \( M \) is a finite-dimensional \( K \)-module.

7. Let \( G \) be a finite group and \( K \) a field, and consider the group algebra \( R = KG \) (that is, \( R \) is a \( K \)-vector space with basis \( \{ g \in G \} \), and multiplication determined by the group product \( g \cdot h \), for \( g, h \in G \)).
   If \( G \) is the dihedral group of order 8, find the dimensions of all of the simple (left) modules for \( R = \mathbb{F}_2 G \).
   (Hint: remember that \( KG \) always has the “trivial representation” \( V_0 = Kv \), such that for any \( g \in G, a \in K, ag \cdot v = av \).)
1. Let \( f(x) = x^3 + 3 \in \mathbb{Q}[x] \). Show that the Galois group of \( f \) is \( S_3 \).

2. (a) Let \( G \) be a group of order \( pqr \), where \( p \leq q \leq r \) are primes. Show that \( G \) contains a normal subgroup of index \( p \).
   (b) Determine up to isomorphism all groups of order \( 3 \cdot 7 \cdot 13 \).

3. Let \( R \) be a commutative Noetherian ring, and let \( I, J, \) and \( K \) be ideals of \( R \). We say \( I \) is irreducible if \( I = J \cap K \Rightarrow I = J \) or \( I = K \).
   (a) Show that every ideal of \( R \) is a finite intersection of irreducible ideals.
   (b) Show that every irreducible ideal is primary. (An ideal \( I \) of \( R \) is primary if \( R/I \neq 0 \), and every zero-divisor in \( R/I \) is nilpotent.)

4. Let \( A \) be a finite-dimensional algebra over a field \( K \), such that for every \( a \in A \), \( a^2 = a \). Show that \( A \) is a direct product (sum?) of fields. Which fields can arise?

5. Let \( G \) and \( H \) be finitely generated abelian groups such that \( G \oplus H = 0 \). Show that \( G \) and \( H \) are finite and have relatively prime orders.

6. Let \( S \) and \( T \) be diagonalizable endomorphisms of a finite-dimensional complex vector space. If \( S \) and \( T \) commute show that they are polynomials in each other.

7. What are the prime ideals of \( \mathbb{Z}[x] \)? What are the maximal ideals? Carefully explain your answers.
ALGEBRA QUALIFYING EXAM FALL 2010

Do all six problems. Each problem is worth 4 points and partial credit may be awarded.

1. Use Sylow’s Theorems to show that any group of order $(9^2 - 4)^3$ is solvable.

2. For any finite group $G$ and positive integer $m$, let $n_G(m)$ be the number of elements $g$ of $G$ that satisfy $g^m = e_G$. If $A$ and $B$ are finite abelian groups so that $n_A(m) = n_B(m)$ for all $m$, show that as groups $A = B$.

3. If $g(x) = x^5 + 2 \in \mathbb{Q}[x]$, for $\mathbb{Q}$ the field of rational numbers, compute the Galois group of a splitting field $L$ over $\mathbb{Q}$ of $g(x)$. How many subfields of $L$ containing $\mathbb{Q}$ are Galois over $\mathbb{Q}$?

4. Let $P$ be a minimal prime ideal in the commutative ring $R$ with 1; that is, if $Q$ is a prime ideal in $R$ and if $Q \subseteq P$, then $Q = P$. Show that each $x \in P$ is a zero divisor in $R$.

5. Set $R = \mathbb{C}[x_1, \ldots, x_n]$ with $n \geq 3$ and $\mathbb{C}$ the field of complex numbers. For any subset $S \subseteq R$, let $\mathcal{V}(S) = \{ \alpha \in \mathbb{C}^n \mid g(\alpha) = 0 \text{ for all } g \in S \}$. Consider the ideal $I$ of $R$ defined by $I = (x_1 \cdots x_{n+1} - x_n, x_1 \cdots x_n - x_{n-2} \cdots, x_2 \cdots x_n - x_1)$, so the generators of $I$ are obtained by subtracting each $x_i$ from the product of the others. Show that there are fixed positive integers $s$ and $t$ so that for each $0 \leq i \leq n$, $(x_i^s - x_i)^t \in I$. (Hint: Consider the product of the generators of $I$.)

6. Let $R$ be a right artinian algebra over an algebraically closed field $F$. Show that $R$ is algebraic over $F$ of bounded degree. That is, show there is a fixed positive integer $m$ so that for any $r \in R$ there is a nonzero $g_r(x) \in F[x]$ with $g_r(r) = 0$ and with $\deg g \leq m$. 
1. Let $G$ be a finite group with a cyclic Sylow $2$-subgroup $S$.
   
   (a) Show that any element of odd order in $N_G(S)$ centralizes $S$.
   
   (b) Show that $N_G(S) = C_G(S)$.
   
   (c) Give an example to show that (a) can fail if $S$ is abelian.

2. Let $G$ be a finite group with a cyclic Sylow $2$-subgroup $S 
eq 1$.
   
   (a) Let $\rho : G \to S_n$ be the regular representation with $n = |G|$. Show that $\rho(G)$ is not contained in $A_n$.
   
   (b) Show that $G$ has a normal subgroup of index $2$.
   
   (c) Show that the set of elements of odd order in $G$ form a normal subgroup $N$ and $G = NS$.

3. For a group $G$ and $p$ a prime let $G(p) = \{g \in G | g^p = 1\}$.
   
   (a) Show that if $G$ is Abelian, then $G(p)$ is a subgroup of $G$. Give an example to show that $G(p)$ need not be a subgroup in general.
   
   (b) Let $G, H$ be finitely generated Abelian groups with $G/G(p) \cong H/H(p)$ and $G/G(q) \cong H/H(q)$ for different primes $p, q$. Show that $G \cong H$.

4. Let $R$ be a prime ring with only finitely many right ideals.
   
   (a) Show that $R$ is a simple ring.
   
   (b) Prove that either $R$ is finite or $R$ is a division ring.

5. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let $J$ be a nonzero proper ideal of $R$. Let $A = A(X), B = B(X) \in M_r(R)$ and assume that $\det(A)$ is a product of distinct monic irreducible polynomials in $R$. Assume that for each $\alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $B(\alpha) \in M_r(\mathbb{C})$ invertible implies that $A(\alpha)$ is invertible. Show that $\det(A)$ divides $\det(B)$ in $R$.

6. Let $L$ be a splitting field over $\mathbb{Q}$ for $p(x) = x^{10} + 3x^5 + 1$. Let $G = \text{Gal}(L/\mathbb{Q})$.
   
   (a) Show that $G$ has a normal subgroup of index $2$.
   
   (b) Show that $4$ divides $|G|$.
   
   (c) Show that $G$ is solvable.
ALGEBRA QUALIFYING EXAM FALL 2011

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of integers is \( \mathbb{Z} \), the set of rational numbers is \( \mathbb{Q} \), and set of the complex numbers is \( \mathbb{C} \).

1. Let \( I \) and \( J \) be ideals of \( R = \mathbb{C}[x_1, x_2, \ldots, x_n] \) that define the same variety of \( \mathbb{C}^n \). Show that for any \( x \in (I + J)/I \) there is \( m = m(x) > 0 \) with \( x^m = 0_{R/I} \). Show there is an integer \( M > 0 \) so that for any \( y_1, y_2, \ldots, y_M \in (I + J)/I \), \( y_1 y_2 \cdots y_M = 0_{R/I} \).

2. If \( K \subseteq L \) are finite fields with \( |K| = p^n \) and \( [L : K] = m \) then show that for each \( 1 \leq i < nm \), any \( a \in L - K \) has a \( p^i \)-th root in \( L \). When \( m = 3 \), show that every \( b \in K \) has a cube root in \( L \).

3. Let \( F \) be an algebraically closed field and \( A \) an \( F \)-algebra with \( \dim_{F} A = n \). If every element of \( A \) is either nilpotent or invertible, show that the set of nilpotent elements of \( A \) is an ideal \( M \) of \( A \), that \( M \) is the unique maximal ideal of \( A \), and that \( \dim_{F} M = n - 1 \).

4. Let \( M \) be a finitely generated \( F[x] \) module, for \( F \) a field.
   i) Show that if \( f(x)m = 0 \) for \( f(x) \neq 0 \) forces \( m = 0 \), then \( M \) is a projective \( F[x] \) module.
   ii) If \( H \) is an \( F[x] \) submodule of \( M \) show that \( M = H \oplus K \) for a submodule \( K \) of \( M \) if and only if: \( f(x)m \in H \) for \( f(x) \neq 0 \) implies that \( m \in H \).

5. Up to isomorphism, describe the possible structures of any group of order 987 = 3·7·47.

6. Let \( R = \mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots] \) and let \( \{f_i(X) \mid i \geq 1\} \subseteq R \) satisfy
   \[ f_i(X)R \subseteq f_2(X)R \subseteq \cdots \subseteq f(X)R \subseteq \cdots \]
   Show \( f_i(X)R = f_m(X)R \) for some \( m \) and all \( s \geq m \).

7. Let \( U \) be the set of all \( n \)-th roots of unity in \( \mathbb{C} \), for all \( n \geq 3 \), and set \( F = \mathbb{Q}(U) \). For primes \( p_1 < \cdots < p_i \) and nonzero \( a_1, \ldots, a_i \in \mathbb{Q} \), set \( M = F(a_1^{1/p_1}, \ldots, a_i^{1/p_i}) \subseteq \mathbb{C} \). Show that \( M \) is Galois over \( F \) with a cyclic Galois group. For any subfield \( F \subseteq L \subseteq M \), show that there is a subset \( T \) of \( \{a_j^{1/p_j}\} \) so that \( L = F(T) \).