1. Let $G$ be a finite group and $p$ a prime number. Let $P$ be a $p$-Sylow subgroup of $G$ and denote the normalizer of $P$ in $G$ by $N_G(P)$.
   i) Show that $N_G(P) = N_G(N_G(P))$.
   ii) If $K$ is a normal subgroup of $G$ and $K$ contains $P$, show that $G = KN_G(P)$.
   iii) If no proper subgroup of $G$ is its own normalizer, show that the center of $G$ is not trivial.

2. Up to isomorphism describe all finitely generated Abelian groups which satisfy all of the following properties: i) $G \otimes \mathbb{Q} \cong \mathbb{Q}^2$; ii) $G \otimes \mathbb{Z}/7\mathbb{Z} \cong (\mathbb{Z}/7\mathbb{Z})^3$; and iii) for any prime $p \neq 7$, $G \otimes \mathbb{Z}/p\mathbb{Z} \cong (\mathbb{Z}/p\mathbb{Z})^2$.

3. Let $R$ be a left Artinian ring with Jacobson radical $J(R)$. If $R \neq J(R)$ show that $R$ is a left Noetherian ring.

4. Determine if each of the following polynomials is irreducible, and justify your answer.
   i) $x^2 + 1 \in \mathbb{Q}[x]$.
   ii) $x^n + x^{n-1} + \cdots + x^2 + x + 1 \in F[x_1, \ldots, x_n]$ for $F$ any field.
   iii) $x^4 + 1 \in F_p[x]$, $p$ an odd prime (note that $p^2 \equiv 1 \pmod{8}$).
   iv) $x^p + x^{p-1} + \cdots + x + 1 \in F_p[x]$, $p$ an odd prime.

5. Let $R$ be a commutative ring with $1$, and let $r_1, \ldots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$.
   If $M = \{(a_1, \ldots, a_n) \in R^n | a_1r_1 + \cdots + a_nr_n = 0\}$, show that $M$ is a projective $R$ module.

6. Let $K$ be a finite Galois extension of $\mathbb{Q}$ with $\text{Gal}(K/\mathbb{Q}) \cong A_4$. How many subfields does $K$ contain, what are their dimensions over $\mathbb{Q}$, and which are Galois over $\mathbb{Q}$?
(1) Let $G$ be a group with $|G| = 585$. Show that $G$ contains a normal cyclic subgroup of prime index. Describe $G$ up to isomorphism. Show that $Z(G) \neq e$ and has composite order.

(2) For any $n > 3$, show that each element of the symmetric group $S_n$ is a product of permutations, each having no fixed point in $\{1, 2, \ldots, n\}$.

(3) Show that any cyclic group $G$ with square free order ($a > 0$ and $a^2 | G |$ implies that $a = 1$) is the Galois group over $\mathbb{Q}$ of some field extension $K \supseteq \mathbb{Q}$. (Hint: For a suitable $n > 0$, consider $x^n - 1 \in \mathbb{Q}[x]$.)

(4) Let $(\mathbb{Q}, +)$ be the rational numbers under addition.
   (a) Show that $(\mathbb{Q}, +)$ is not a finitely generated Abelian group.
   (b) Show that any finitely generated $\mathbb{Z}$ submodule of $\mathbb{Q}$ is free.
   (c) Determine if $(\mathbb{Q}, +)$ is a free $\mathbb{Z}$ module.
   (d) Is $(\mathbb{Q}, +)$ a projective $\mathbb{Z}$ module.

(5) Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let $I$ and $J$ be ideals of $R$ satisfying: for all $\alpha \in \mathbb{C}^n$, $f(\alpha) = 0$ for all $f \in I$ in and only if $g(\alpha) = 0$ for all $g \in J$.
   (a) Show that $(I + J)/I$ is a nil ring.
   (b) Show that $(I + J)/I$ is a nilpotent ring. (Note that $(I + J)/I$ is an ideal of $R/I$.)

(6) If $R$ is a finite ring and $x^5 = x$ for all $x \in R$, describe the structure of $R$. 
ALGEBRA QUALIFYING EXAM

Partial credit is given for partial solutions.

1. Up to isomorphism, describe all groups of order 495.

2. Let \( x^4 - 7 \in \mathbb{F}[x] \) for \( \mathbb{F} \subseteq \mathbb{C} \). If \( \mathbb{F} \subseteq \mathbb{M} \subseteq \mathbb{C} \) and \( \mathbb{M} \) is a splitting field for \( x^4 - 7 \) over \( \mathbb{F} \), find \( \text{Gal}(\mathbb{M}/\mathbb{F}) \): when \( \mathbb{F} = \mathbb{Q} \); when \( \mathbb{F} = \mathbb{Q}[\sqrt[4]{7}] \); and when \( \mathbb{F} = \mathbb{Q}[i] \), with \( i^2 = -1 \).

3. Let \( \mathbb{M} \) be a finitely generated \( \mathbb{F}[x] \) module (\( \mathbb{F} \) a field). If every submodule of \( \mathbb{M} \) has a complement, describe the structure of \( \mathbb{M} \) in terms of \( \mathbb{F}[x] \). (Recall that a submodule \( \mathbb{H} \) of a module \( \mathbb{M} \) has a complement if there is a submodule \( \mathbb{H}' \) so that \( \mathbb{M} \cong \mathbb{H} \oplus \mathbb{H}' \); i.e. \( \mathbb{H} + \mathbb{H}' = \mathbb{M} \) and \( \mathbb{H} \cap \mathbb{H}' = \{0\} \).

4. Show that some power of \((x + y)(x^2 + y^4 - 2)\) is in the ideal of \( \mathbb{C}[x,y] \) generated by \( x^3 + y^2 \) and \( y^3 + xy \).

5. Let \( \mathbb{R} \) be a commutative Noetherian ring with no nonzero nilpotent element.
   
   Set \( \mathcal{A} = \{ \text{ann} I \mid I \text{ is a nonzero ideal of } \mathbb{R} \} \) and \( \mathcal{M} = \{ \text{maximal elements in } \mathcal{A} \} \). Prove that \( \mathbb{R} \) embeds in a direct sum of finitely many domains as follows:
   
   a) Show that the elements of \( \mathcal{M} \) are prime ideals in \( \mathbb{R} \).
   
   b) For \( P \neq Q \) in \( \mathcal{M} \), show \( \text{ann} Q \subseteq P \).
   
   c) Show that \( \mathcal{M} \) is finite (consider sums of \( \text{ann} P_i \) for \( P_i \in \mathcal{M} \)).
   
   d) Show that the intersection of the elements in \( \mathcal{M} \) is zero.

6. Let \( \mathbb{R} \) be a finite dimensional algebra over the field \( \mathbb{F} \). Assume that for every \( r \in \mathbb{R} \) there some \( g(x) \in \mathbb{F}[x] \), depending on \( r \), so that \( r + g(r)r^2 = 0 \). Determine the structure of \( \mathbb{R} \).
Directions: Work any 5. Partial credit in units of 1/4 is given for partial solutions.

1. Let $G$ be a finite group, $N$ a normal subgroup, and $P$ a $p$-Sylow subgroup of $N$.
   a. Show that $G = N_G(P)$, $N_G(P) = \{g \in G : gPg^{-1} = P\}$ the normalizer of $P$ in $G$.
   b. Let $\Phi(G)$ denote the intersection of the proper maximal subgroups of $G$. Show that $\Phi(G)$ is normal in $G$ and if $H$ is a subgroup such that $G = \Phi(G)H$ then $H = G$.
   c. Show that $\Phi(G)N_G(P) = G$ and that every $p$-Sylow subgroup $P$ of $\Phi(G)$ is normal in $G$.

2. View the $n \times n$ matrix $T$ over the ring of integers $\mathbb{Z}$ as a linear transformation on $\mathbb{Z}^n$; that is, $T(X) = XT$, the matrix product of $X = (x_1, \ldots, x_n)$ and $T$. Set $\text{Im}(T) = \{T(X) : X \in \mathbb{Z}^n\}$, $\text{Ker}(T) = \{X \in \mathbb{Z}^n : T(X) = 0\}$.
   a. Show that $\mathbb{Z}^n = \text{Ker}(T) \oplus \text{Im}(T)$.
   b. What is the structure of the abelian group $\mathbb{Z}^n/\text{Im}(T)$ if $T^2 = pI_n$, $p$ a prime and $I_n$ the $n \times n$ identity matrix.

3. Let $\mathbb{Q} \subset F \subset \mathbb{C}$ where $F$ is the field generated over the rationals $\mathbb{Q}$ by all roots of unity in the field $\mathbb{C}$ of complex numbers. Let $a_1, \ldots, a_k \in \mathbb{Q}, p_1 < \cdots < p_k$ primes, and set $M = F(a_1^{1/p_1}, \ldots, a_k^{1/p_k})$.
   a. Show that $M$ is a Galois extension of $F$.
   b. Describe the Galois group of $M$ over $F$.
   c. For any subfield $F \subset K \subset M$, show that $K = F(S)$ for some subset $S \subset \{a_1^{1/p_1}, \ldots, a_k^{1/p_k}\}$.

4. Let $\mathbb{F}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$ with $p$ elements, $p$ a prime, and let $\mathbb{F}_{p^n}$ denote the subfield of $\mathbb{F}_p$ with $p^n$ elements.
   a. For $x \in \mathbb{F}_p$, show that $x^{p^n-1} \in \mathbb{F}_p$ if and only if $x \in \mathbb{F}_{p^n}$.
   b. Let $F(n) = \{x \in \mathbb{F}_p : x^n \in \mathbb{F}_p\}$. Show that $F(n)$ is finite and $F(pn) = F(n)$.

5. Let $F$ denote a field, $\sigma, \tau$ automorphisms of $F$ generating an abelian subgroup $H$ of $\text{Aut}(F)$ of finite order $s$. The twisted polynomial ring $F_H[x, y]$ consists of all polynomial expressions $\sum_{i,j=0}^{m} f_{ij} x^i y^j$, $f_{ij} \in F$ in commuting indeterminates $x, y$ over $F$ subject to the usual polynomial addition and multiplication except that $xf = \sigma(f)x, yf = \tau(f)y$. Let $Z$ denote the center of $F_H[x, y]$.
   a. Show that $Z$ contains $F_0[x^s, y^s]$, $F_0$ the subfield of $F$ fixed by $H$.
   b. Show that $F_H[x, y]$ is Noetherian and $Z$ is Noetherian.

6. Let $I$ denote an ideal in $\mathbb{C}[x_1, \ldots, x_n], \mathbb{C}$ the field of complex numbers and suppose that $I$ is the intersection of $k$ maximal ideals. Show that if $k < n$ then $I$ contains a homogeneous linear polynomial $a_1x_1 + \cdots + a_nx_n$ with $a_i \neq 0$ for some $i, 1 \leq i \leq n$. Give an example to show that this can fail for $k \geq n$.

7. Let $R$ be a finite dimensional, semisimple $\mathbb{C}$-algebra, $\mathbb{C}$ the field of complex numbers, and for $r \in R - \{0\}$, let $m_r(x) \in \mathbb{C}[x]$ denote the monic polynomial of least degree such that $m_r(r) = 0$; i.e., the minimal polynomial for $r$. Show that $R$ is commutative if and only if $m_r(x)$ has no multiple roots $\forall r \in R - \{0\}$, and $R$ is noncommutative if and only if $\deg m_r(x) < \dim R/k \forall r \in R - \{0\}$.
1. Let $G$ be a group of order 105. Show that $G$ contains a normal subgroup of index 3 and determine how many possibilities there are for the structure of $G$, up to isomorphism. Show that $G$ has a nontrivial center.

2. For a prime integer $p$, a group $G$ is called $p$-divisible if the function $f_p: G \rightarrow G$ given by $f_p(g) = g^p$ is surjective (i.e. onto). If $G$ is Abelian and $p$-divisible, show that $G$ is finitely generated if and only if $G$ is finite with order relatively prime to $p$.

3. Let $Q \subseteq M \subseteq C$ with $M$ a finite dimensional Galois extension of $Q$, the rational numbers. If for all subfields $Q \subseteq L \subseteq M$, $[L:Q]$ is even, what can the order of $\text{Gal}(M/Q)$ be? In this case, show that $M$ embeds in a radical extension of $Q$.

4. Let $C[x_1, x_2, \ldots, x_n] = R$ and let $f(X) = f(x_1, x_2, \ldots, x_n) \in R$ be irreducible. Given $g(X), h(X) \in R$ so that $g(\alpha) - h(\alpha) = 0$ for all $\alpha \in C^n$ satisfying $f(\alpha) = 0$, show that $g(X) + (f(X)) = h(X) + (f(X))$ in $R/(f(X))$.

5. Let $R$ be a commutative Noetherian ring with 1. Prove that $R$ is isomorphic to a finite direct sum of fields if and only if every (ring) homomorphic image of $R$ is projective as an $R$ module.

6. Let $F$ be a finite field and let $A$ be an $F$ subalgebra of $M_n(F)$.
   a) If $A$ is a domain, show that $\dim_F A \leq n$.
   b) If $A$ is simple with $F-I_n$ as its center, show that $\sqrt{\dim_F A}$ is an integer and divides $n$. 
Written Qualifying Exam, Algebra, Nov. 1998

Directions. Partial credit in units of 1/4 is given for partial solutions.

1. Let $G$ be a group of order $p^aq^b$, $p, q$ distinct primes and $a, b$ positive integers. Prove that if $q < p$ and the order of $q$ mod $p$ exceeds $b$ then $G$ is solvable.

2. Let $G$ be a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module).
   a). Prove that $G$ has no elements of order $p$, $p$ a prime, if and only if $G \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p^r$, for some positive integer $r$, $\mathbb{Z}_p$ is the local ring of rational numbers with denominator prime to $p$.
   b). Prove that $G$ is projective if and only if there is an integer $r$ such that $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian groups $H$.

3. Let $\mathbb{F}_{p^n}$ be a finite field with $p^n$ elements, $p$ a prime. Recall that the norm map $N : \mathbb{F}_{p^n} \to \mathbb{F}_p$ is defined by $N(x) = \prod_{g \in \text{Gal}_{\mathbb{F}_p^{p^n}}} g(x)$ and the trace map is defined by $T(x) = \sum_{g \in \text{Gal}_{\mathbb{F}_p^{p^n}}} g(x)$. Determine the image of each of these maps, show that the kernel of the norm map is $\{x/g(x) : x \in \mathbb{F}_{p^n}^*, g \in \text{Gal}_{\mathbb{F}_p^{p^n}} \}$ and that the kernel of the trace map is $\{x - g(x) : x \in \mathbb{F}_{p^n}^*, g \in \text{Gal}_{\mathbb{F}_p^{p^n}} \}$.

4. Let $R$ be a subring of $\mathbb{C}[x_1, \ldots, x_n]$, containing $\mathbb{C}$ and assume that the field of quotients of $R$ is $\mathbb{C}(x_1, \ldots, x_n)$. Show that there are polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ such that $d\mathbb{C}[x_1, \ldots, x_n] \subset R$ if and only if $d \in I = f_1 \mathbb{C}[x_1, \ldots, x_n] + \cdots + f_s \mathbb{C}[x_1, \ldots, x_n]$. In addition, show that $I$ cannot be a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$.

5. Maschke's theorem implies that the group algebra $k[G]$ over a field $k$ of characteristic zero is semisimple when $G$ is a finite group. Using this fact,
   a). Determine the structure of $\mathbb{C}[S_3]$, $S_3$ the symmetric group on three symbols.
   b). An epimorphism of groups, $\phi : G \to H$, induces an epimorphism $\Phi : k[G] \to k[H]$ on the corresponding group rings over $k$. Prove that if $k$ has characteristic 0 and $G$ is finite then $k[H]$ is a ring direct summand of $k[G]$.

6. Determine the galois group of $x^n - p$ over the rationals, $p$ a prime, and determine all subfields of its splitting field which are normal over the rational numbers.
Partial credit is given for partial solutions.

1. Let $G$ be a group of order $1705 = 5 \cdot 11 \cdot 31$. Describe the possible structures of $G$ up to isomorphism.

2. Let $G$ be a group and $N < G$. Show: i) $G$ is solvable $\iff N$ and $G/N$ are solvable; and ii) if $|G| = p^n$ for $p$ a prime then $G$ is solvable.

3. Let $f(x) \in \mathbb{Q}[x]$ be irreducible with $\deg f = p$, an odd prime, and let $K \subseteq \mathbb{C}$ be a splitting field for $f(x)$ over $\mathbb{Q}$. Suppose that $f(x)$ has exactly two roots in $\mathbb{C} - \mathbb{R}$. Prove that $\text{Gal}(K/\mathbb{Q}) \cong S_p$.

4. Using methods of algebraic geometry show that there is a fixed $m > 0$ so that for any linear polynomial $f(x,y,z,t) = ax + by + cz + dt$, $f(x,y,z,t)^m \in (x^{19}y^{32}z^{31}, x^3 + y^3, y^3 + z^4, z^{13} + t^7) \subseteq \mathbb{C}[x,y,z,t]$.

5. Let $A = C[x, \sigma]$ be the twisted polynomial ring over $\mathbb{C}$ where $\sigma$ is complex conjugation. The elements of $A$ are the polynomials $p(x) = c_0x^n + \cdots + c_1x + c_0$ which add in the usual way but with multiplication given by $xa = \sigma(a)x = \bar{a}x$, and extended by the associative and distributive laws. The general expression for products is $\sum a_i x^i \sum b_j x^j = \sum (\sum a_i \sigma^j(b_j)) x^k$.
   i) Find the center of $A$.
   ii) Is the center of $A$ a Noetherian ring (and why)?
   iii) Show that $A$ is a left and a right Noetherian ring.

6. Let $R$ be a right Artinian ring so that each $r \in R$ satisfies $r^3 = r$.
   i) Show that $R$ is a finite ring.
   ii) Show that there is some $m \geq 1$ so that $R$ has exactly $2^m$ elements satisfying $x^2 = x$.
   iii) Using $m$ in ii), find the possible values for $|R| = \text{card}(R)$.
1. For $p$ and $q$ distinct primes show that any group of order $p^aq$ is solvable.

2. Let $G$ be a finite Abelian group so that whenever $H$ and $K$ are subgroups of $G$ of the same order then $H \cong K$ as groups. Describe the possible structures of $G$. If $|G| = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$, up to isomorphism how many possibilities are there for $G$?

3. Let $p_1, \ldots, p_k$ be distinct primes in $\mathbb{Z}$ and set $F = \mathbb{Q}(\sqrt[p_1]{1}, \ldots, \sqrt[p_k]{1}) \subseteq \mathbb{R}$. 
   i) Show that $F$ is a Galois extension of $\mathbb{Q}$ with $\text{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^k$.
   ii) Show that $F = \mathbb{Q}(\sqrt[p_1]{1} + \cdots + \sqrt[p_k]{1})$.

4. For $F$ a field and $R = F[x_1, \ldots, x_n]$ let $M$ be a finitely generated $R$ module. Show that there are positive integers $s$ and $t$ and an exact sequence of $R$ modules $0 \rightarrow K \rightarrow R^s \rightarrow R^t \rightarrow M \rightarrow 0$.

5. If $I$ is a nonzero ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$ which is not maximal then if $R/I$ is a domain, show that $\dim_{\mathbb{C}} R/I$ must be infinite.

6. If $R \neq \{0\}$ is a finite ring so that each $r \in R$ satisfies the polynomial $x^t - x$, describe the possible structures of $R$. 
Partial credit is given for partial solutions.

1. Up to isomorphism describe all groups of order 595 (5•7•17).

2. Let M be a finitely generated module over a PID R. If \( M \otimes_R M \cong M \) determine the structure of M.

3. Let \( \rho \in \mathbb{C} \) be a primitive \( p^{\text{th}} \) root of 1 for an odd prime \( p \) and set \( L = \mathbb{Q}(\rho) \).

   What is \( \text{Gal}(L/\mathbb{Q}) \)? If \( m \) is the number of different positive integer divisors of \( p - 1 \), how many fields \( F \) satisfy \( \mathbb{Q} \subseteq F \subseteq L \) and how many of these are Galois extensions of \( \mathbb{Q} \)? What are the \( \text{Gal}(F/\mathbb{Q}) \)? Show that \( [L : \mathbb{R} \cap L] = 2 \). Show that \( N_{L/\mathbb{Q}}(1 - \rho^j) = \rho \) for any \( 1 \leq j \leq p-1 \).

4. Let \( R \) be a commutative Noetherian ring with 1 and let \( \varphi : R[x_1, ..., x_n] \rightarrow R[x_1, ..., x_n] \) be a surjective ring homomorphism. Show that \( \varphi \) is an automorphism.

5. Let \( I \) be an ideal in \( \mathbb{C}[x_1, ..., x_n] \).
   
   i) Show that there is \( k > 0 \) so that \( (\sqrt[1]{I})^k \subseteq I \).
   
   ii) Prove that if I is maximal then \( L/I^k \) is a finite dimensional \( \mathbb{C} \)-vector space for all \( k \geq 0 \).
   
   iii) Show that \( \mathbb{C}[x_1, ..., x_n]/I \) is finite dimensional over \( \mathbb{C} \) \( \iff \{ \alpha \in \mathbb{C}^n \mid f(\alpha) = 0, \text{ all } f \in I \} \) is finite.

6. If \( R \) is a finite ring with 1 and \( x, y \in R \) satisfy \( xy = 1 \), show that \( yx = 1 \).
(1) Describe all groups of order $3 \cdot 17 \cdot 23$ up to isomorphism.

(2) Let $G$ be a finitely generated Abelian group so that every proper homomorphic image of $G$ is cyclic. Prove that $G$ is cyclic or that $|G| = p^3$ for $p$ a prime.

(3) Let $K \subseteq \mathbb{C}$ be a splitting field over $\mathbb{Q}$ of $x^3 - 5$. Describe $\text{Gal}(K/\mathbb{Q})$. Describe those fields $\mathbb{Q} \subseteq M \subseteq K$ with $M$ Galois over $\mathbb{Q}$, and for these find $\text{Gal}(M/\mathbb{Q})$.

(4) Let $\overline{F}$ be an algebraic closure of the field $F$. If $M \subseteq F[x_1, \ldots, x_n]$ is a maximal ideal, show that $V(M) = \{\alpha \in \overline{F} \times \cdots \times \overline{F} | f(\alpha) = 0$ for all $f \in M\}$ is finite and not empty.

(5) Let $M \subseteq \mathbb{Q}$ be Noetherian $\mathbb{Z}$-submodule. For $N$ a $\mathbb{Z}$-submodule of $M$, show $M/N$ is finite (as a set) $\iff M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$.

(6) If $R$ is a right Artinian ring and $x^3 = x$ for all $x \in R$, show: $R$ is commutative; $R$ is finite; and $R$ has $2^a 3^b$ elements for some $a, b \geq 0$. 