1. Symplectic Vector Spaces

**Definition 1.1.** Let \( V \) be a vector space over a field \( k \). A *symplectic form* on \( V \) is a bilinear form \( B : V \times V \to k \) which satisfies the following three properties for \( x, y \in V \):

1. \( B(x, y) = -B(y, x) \) (Skew-symmetric)
2. \( B(x, x) = 0 \) (Totally isotropic)
3. If \( B(x, y) = 0 \) for all \( y \in V \), then \( x = 0 \) (Non-degenerate)

The pair \( (V, B) \) is a *symplectic vector space*. A subspace \( W \subset V \) that satisfies \( B(x, y) = 0 \) for all \( x, y \in W \) is isotropic.

From now on, \( V \) will be finite dimensional. If \( V \) has dimension \( n \), then a choice of basis \( \{v_1, \ldots, v_n\} \) determines an \( n \times n \) matrix, also denoted \( B \), whose \((i, j)\)th entry is \( B(v_i, v_j) \).

**Proposition 1.2.** For any choice of basis, the matrix \( B \) is skew-symmetric and non-singular.

*Proof.* Condition (1) in definition 1.1 shows that \( B \) is skew-symmetric. To see that \( B \) is non-singular, notice that for any \( v \in V \) we have

\[
Bv = \sum_{i=1}^{n} B(v_i, v)v_i
\]

It follows that if \( v \) is in the kernel of \( B \), then \( B(v_i, v) \) is zero for all \( i \). Non-degeneracy implies then that \( v = 0 \) so that \( B \) is non-singular. \( \square \)

**Corollary 1.3.** If \( V \) is a symplectic vector space, then the dimension of \( V \) is even.

*Proof.* Suppose \( V \) were odd dimensional. Any choice of basis of \( V \) yields a skew-symmetric, non-singular matrix \( B \). We show by way of contradiction that any skew-symmetric matrix of odd dimension with zeroes in the main diagonal has determinant zero (notice that these conditions are not redundant if the field has characteristic 2). Notice that by total isotropy, we only need to sum over derangements in \( S_n \) when computing the determinant. Also notice that since \( n \) is odd, there is no derangement of order 2, so we can write the set of derangements as the disjoint union of two sets \( S \) and \( S^{-1} \) where \( \sigma \in S \) if and only if \( \sigma^{-1} \in S^{-1} \). This leads to the following:

\[
\det(B) = \sum_{\sigma \in S} \text{sgn}(\sigma) (b_{1,\sigma(1)} \cdots b_{n,\sigma(n)} + b_{1,\sigma^{-1}(1)} \cdots b_{n,\sigma^{-1}(n)})
\]

\[= \sum_{\sigma \in S} \text{sgn}(\sigma) (b_{1,\sigma(1)} \cdots b_{n,\sigma(n)} - b_{\sigma^{-1}(1),1} \cdots b_{\sigma^{-1}(n),n}) = 0\]

(Here we have used the fact that \( \sigma(i) = j \) if and only if \( \sigma^{-1}(j) = i \), and that \( \text{sgn}(\sigma) = \text{sgn}(\sigma^{-1}) \)). \( \square \)

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The following theorem given without proof can be thought of as an analogue of the Gram-Schmidt orthogonalization process for inner-product spaces.

**Theorem 1.4.** Let \( \dim(V) = 2n \). There exists a choice of basis of \( V \) such that \( B \) has the form

\[
B = \begin{pmatrix}
O & I_n \\
-I_n & O
\end{pmatrix}
\]

Suppose \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) is a basis giving \( B \) the desired form. Letting \( X \) and \( Y \) be the subspaces spanned by the \( x_i \) and \( y_i \) respectively, we see that \( V = X \oplus Y \), with \( X \) and \( Y \) both isotropic. The following proposition shows that \( X \) and \( Y \) are maximal isotropic subspaces.

**Proposition 1.5.** Let \( \dim(V) = 2n \). If \( W \subset V \) is an isotropic subspace, then \( \dim(W) \leq n \).

**Proof.** Suppose by way of contradiction that \( \dim(W) = m > n \). Extending any basis of \( W \) to a basis of \( V \) gives \( B \) the form of a matrix with an \( m \times m \) zero matrix in its upper left corner. Hence, the first \( m \) columns can span a vector space of dimension at most \( 2n - m < n \), which contradicts the non-singularity of \( B \). \( \square \)

**Definition 1.6.** Let \( \dim(V) = 2n \). An isotropic subspace of dimension \( n \) is called *Lagrangian*.

Hence, any symplectic vector space splits as the direct sum of two Lagrangian subspaces.

2. **Symplectic Complements**

**Definition 2.1.** Let \( W \) be a subspace of a symplectic vector space \( V \). Define \( W^\perp \), the *symplectic complement* of \( W \), to be the set:

\[
\{v \in V \mid B(w, v) = 0, \forall w \in W\}
\]

Notice that a subspace \( W \) is isotropic if and only if \( W \subset W^\perp \). We have the following standard result of complements defined by forms on vector spaces.

**Proposition 2.2.** Let \( W \) be a subspace of \( V \). Then \( \dim(W) + \dim(W^\perp) = \dim(V) \).

**Proof.** We wish to show that any element of \( W^* \) has the form \( B(w, \cdot) \) for some \( w \in W \). The nondegeneracy of \( B \) shows that the map sending \( w \) to \( B(w, \cdot) \) is an injection. Then a dimension argument shows it is an isomorphism. Next, define a linear map \( T : V \to W^* \) by sending \( v \) to \( B(v, \cdot) \). The image of \( T \) is all of \( W^* \), and the kernel of \( T \) is precisely \( W^\perp \). The result follows from the rank-nullity theorem. \( \square \)

3. **Applications**

We give a brief application of symplectic vector spaces.

**Definition 3.1.** Let \( g \) be a restricted lie algebra over \( k \), a field of characteristic \( p \). \( g \) is called *extraspecial* if the dimension of its center \( z \) is one and the restricted lie algebra \( g/z \) is an elementary lie algebra.

The motivation for the definition comes from that of an extraspecial group.
Theorem 3.2 (Carlson, Friedlander, Pevtsova, 2012). Let $\mathfrak{g}$ be an extraspecial lie algebra. Then the following are true:

(i) The dimension of $\mathfrak{g}$ is odd.

(ii) There is a basis of $\mathfrak{g}$, given by $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$, such that $\langle z \rangle = \mathfrak{z}$, $[x_i, x_j] = [y_i, y_j] = 0$, and $[x_i, y_j] = \delta_{ij} z$ for all $i, j = 1, \ldots, n$.

Proof. If $\mathfrak{g}$ is one dimensional, then the theorem holds. Now, note that since $\mathfrak{g}/\mathfrak{z}$ is abelian, we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$. Then, since $\mathfrak{g}$ is not abelian, it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{z}$.

Now, choose $0 \neq c \in \mathfrak{z}$, and fix the isomorphism of $\mathfrak{z}$ with $k$ by sending $c$ to $1 \in k$. Let $V = \mathfrak{g}/\mathfrak{z}$, and let $\pi : \mathfrak{g} \to V$ be the canonical projection. Let $\sigma : V \to \mathfrak{g}$ be any $k$-linear splitting of $\pi$, i.e., choose a basis of $V$ given by $\{v_i\}$, choose a representative $a_i \in \mathfrak{g}$ of each basis element, let $\sigma(v_i) = a_i$ and extend linearly.

Define on $V$ a bilinear form $B$ by $B(u, v) = [\sigma(u), \sigma(v)] \in [\mathfrak{g}, \mathfrak{g}] = \mathfrak{z} \cong k$. It can be shown that $B$ is independent of the choice of splitting of $\pi$, and that $B$ is skew-symmetric, totally isotropic, and non-degenerate. To see non-degeneracy, suppose that $u \in V$ such that $B(u, v) = 0$ for all $v \in V$. Then for any $a \in \mathfrak{g}$, $[\sigma(u), a] = [\sigma(u), \sigma(v)] = 0$ so that $\sigma(u) \in \mathfrak{z}$, and thus $u = 0$.

Hence $V$ is a symplectic vector space, which is obtained from $\mathfrak{g}$ as a quotient space of the one dimensional center. Since $V$ has dimension $2n$, it follows that $\mathfrak{g}$ has dimension $2n + 1$.

Next, find a basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of $V$ which gives $B$ the form of (1.1). Then set $x_i = \sigma(u_i)$ and $y_i = \sigma(v_i)$. The form of $B$ shows that this is the desired basis. \qed