ALGEBRA EXAM    SEPTEMBER 2004

Do as many problems as you can

1. Up to isomorphism describe all groups of order 399 = 3 · 7 · 19. For each group find the order of its center and the order of its commutator subgroup.

2. Suppose $R$ is a finite dimensional algebra over a field $F$ with 1 and $U(R)$, the group of units in $R$, is abelian. Show that the Jacobson radical $J(R)$ and $R/J(R)$ are commutative.

3. Let $L$ be a subfield of the finite field $K$ of characteristic $p$. Let $\alpha \in K$ with minimal polynomial $v(z)$ of degree $d$ over $L$. Show that $v(z)$ splits over $K$ and that for some $q = p^m$ the roots of $v(z)$ in $K$ are $\{\alpha, \alpha^q, \ldots, \alpha^{q^{d-1}}\}$.

4. Let $R$ be a commutative ring with 1 and $M, N, V$ all $R$-modules.

(a) If $M$ and $N$ are projective show that $M \otimes_R N$ is also a projective $R$-module.

(b) Let

$$\text{Tr}(V) = \{\sum_{i=1}^{n} \phi_i(v_i) | \phi_i \in \text{Hom}_R(V, R), v_i \in V, n = 1, 2, \ldots\}.$$ 

If $1 \in \text{Tr}(V)$ show that up to isomorphism some finite direct sum $V^k$ contains $R$ as an $R$-module direct summand.

5. Show that any surjective ring homomorphism $f : R \rightarrow R$ of a left Noetherian ring $R$ must be an isomorphism. Give an example to show this may be false if the ring is not noetherian.

6. In $\mathbb{C}[x, y]$ show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal $(x^3 + y^2, y^3 + xy)$. 