ALGEBRA QUALIFYING EXAM SPRING 2012

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of rational numbers is $\mathbb{Q}$ and set of the complex numbers is $\mathbb{C}$.

1. Let $I$ be an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$. Show that $\dim_{\mathbb{C}} R/I$ is finite $\iff I$ is contained in only finitely many maximal ideals of $R$.

2. If $G$ is a group with $|G| = 7^2 \cdot 11^2 \cdot 19$, show that $G$ must be abelian and describe the possible structures of $G$.

3. Let $F$ be a finite field and $G$ a finite group with $\gcd\{\text{char } F, |G|\} = 1$. The group algebra $F[G]$ is an algebra over $F$ with $G$ as an $F$-basis, elements $\alpha = \sum_{g \in G} a_g g$ for $a_g \in F$, and multiplication that extends $ag \cdot bh = abgh$. Show that any $x \in F[G]$ that is not a zero left divisor (i.e. if $xy = 0$ for $y \in F[G]$ then $y = 0$) must be invertible in $F[G]$.

4. If $p(x) = x^8 + 2x^6 + 3x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$ and if $\mathbb{Q} \subseteq M \subseteq \mathbb{C}$ is a splitting field for $p(x)$ over $\mathbb{Q}$, argue that $\text{Gal}(M/\mathbb{Q})$ is solvable.

5. Let $R$ be a commutative ring with 1 and let $x_1, \ldots, x_n \in R$ so that $x_1 y_1 + \cdots + x_n y_n = 1$ for some $y_j \in R$. Let $A = \{(r_1, \ldots, r_n) \in R^n \mid x_1 r_1 + \cdots + x_n r_n = 0\}$. Show that $R^n \cong_R A \oplus R$, that $A$ has $n$ generators, and that when $R = F[x]$ for $F$ a field then $A_R$ is free of rank $n - 1$.

6. For $p$ a prime let $F_p$ be the field of $p$ elements and $K$ an extension field of $F_p$ of dimension 72.
   i) Describe the possible structures of $\text{Gal}(K/F_p)$.
   ii) If $g(x) \in F_p[x]$ is irreducible of degree 72, argue that $K$ is a splitting field of $g(x)$ over $F_p$.
   iii) Which integers $d > 0$ have irreducibles in $F_p[x]$ of degree $d$ that split in $K$?