(1) Let $G$ be a finite group.
   (a) If $|G| = 2^s \cdot 7$ for $0 \leq s \leq 3$, then $G$ is solvable.
   (b) Suppose $|G| = 112$ and $G$ is not simple. Show that $G$ is solvable.
   (c) If $|G| = 112$, show that $G$ is not simple. \textit{Hint:} If $G$ were simple, show that $G$
   embeds into $S_7$, and then show that $G$ embeds, in fact, into $A_7$.
(2) Suppose that a group $G$ is the direct sum of $k$ cyclic groups, each of prime power order. If $H$
   is a subgroup of $G$ containing nontrivial subgroups $H_1, \ldots, H_s$, whose sum is direct, show that $s \leq k$.
(3) Let $f(x) = (x^3 - 3)(x^2 + 1) \in \mathbb{Q}[x]$. Denote by $K$ the splitting field of $f(x)$ over $\mathbb{Q}$,
and by $G$ the Galois group of $K$ over $\mathbb{Q}$.
   (a) Find $|G|$.
   (b) Show that $G$ has a normal subgroup of order 2.
   (c) Show that the 3-Sylow subgroup of $G$ is normal. \textit{Hint:} $K$ contains all 12-th roots of unity.
   (d) Show that $G$ has a central element of order 2.
(4) Let $R$ be a right Noetherian ring (with 1). Prove that $R$ has a unique maximal nilpotent ideal $P(R)$.
   Show that the polynomial ring $R[x]$ must also have a unique maximal nilpotent ideal $P(R[x])$, and that $P(R[x]) = P(R)[x]$.
(5) Let $M$ be a maximal ideal in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Show that there are only finitely many maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ containing $M$. \textit{Hint:} Show first
   that for each $i$ there is a polynomial $f_i(y) \in \mathbb{Q}[y]$ such that $f_i(x_i) \in M$.
(6) Let $G$ be a finite group, and $\mathbb{C}[G]$ it group algebra. Define a bijection $*: \mathbb{C}[G] \rightarrow \mathbb{C}[G]$
as follows: For $x = \sum_{g \in G} a_g g \in \mathbb{C}[G]$, set $x^* = \sum_{g \in G} \bar{a}_g g^{-1}$. One calls $x$ symmetric
   if $x = x^*$.
   (a) Given $x, y \in \mathbb{C}[G]$, show that $(x^*)^* = x$, and $(xy)^* = y^*x^*$.
   (b) Given $x \in \mathbb{C}[G]$, show that $xx^*$ is symmetric, and that $xx^* = 0$ if and only if
   $x = 0$.
   (c) Show that nonzero symmetric elements are not nilpotent.
   (d) Assume that $\mathbb{C}[G]$ has no nonzero nilpotent elements. Show that $G$ is abelian.
(7) Let $K$ be a field extension of $k = \mathbb{F}_p$ of degree $n$. Let $\sigma$ be the automorphism of $K$
given by $\sigma(a) = a^p$ for $a \in K$.
   (a) Let $x$ be an element of $K$ such that both $x + \sigma(x)$ and $x \sigma(x)$ belong to $k$. Show
   that $[k(x) : k] \leq 2$. Moreover, show that $[k(x) : k] = 2$ if and only if $\sigma(x) \neq x$.
   (b) Set
   $$F = \{x \in K \mid x + \sigma(x), x \sigma(x) \in k\}$$
   Show that $F$ is a subfield of $K$, and that $[F : k] \leq 2$. Moreover, show that
   $[F : k] = 2$ if and only if $2 | n$. 