1. Rational Points

Let \( X \) be a scheme. To each \( x \in X \) is attached a local ring, the stalk \( \mathcal{O}_x \). Let \( \mathfrak{m}_x \) be the unique maximal ideal contained in \( \mathcal{O}_x \).

**Definition 1.1.** The residue field of \( x \) is the field \( k(x) := \mathcal{O}_x / \mathfrak{m}_x \). In the case that \( X \) is a scheme over a field \( k \), then \( x \) is a rational point if \( k(x) \cong k \).

Now suppose \( X \) is an affine scheme over \( k \), with coordinate algebra \( k[X] \). We now show there is a one-to-one correspondence between the rational points of \( X = \text{Spec} \ k[X] \) and \( X(k) := \text{Hom}_k(k[X], k) \).

Suppose \( p \in \text{Spec} \ k[X] \) is rational, ie, \( p \subset k[X] \) is a prime ideal and \( k(p) = k[X]_p / \mathfrak{m}_p \cong k \) where \( k[X]_p \) is the localization of \( k[X] \) at the prime ideal \( p \), and \( \mathfrak{m}_p \) is the unique maximal ideal \( pk[X]_p \) in \( k[X]_p \). Remember that in the case of an affine scheme \( X = \text{Spec} \ A \), we have \( \mathcal{O}_p \cong A_p \) and \( \mathfrak{m}_p \cong pA_p \) where

\[
pA_p = \{ \frac{a}{b} \mid a \in p, b \notin p \}\]

Define \( f_p \) to be the following map of \( k \)-algebras:

\[
k[X] \to k[X]_p \to k[X]_p / \mathfrak{m}_p
a \mapsto \frac{a}{1} \mapsto \frac{a}{1} + \mathfrak{m}_p
\]

Thus given a rational point \( p \), we have defined a \( k \)-algebra map \( f_p : k[X] \to k \). Notice that the kernel of \( f_p \) is \( p \).

Next, let \( f : k[X] \to k \) be a map of \( k \)-algebras. Since \( f \) is a map of \( k \)-algebras, it must be surjective, and hence its kernel, \( p_f \) is a maximal ideal. I claim that \( p_f \in X \) is a rational point. To prove the claim, we must exhibit an isomorphism of fields between \( k(p_f) \) and \( k \).

Consider the following map:

\[
k(p_f) \longrightarrow k
\frac{a}{b} + \mathfrak{m}_{pf} \mapsto \frac{f(a)}{f(b)}
\]

The map is well defined because \( b \notin p_f = \ker f \) so the denominator on the right is not zero, and also because \( f(a) = 0 \) for all \( a \in p_f \) so that the map factors through \( \mathfrak{m}_{pf} \), ie, the map doesn’t depend on the choice of representative on the left. It is a map of fields because \( f \) is a map of \( k \)-algebras. Surjectivity follows from that of \( f \), and injectivity follows from:

\[
\frac{f(a)}{f(b)} = 0 \iff f(a) = 0 \iff a \in p_f \iff \frac{a}{b} \in \mathfrak{m}_{pf} \iff \frac{a}{b} + \mathfrak{m}_{pf} = 0
\]

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Now consider the map of \( k \)-algebras:
\[
k[X] \to k[X]_{p_f} \to k[X]_{p_f}/\mathfrak{m}_{p_f} \to k
\]
so that given \( p_f \), we recover \( f \). This observation finishes the proof of the bijection between rational points of an affine scheme \( X \) over \( k \) and \( X(k) \), the set of \( k \)-algebra maps from \( k[X] \) to \( k \).

**Conjecture 1.2.** The functor from affine schemes over \( k \) to sets assigning to a scheme \( X \) the set of rational points of \( X \) is naturally isomorphic to the functor assigning to \( X \) the set of \( k \)-algebra maps from \( k[X] \) to \( k \).

For any field extension \( K/k \), notice that a point \( p \in X \) with \( k(p) \cong K \) yields a surjection \( k[X] \to K \) in a similar manner to our definitions above, but that the kernel of a map of \( k \)-algebras \( k[X] \to K \) does not necessarily have residue field isomorphic to \( K \), because the map \( k[X] \to K \) need not be surjective.

Let’s consider some examples.

**Example 1.3.** Let \( X = \text{Spec } k[x] = \mathbb{A}^1_k \) be the affine line over \( k \). \( k \)-algebra homomorphisms \( k[x] \to k \) are determined by a choice of image for \( x \), and are thus the same as evaluation maps. Let \( f_x : k[x] \to k \) be evaluation at \( a \in k \), so that \( \ker f_x = (x - a) \in X \). The above correspondence says that \( k(\ker f_x) \cong k \), so that the rational points of \( X \) are the ideals \( (x - a) \).

What about the prime ideal \( (0) \in X \)? Localizing at \( (0) \) gives the fraction field, so that \( k((0)) \cong k(x) \), the field of rational functions. In general, if \( k[X] \) is an integral domain, then \( (0) \) is a prime ideal, and \( k((0)) \cong \text{Frac}(k[X]) \).

Since \( k[x] \) is a PID, the other points of \( X \) (prime ideals of \( k[x] \)) correspond to monic irreducible non-constant polynomials of degree \( \geq 2 \). If \( k \) is algebraically closed, no such polynomials exist. Suppose \( k = \mathbb{R} \) and consider the prime ideal \( p = (x^2 + 1) \). What is \( k(p) \)? I claim that \( k(p) \cong \mathbb{C} \). Consider the following map:
\[
\mathbb{R}[x]_{p}/\mathfrak{m}_{p} \to \mathbb{C}
\]
\[
f + \mathfrak{m}_{p} \to \frac{f(i)}{g(i)}
\]
This map is an isomorphism of fields, which can be checked with similar reasoning used in the correspondence shown above.

**Conjecture 1.4.** Let \( X = \text{Spec } k[x] \) be the affine line. Then for any monic irreducible non-constant polynomial \( f \), \( k(p) \cong k(f) \subset \bar{k} \) where \( p = (f) \) and \( k(f) \) is the splitting field of \( f \).

If the above conjecture is true, in particular it would show that every monic irreducible non-constant polynomial of degree \( \geq 2 \) in \( \mathbb{R}[x] \) defines a prime ideal whose residue field is \( \mathbb{C} \). This follows from the fact that there are no intermediate fields between \( \mathbb{R} \) and \( \mathbb{C} \), which results from Galois theory and the fact that \( [\mathbb{C}, \mathbb{R}] = 2 \).

**Example 1.5.** Let \( X = \text{Spec } k[x, y] \cong \mathbb{A}^2_k \) be the affine plane. As in the previous example, \( k \)-algebra maps \( k[x, y] \to k \) are evaluation maps, and we see that the rational points are the ideals \( (x - a, y - b) \) for \( a, b \in k \). Also as in the previous example, the residue field of the zero ideal is the fraction field, ie, \( k((0)) \cong k(x, y) \).
Conjecture 1.6. Let $X = \text{Spec } k[x, y]$ be the affine plane. Let $p_1 \subset p_2$ be two prime ideals in $k[x, y]$. Then $k(p_2) \subset k(p_1) \subset k(x, y)$, i.e., there is an order-reversing Galois correspondence between prime ideals in $k[x, y]$ and residue fields in $k(x, y)$. 