1. For $G$ a finite group with $|G| > 1$ and $p$ a prime dividing the order of $G$, let $O_p(G) = \bigcap \{P \in \text{Syl}_p(G)\}$.
   a) Show that $O_p(G)$ is a normal subgroup of $G$.
   b) Show that if $N$ is a normal subgroup of $G$ with $|N| = p^k$, then $N \subseteq O_p(G)$.
   c) Prove that if $G$ is solvable then for some $p$, $|O_p(G)| \neq 1$.

2. Let $F = GF(p^n)$ be a field of (exactly) $p^n$ elements. Suppose that $k$ is a positive integer dividing $n$, and set $B = \{a^{p^k} + a^{p^{2k}} + \cdots + a^{p^n} \mid a \in F\}$.
   i) Show that $B \subseteq E$, a subfield of $F$ with $p^k$ elements.
   ii) Show that $B = E$.

3. Let $A \in M_n(Q)$ with $A^k = I_n$. If $j$ is a positive integer with $(j, k) = 1$, show that $\text{tr}(A) = \text{tr}(A^j)$.
   (Hint: Consider $A \in M_n(Q(e))$ for $e = e^{2\pi i/k}$, where $i^2 = -1$.)

4. Let $R$ be a commutative ring with $1$ and let $M$ be a Noetherian $R$-module.
   If $f \in \text{Hom}_R(M_R, M_R)$ is surjective, show that $f$ is an automorphism of $M_R$.

5. Let $f, g \in C[x, y]$ so that $(0, 0) \in C^2$ is the only common zero of $f$ and $g$. Prove that there is a positive integer $m$ so that whenever $h \in C[x, y]$ has no monomial of degree less than $m$, then $h \in fC[x, y] + gC[x, y]$.

6. For a fixed positive integer $n > 1$, describe all finite rings $R$ so that $x^n = x$ for all $x \in R$. 