Topics for the Graduate Exam in Algebra

Most of the following topics are normally covered in a two semester graduate sequence in algebra (510ab).

This is a two hour exam.

Groups: Review of elementary group theory, isomorphism theorems, group actions, orbits, stabilizers, simplicity of $A_n$, Sylow’s theorems, direct products and direct sums, semi-direct products, Fundamental Theorem of Abelian Groups, solvable groups.

Fields: Relative dimensions, automorphisms, splitting fields, isomorphism extension theorem, separable extensions, Galois correspondence, Fundamental Theorem of Galois Theory, principal element theorem, traces and norms, radical extensions, finite fields, cyclotomic extensions, algebraic closure.


Modules: Irreducible modules, torsion modules, free modules, projective modules, modules over PIDs, chain conditions, tensor products, exact sequences.


References:

D. Rotman, The Theory of Groups
S. Lang, Algebra
T. Hungerford, Algebra
T.Y. Lam, A first course in non-commutative rings
M. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra
I. Stewart, Galois theory
H.M. Edwards, Galois theory
(1) Let $G$ be a group with $|G| = 5 \cdot 7^2 \cdot 17$. Determine the possible structures for $G$.

(2) Let $G$ be a finitely generated abelian group $G \neq \{1\}$, $n$ a positive integer and let 
$\varphi_n : G \to G$ be the homomorphism defined by $\varphi_n(g) = g^n$. Prove
(a) $G$ is not divisible, i.e. $\exists n$ such that $\varphi_n$ is onto.
(b) $G$ is finite if and only if $\exists n$ such that $\varphi_n$ is the trivial map, i.e. $\varphi_n(g) = 1$ for all 
$g \in G$.
(c) $G$ is free abelian group if and only if $\varphi_n$ is 1-1 for all positive integers $n$.
(d) $G$ is finite if and only if $\varphi_n$ is an isomorphism for some positive integer $n > 1$.

(3) Let $k = \mathbb{Q}(\zeta_{15})$, where $\zeta_{15}$ is a primitive 15-th root of unity. What is the galois group 
of $k/\mathbb{Q}$? How many subfields does $k$ have? List all subfields (recall that the field 
$\mathbb{Q}(\zeta_p)$ of $p$-th roots of unity, $p$ a prime, contains the subfield $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 \mod 4$ 
and $\mathbb{Q}(\sqrt{-p})$ if $p \not\equiv 1 \mod 4$).

(4) Let $F = \mathbb{F}_{p^n}$ be a field of $p^n$ elements. For $1 \leq k \leq n$ set $L_k = \{a \in F : a^{p^k} = a\}$. 
Show that each $L_k$ is a subfield of $F$, that $\{L - k : 1 \leq k \leq n\}$ is the set of all subfields of $F$, 
and for $n$ greater than 2, $L_i = L_j$ for some $1 \leq i < j \leq n$.

(5) Let $A$ be a commutative noetherian ring, $M$ a noetherian $A$-module.
(a) Prove that $M \otimes_A A[x]$ is a noetherian $A[x]$-module.
(b) If $A$ is a commutative Noetherian domain with 1, and $0 \neq y \in A$, a nonunit.
Show that $y = a_1a_2 \ldots a_k$ with each $a_i \in A$ irreducible.
(c) Let $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over the complex 
numbers. An (irreducible) hypersurface in $\mathbb{C}^n$ is the solution set $Z(f)$ of $f(x) = 0$, $f$ an irreducible polynomial in $\mathbb{C}[X]$. Let $\mathcal{F}(Z(f))$ denote the ring of complex valued polynomial functions on the hypersurface $Z(f)$; i.e. $h \in \mathcal{F}(Z(f))$ if 
and only if $\exists g \in \mathbb{C}[X]$ such that $h(T) = g(T)$ for all $T \in Z(f)$. Prove that 
$\mathcal{F}(Z(f)) \cong \mathbb{C}[X]/f(X)\mathbb{C}[X]$.
(d) Let $A$ be a finite dimensional semi-simple algebra over $\mathbb{C}$, and set $M_n(A) = \text{ring}$
of $n \times n$ matrices over $A$. Not that $M_n(M_n(A)) \cong M_{nn}(A)$ and $M_n(A \oplus B) \cong$ 
$M_n(A) \oplus M_n(B)$.
(a) Show that $M_2(A)$ is semi-simple.
(b) If $\dim_{\mathbb{C}}(A)$ is prime, show that $M_2(A)$ is not simple.
(c) If $A$ is not commutative, there is a $t \in M_2(A)$ with $t^3 \neq 0$ and $t^4 = 0$. 
ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 1992

(1) If $G$ is a group of order $2^4 \cdot 19 \cdot 23$ show that $G$ has a normal subgroup of order $4 \cdot 19 \cdot 23$ and the center of $G$ contains an element of order 2.

(2) Let $T : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism.
   (a) If $T$ is onto, show that $\mathbb{Z}^n \cong \text{Ker } T \oplus \mathbb{Z}^m$.
   (b) Prove that $T$ is injective if and only if $m \geq n$ and $\dim_\mathbb{Q} \mathbb{Q} \otimes \mathbb{Z}(\mathbb{Z}/T(\mathbb{Z})^n) = m - n$.

(3) Let $F$ be a finite Galois extension of the field $k$. A subfield $k \subset L \subset F$ is abelian if $L$ is a Galois extension of $k$ and $\text{Gal}(L/k)$ is abelian.
   (a) Prove there is a unique maximal abelian subfield of $F$.
   (b) Prove that if $L, J$ are abelian extensions of $k$ then their composite is an abelian extension of $k$.

(4) Let $L$ be the splitting field of $x^{68} - 1$ over $\mathbb{Q}$. Find $[L : \mathbb{Q}], |\text{Gal}(L/\mathbb{Q})|$, the structure of $\text{Gal}(L/\mathbb{Q})$, the number of subfields of $L$ and the subfields which are normal over $\mathbb{Q}$.

(5) (a) Let $M_2 \stackrel{g}{\to} M_3 \stackrel{h}{\to} M_4 \to 0$ be an exact sequence of $A$-modules, $A$ a ring and $f : M_1 \to M_2$ an $A$-module homomorphism. Prove that $M_1 \xrightarrow{g \circ f} M_3 \xrightarrow{h} M_4 \to 0$ is exact if and only if $f(M_1) + \ker g = M_2$.
   (b) Show that whenever $A = k[x, y]$, $k$ a field, and $I$ is an ideal in $A$ then there is an exact sequence $A^m \to A^n \to I \to 0$ for some positive integers $m, n$.

(6) For an ideal $I$ in $A = \mathbb{C}[x, y, z]$ set $Z_{xy}(I) = \{(a, b) \in \mathbb{C}^2 : f(a, b, z) = 0 \text{ for all } f \in I\}$
   (a) Prove that $I$ maximal implies $Z_{xy}$ is empty.
   (b) Prove that $Z_{xy}(I) \times \mathbb{C} = Z(I) = \{(a, b, c) \in \mathbb{C}^3 : f(a, b, c) = 0 \forall f \in I\}$ if and only if $\text{rad } I = JA$, where $J = \{f(x, y) : f(a, b) = 0 \forall (a, b) \in Z_{xy}\}$.

(7) Describe up to isomorphism all semi-simple $\mathbb{C}$-subalgebras of $M_4(\mathbb{C})$, the ring of $4 \times 4$ matrices over $\mathbb{C}$. (Note that if $A, B$ are $\mathbb{C}$-algebras and $\alpha \in A, \beta \in B$ have minimal polynomials $f, g$ respectively then $(\alpha, \beta) \in A \oplus B$ has minimal polynomial $h = \text{lcm}(f, g)$.)
(1) Find up to isomorphism all groups of order $3 \cdot 7 \cdot 19 \cdot 37$.

(2) Let $D$ be a commutative domain with multiplicative identity $1$ and assume that the additive group $D$ is finitely generated. Prove
   (a) Characteristic $K = 0$ if and only if $(D, +)$ is a free abelian group.
   (b) If there exists an integer $n > 1$ such that $f : D \rightarrow D$ defined by $x \mapsto nx$ is onto then $D$ is a finite field.
   (c) If $M$ is a maximal ideal in $D$ then $M \cap i(Z) = \pi(Z)$ for some prime $p$, $i(Z) = \{ n \cdot 1, n \in \mathbb{Z} \}$.

(3) Let $f(x) \in \mathbb{Q}(x)$ be irreducible with $\deg f = n$. Let $M \subset \mathbb{C}$ be a splitting field for $f(x)$ over $\mathbb{Q}$.
   (a) Show that if $\text{Gal}(M/\mathbb{Q})$ is abelian then every subfield field of $M$ is Galois over $\mathbb{Q}$.
   (b) Show that if $\text{Gal}(M/\mathbb{Q})$ is abelian then $[M : \mathbb{Q}] = n$.

Let $f(x) \in \mathbb{F}_p[x]$ be irreducible of degree $t$, $\mathbb{F}_p$ a field with $p^n$ elements, $p$ a prime.
   (a) Show that $\mathbb{F}_p[t]$ is a splitting field of $f$ over $\mathbb{F}_p$.
   (b) For $n = 1$, show that $f(x)$ divides $x^{p^n} - x$ if and only if $t$ divides $n$.
   (c) How many distinct irreducibles in $\mathbb{F}_2[x]$ have degree 5?

(4) Let $f_i(x, y) = a_i x^2 + b_i x y + c_i y^2 \in \mathbb{C}[x, y]$, $1 \leq i \leq n$. Show that there exists $(u, v) \in \mathbb{C}^2$ such that $u^2 + v^2 = 1$, but $f_i(u, v) \neq 0 \forall i = 1, \ldots, n$.

(5) Given the linear equation $a_1 X_1 + \ldots + a_t X_t = 0$, $a_i \in A = k[x_1, \ldots, x_m]$ and $k$ a field, prove that there are solutions $Y_1, \ldots, Y_t \in A^t$ such that for each solution $Y$, there exists $b_1, \ldots, b_t \in A$ such that $Y = \sum_{i=1}^{t} b_i Y_i$. If $A = \mathbb{Z}$, prove that you can take $q = t - 1$.

(6) Let $x$ denote a fixed non zero vector in $\mathbb{C}^3$ and $A_x$ denote the ring of matrices $T \in M_3(\mathbb{C})$ such that $xT = 0$.
   (a) Prove that $A_{xU} \cong A_x$ for any $U \in \text{GL}_2(\mathbb{C})$, hence $A_x \cong A_y$ for any non zero $y \in \mathbb{C}^3$.
   (b) Prove that $\{(a_{ij}) \in A_{(1,0)} : a_{ij} = 0 \text{ for } j > 1 \}$ is nilpotent ideal in $A_{(1,0,0)}$.
   (c) Prove that the Jacobson radical $J(A_x)$ is not zero and that $A_x/J(A_x) \cong M_2(\mathbb{C})$. 
(1) Let $G$ be a finite group.
   (a) If $|G| = 2^s \cdot 7$ for $0 \leq s \leq 3$, then $G$ is solvable.
   (b) Suppose $|G| = 112$ and $G$ is not simple. Show that $G$ is solvable.
   (c) If $|G| = 112$, show that $G$ is not simple. \textit{Hint:} If $G$ were simple, show that $G$
   embeds into $S_7$, and then show that $G$ embeds, in fact, into $A_7$.

(2) Suppose that a group $G$ is the direct sum of $k$ cyclic groups, each of prime power
   order. If $H$ is a subgroup of $G$ containing nontrivial subgroups $H_1, \ldots, H_k$, whose
   sum is direct, show that $s \leq k$.

(3) Let $f(x) = (x^3 - 3)(x^2 + 1) \in \mathbb{Q}[x]$. Denote by $K$ the splitting field of $f(x)$
   over $\mathbb{Q}$, and by $G$ the Galois group of $K$ over $\mathbb{Q}$.
   (a) Find $|G|$.
   (b) Show that $G$ has a normal subgroup of order 2.
   (c) Show that the 3-Sylow subgroup of $G$ is normal. \textit{Hint:} $K$ contains all 12-th
   roots of unity.
   (d) Show that $G$ has a central element of order 2.

(4) Let $R$ be a right Noetherian ring (with 1). Prove that $R$ has a unique maximal
   nilpotent ideal $P(R)$. Show that the polynomial ring $R[x]$ must also have a unique
   maximal nilpotent ideal $P(R[x])$, and that $P(R[x]) = P(R)[x]$.

(5) Let $M$ be a maximal ideal in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Show that there are
   only finitely many maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ containing $M$. \textit{Hint:} Show first
   that for each $i$ there is a polynomial $f_i(y) \in \mathbb{Q}[y]$ such that $f_i(x_i) \in M$.

(6) Let $G$ be a finite group, and $\mathbb{C}[G]$ it group algebra. Define a bijection $\ast : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$
   as follows: For $x = \sum_{g \in G} a_g g \in \mathbb{C}[G]$, set $x^* = \sum_{g \in G} \overline{a_g} g^{-1}$. One calls $x$
   symmetric if $x = x^*$.
   (a) Given $x, y \in \mathbb{C}[G]$, show that $(x^*)^* = x$, and $(xy)^* = y^* x^*$.
   (b) Given $x \in \mathbb{C}[G]$, show that $xx^*$ is symmetric, and that $xx^* = 0$ if and only if
   $x = 0$.
   (c) Show that nonzero symmetric elements are not nilpotent.
   (d) Assume that $\mathbb{C}[G]$ has no nonzero nilpotent elements. Show that $G$ is abelian.

(7) Let $K$ be a field extension of $k = \mathbb{F}_p^n$ of degree $n$. Let $\sigma$ be the automorphism of $K$
   given by $\sigma(a) = a^p$ for $a \in K$.
   (a) Let $x$ be an element of $K$ such that both $x + \sigma(x)$ and $x \sigma(x)$ belong to $k$. Show
   that $|k(x) : k| \leq 2$. Moreover, show that $|k(x) : k| = 2$ if and only if $\sigma(x) \neq x$.
   (b) Set $F = \{x \in K | x + \sigma(x), x \sigma(x) \in k\}$
   Show that $F$ is a subfield of $K$, and that $|F : k| \leq 2$. Moreover, show that
   $|F : k| = 2$ if and only if $2|n$. 

ALGEBRA QUALIFYING EXAM  
NOVEMBER, 1995

Partial credit is given for partial solutions.

1. Up to isomorphism, determine all groups of order $7^2 \cdot 11^2 \cdot 19$.

2. Let $G$ be a finite Abelian group and recall that the exponent of $G$ is the smallest positive integer $n$ so that $g^n = e_G$ for all $g \in G$. Show that $G$ is a cyclic group if and only if the order and exponent of $G$ are equal.

3. An ideal in a commutative ring is called irreducible if it cannot be written as an intersection of finitely many properly larger ideals. If $A = \mathbb{Z}[x_1, ..., x_n]$ is the polynomial ring in $n$ variables over $\mathbb{Z}$, show that any ideal of $A$ is an intersection of finitely many irreducible ideals.

4. Let $K$ be a splitting field over $\mathbb{Q}$ of the polynomial $x^{11} - 17$. Show that $Gal(K/\mathbb{Q})$ is isomorphic to the group of matrices $G = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}/11\mathbb{Z}) \}$.

5. Let $F$ be a field and $M$ an irreducible (i.e. simple and nontrivial) $F[x_1, ..., x_n]$ module.
   i) If $F$ is algebraically closed, show that $\dim_F M = 1$.
   ii) For any $F$ show that $\dim_F M$ is finite.

6. Let $R$ be a finite ring in which every element is a sum of nilpotent elements. Show that $R$ is nilpotent. (Hint: what is the trace of a nilpotent element in $M_n(F)$ for $F$ a field?)