1. Representation Theory of Finite Groups

$k$ - field, algebraically closed $G$ - finite group Definitions

1. A representation of $G$ over $k$ of degree $n$ is a homomorphism of groups $\rho : G \to \text{GL}_n(k)$, or equivalently, a $kG$-module structure for an $n$-dimensional $k$ vector space $M$, where $kG$ is the group algebra.

2. A representation $M$ is irreducible if it is a simple $kG$-module, and indecomposable if $M \neq N_1 \oplus N_2$ for any non-trivial submodules $N_1, N_2$. Notice that irreducible $\implies$ indecomposable.

Representation Theory of Finite Groups is thus the study of groups acting on vector spaces, ie, sending abstract group elements to concrete matrices. It has been used to prove group-theoretic results of Frobenius and Burnside. Classically there are two cases.

Case 1: - char$k$ does not divide $|G|$.
Maschke’s Theorem then says that $kG$ is semisimple, ie, all submodules are summands, ie, irreducibles = indecomposables. Here all irreducible representations are summands of the regular representation, $kG$ considered as a module over itself, so in particular, there are only finitely many indecomposable representations.

Case 2: - char$k||G|... things aren’t so simple, or maybe I should say things aren’t so semisimple.

2. Representable Functors and Yoneda’s Lemma

Before defining Affine Group Schemes, we need to know a bit about representable functors and Hopf Algebras. The next two sections are devoted to these topics.

Definition

Let $C$ be a locally small category (ie $\text{Hom}(A, B)$ is a set for all $A, B \in \text{ob}(C)$). A functor $F$ from $C$ to $\text{Sets}$ is called representable if there exists $A \in \text{ob}(C)$ such that $F$ is naturally isomorphic to the functor $\text{Hom}(A, -)$. We say that $A$ is the representing object of $F$.

The following lemma says that

Lemma - (Yoneda)

Let $F$ be a representable functor from $C$ to $\text{Sets}$ with representing object $A$, and let $X$ be any other functor from $C$ to $\text{Sets}$. Then natural transformations $\Phi : F \to X$ are in one-to-one
correspondence with elements of $X(A)$. In particular, if $X$ is representable, with representing object $B$, then there is a one-to-one correspondence between natural transformations from $F$ to $X$ and $\text{Hom}(B, A)$.

Proof:
Let $\Phi : \text{Hom}(A, \cdot) \to X$ be a natural transformation, so that $\Phi(A) : \text{Hom}(A, A) \to X(A)$ is a morphism in $\text{Sets}$. Consider $id_A \in \text{Hom}(A, A)$, and $x_{\Phi} := \Phi(A)(id_A) \in X(A)$. Thus, to each natural transformation $\Phi$, we’ve assigned an element $x_{\Phi} \in X(A)$.

For the reverse direction, start with any element $x \in X(A)$. To define a natural transformation $\Phi_x : \text{Hom}(A, \cdot) \to X$, for every $B \in C$ we wish to define a morphism of sets $\text{Hom}(A, B) \to X(B)$. Suppose $f \in \text{Hom}(A, B)$. Define $\Phi_x(B)(f) = X(f)(x)$.

3. **Hopf Algebras**

Let $k$ be a field.

**Definitions**

1. A unital associative algebra over $k$, henceforth called an algebra, is a $k$-space $A$ together with two maps $m : A \otimes A \to A$, called multiplication, and $u : k \to A$, called unit, that satisfy:
   - $m \circ (m \otimes id_A) = m \circ (id_A \otimes m)$ on $A \otimes A \otimes A$
   - $m(u(b) \otimes a) = b \cdot a$ and $m(a \otimes u(b)) = b \cdot a$ for all $a \in A$, $b \in k$, where $\cdot$ is scalar multiplication.

2. A coalgebra is a $k$-space $C$ together with two maps $\Delta : C \to C \otimes C$, called comultiplication, and $\varepsilon : C \to k$, called counit, that satisfy:
   - $(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta$ on $C$
   - $(\varepsilon \otimes id_C) \circ \Delta(c) = 1_k \otimes c$ and $(id_C \otimes \varepsilon) \circ \Delta(c) = c \otimes 1_k$ for all $c \in C$, where $1_k$ is the identity in $k$.

3. A bialgebra is both an algebra and a coalgebra such that one of the following two equivalent conditions holds:
   - $\Delta$ and $\varepsilon$ are maps of algebras
   - $m$ and $u$ are maps of coalgebras

A Hopf algebra is a bialgebra with one additional piece of information, which we now motivate using convolution

4. **Affine Group Schemes**

All $k$-algebras are commutative.

**Definitions**

1. An affine group scheme, $G$, is a representable functor from $k$-algebras to groups. The representing object is called the coordinate algebra of $G$, denoted $k[G]$, and the group algebra, $kG$, is defined to be the linear dual of $k[G]$, i.e., $kG := (k[G])^* = \text{Hom}_k(k[G], k)$.

2. An affine group scheme $G$ is finite if $k[G]$ is a finite dimensional $k$-space. $G$ is called infinitesimal if it is finite and $k[G]$ is a local ring. If $G$ is infinitesimal, the maximal ideal $\mathfrak{m}$ of $k[G]$ consists of the nilpotent elements of $k[G]$, so for char $k = p$ we may
define the height of $G$ to be the smallest positive integer $r$ such that $x^{p^r} = 0$ for all $x \in \mathfrak{m}$.

Yoneda’s Lemma says that natural transformations between affine group schemes correspond to $k$-algebra morphisms on coordinate algebras in the opposite direction. Since $G$ takes values in groups, the natural transformations corresponding to the group laws gives a commutative Hopf algebra structure to $k[G]$. If $k[G]$ is cocommutative, then $kG$ is also a commutative Hopf algebra, and defines the Cartier dual group scheme, $G^D$. This discussion shows there is a one-to-one correspondence between finite dimensional cocommutative Hopf algebras $H$ and finite group schemes $G$ given by $G \to kG$ and $H \to G$ such that $G(R) = \operatorname{Hom}_{k-alg}(H^*, R)$ for all $k$-algebras $R$.

**Examples**

(1) The Constant Group Scheme. Let $\Gamma$ be any finite group. Define a group scheme $G_{\Gamma}$ by constructing its coordinate algebra as follows: Let $k[\Gamma]$ have a $k$-basis of orthogonal idempotents indexed by the elements of $\Gamma$, i.e., for each $\sigma \in \Gamma$, let $e_\sigma$ be an idempotent and $e_\sigma e_\tau = 0$ if $\sigma \neq \tau$. Extend multiplication to all of $k[\Gamma]$ linearly. The Hopf algebra structure on $k[\Gamma]$ is given by $\Delta(e_\sigma) = \sum_{\rho \sigma = \sigma} e_\rho \otimes e_\tau$, $S(e_\sigma) = e_{\sigma^{-1}}$ and $\varepsilon(e_\sigma) = \delta_{\sigma,1}$ where 1 denotes the identity of $\Gamma$.

With this coordinate algebra it can be shown that for any $k$-algebra $R$, $G_{\Gamma}(R) = \Gamma^{\pi_0(R)}$, where $\pi_0(R)$ is the number of connected components of $R$, i.e., the number of pairs of orthogonal idempotents $e$ and $1 - e$ in $R$. So $G_{\Gamma}$ is constant on connected $k$-algebras, with value $\Gamma$.

Notice that $G_{\Gamma}$ is a finite group scheme because $\dim(k[G_{\Gamma}]) = |\Gamma|$. $G_{\Gamma}$ is infinitesimal if and only if $\Gamma$ is the trivial group. This follows because as a ring, $k[G_{\Gamma}]$ is just $|\Gamma|$ copies of $k$, and thus has more than one maximal ideal if and only if $|\Gamma| > 1$.

It can also be shown that $kG_{\Gamma}$ is isomorphic as a Hopf algebra to $k\Gamma$, the usual Hopf algebra associated to $\Gamma$. However, as noted above, $kG_{\Gamma}$ represents the Cartier dual of $G$ iff $k[G]$ is cocommutative iff $\Gamma$ is abelian.

(2) $GL_n$. For any $k$-algebra $R$, let $GL_n(R)$ be the group of invertible $n \times n$ matrices under usual matrix multiplication. With this definition on $k$-algebras, it follows that $k[GL_n(R)] = \frac{k[x_{ij}, 1 \leq i, j \leq n, t]}{(\det(x_{ij})t - 1)^n = k[x_{ij}, 1 \det(x_{ij})]}$, and the coproduct that corresponds to matrix multiplication is $\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}$. The necessary counit giving the proper identity in our familiar matrix groups is $\varepsilon(x_{ij}) = \delta_{i,j}$. The antipode map is given by Cramer’s Rule.

$GL_n$ is not finite because $k[GL_n]$ is infinite dimensional, and it is not infinitesimal because each invertible $n \times n$ matrix with entries in $k$ defines a surjection onto $k$ whose kernel is a maximal ideal in $k[GL_n]$. These maximal ideals are distinct for distinct matrices; they correspond to the point of $kn^2$ defining the matrix, i.e., they are of the form $(x_{ij} - a_{ij}, t - \det(a_{ij}))$ where $(a_{ij})$ is the given matrix.

(3) Restricted enveloping algebras. Let $(L, [\cdot, \cdot])$ be a finite dimensional Lie algebra over $k$, and let $\operatorname{char} k = p$. $L$ is called a restricted Lie algebra if there is a map $(-)^{[p]} : L \to L$,
called the $p$-operation satisfying:

- $(\alpha x)^p = \alpha^p x^p, \forall \alpha \in k, x \in L$
- $[x, y]^p = (ad y)^p(x), \forall x, y \in L$
- $(x + y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y) \lambda^{i-1}$, $\forall x, y \in L$ where $s_i(x, y)$ is the coefficient of $\lambda^{i-1}$ in the expression $(ad(\lambda x + y))^p(x)$.

If $A$ is an associative $k$-algebra, not necessarily commutative, we can form the Lie algebra associated to $A$, denoted $A^-$, by defining $[a, b] = ab - ba, \forall a, b \in A$. Using normal $p^{th}$ power for the the $p$-operation, it can be seen that $A^-$ becomes a restricted Lie algebra.

If $U(L)$ is the universal enveloping algebra of $L$, give $U(L)$ a Hopf algebra structure as follows: $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$, and $S(x) = -x, \forall x \in L$. (Here we have identified $L$ with the degree one elements of $U(L)$). This makes $U(L)$ into a cocommutative Hopf algebra, but $U(L)$ is not finite dimensional, so it does not correspond to a finite group scheme.

We thus form the restricted enveloping algebra as follows: let $I$ be the two-sided ideal generated by elements of the form $x^p - x^{|p|}, \forall x \in L$, and define the restricted enveloping algebra, $u(L)$, to be the quotient $U(L)/I$. Since $I$ is a Hopf ideal, $u(L)$ inherits a Hopf algebra structure from $U(L)$, and is thus itself a cocommutative Hopf algebra. Notice that if $\{e_1, ..., e_n\}$ is a basis for $L$, then $\{e_1^i ... e_n^i\}, 1 \leq i_j \leq p - 1$ for $j = 1, ..., n$ is a basis for $u(L)$ (again identifying $L$ with degree one elements in $u(L)$). So $\dim u(L) = p^{\dim L}$. In particular, $u(L)$ is finite dimensional, and so it gives rise to a finite group scheme with coordinate algebra equal to $u(L)^*$. How is this infinitesimal?

Examples (1) and (2) above are important for seeing how the representation theory of finite group schemes generalizes that of finite groups, which we illustrate in the next section. Before doing so, let us discuss the concept of a base change of an affine group scheme.

Suppose $G$ is an affine group scheme over $k$ and $k \to k'$ is a ring map. This map allows us to view all $k'$ algebras as $k$-algebras, and so we can define an affine group scheme $G_{k'}$ over $k'$ by $G_{k'}(R) = G(R)$ for $R$ a $k'$ algebra, viewed as a $k$ algebra on the right hand side. What is $k'[G_{k'}]$? Well, since $G(R) = \text{Hom}_{k-alg}(k[G], R) \cong \text{Hom}_{k'-alg}(k[G] \otimes k', R)$ we see that $k'[G_{k'}] = k[G] \otimes k'$. The isomorphism above can be seen by sending a $k$-algebra map $f : k[G] \to R$ to the $k'$ algebra map $f' : k[G] \otimes k' \to R$ where $f' = f \otimes id_{k'}$, ie, for any $a \in k[G], b \in k'$, $f'(a \otimes b) = b \cdot f(a)$. In the reverse direction, a $k'$-algebra map $f' : k[G] \otimes k' \to R$ restricts to a $k$-algebra map $f'|_{k[G] \otimes 1} = f : k[G] \to R$.

5. Representation Theory of Finite Group Schemes

In analogy with the first section, we now define representations of finite group schemes. In this section, $G$ will denote a finite group scheme.
Definitions

(1) A representation of $G$ over $k$ of degree $n$ is a homomorphism of group schemes $\rho : G \rightarrow \text{GL}_n$, or equivalently, a $k[G]$-comodule structure for an $n$-dimensional $k$ vector space $M$, or equivalently, a $kG$-module structure for $M$.

(2) A representation $M$ is irreducible if it is a simple $kG$-module, and indecomposable if $M \neq N_1 \oplus N_2$ for any non-trivial submodules $N_1, N_2$. Notice that irreducible $\implies$ indecomposable.

Representation Theory of finite group schemes is thus a generalization of that of finite groups: for the latter, we study $kG$-modules only for constant group schemes $G$.

As in the case of finite groups, we are interested in when $kG$ is semisimple, or equivalently when $k[G]$ is cosemisimple. Thus we seek a generalization of Maschke’s Theorem. To state the generalization properly, we need to discuss the concept of integrals in Hopf algebras.

Definition

Let $H$ be a Hopf algebra. An element $t \in H$ is called a left integral if it is "invariant" under multiplication on the left, ie, if $ht = \epsilon(h)t, \forall h \in H$. The vector space spanned by all left integrals is denoted $\int_H^l$. Right integrals are defined similarly.

It is a fact of finite dimensional Hopf algebras that $\int_H^l$ is always one-dimensional.

Examples

Let’s find integrals in the finite dimensional Hopf algebras we listed above.

(1) For the constant group scheme $G_\Gamma$ associated to a finite group $\Gamma$, the space of left and right integrals for $k[G_\Gamma]$ is spanned by $e_1$ where 1 denotes the identity in $\Gamma$. The space of left and right integrals for $kG_\Gamma = k\Gamma$ is spanned by $\sum_{g \in \Gamma} g$. Hopf algebras for which the left and right integrals coincide are called unimodular.

(2) Restricted Enveloping Algebra

With the concept of an integral in hand, the following generalization of Mashcke’s Theorem now answers our question of when $kG$ is semisimple:

Theorem (reference)

A finite-dimensional Hopf algebra $H$ is semisimple $\iff$ $\epsilon(\int_H^l) \neq 0$ $\iff$ $\epsilon(\int_H^r) \neq 0$ $\iff$ $H^*$ is cosemisimple $\iff$ there is an integral $T \in H^*$ such that $\langle T, 1_H \rangle = 1$.

That this theorem generalizes Maschke’s theorem is even more clear upon noting that in the case of a finite group $\Gamma$, the integral, $t$ found above for the Hopf Algebra $k\Gamma$, satisfies $\epsilon(t) = |\Gamma|$.

6. Modular Representation Theory and Representation Type

$k$ - field, char$k=p$
$G$ - finite group, $p||G|$

$kG$ is not semisimple $\implies$ there are indecomposables that are not irreducible.
Example (Benson)

\[ G = \mathbb{Z}/p \times \mathbb{Z}/p = \langle g_1, g_2 \rangle, \lambda \in k. \] Define \( \rho_\lambda : G \rightarrow \text{GL}_2(k) \) by:

\[
g_1 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}
\]

It is easy to check that the matrices have order \( p \) and commute. Then \( \rho_\lambda \) is indecomposable (neither matrix is diagonalizable) but not irreducible (there are clear invariant subspaces). Also, \( \rho_\lambda \cong \rho_\mu \iff \lambda = \mu \). So if \( k \) is infinite, we've just produced infinitely many non-isomorphic indecomposable \( kG \)-modules.

Fact: The only irreducible representation of a \( p \)-group is the trivial representation. Also, \( \mathbb{Z}/p \times \mathbb{Z}/p \) is a quotient of every non-cyclic finite \( p \)-group, so this construction generalizes.

Thus, in the extreme case of a \( p \)-group, indecomposability is far less rigid than irreducibility.

Definitions - Let \( \Lambda \) be a finite dimensional algebra over \( k = \overline{k} \).

1. \( \Lambda \) has finite representation type if there exist finitely many (isomorphism classes of) indecomposable \( \Lambda \)-modules.
2. \( \Lambda \) has tame representation type if in any given dimension the indecomposable modules can be classified by finitely many one parameter families, with finitely many exceptions.
3. \( \Lambda \) has wild representation type if it "includes" representations of the free algebra on two variables. Classifying representations of the free algebra on two variables is equivalent to the problem of finding a canonical form for pairs of matrices under conjugation, which is thought to be unsolvable.

Trichotomy Theorem (Drozd, 1980)

Every finite dimensional algebra over an algebraically closed field is of finite, tame, or wild representation type, and these types are mutually exclusive.

Here is the classification of representation type for group algebras of finite groups.

Theorem (Bondarenko and Drozd, 1977)

- \( kG \) has finite representation type if and only if \( G \) has cyclic Sylow \( p \)-subgroups.

- \( kG \) has tame representation type if and only if the Sylow \( p \)-subgroups are dihedral, semidihedral or generalised quaternion.

- In all other cases, \( kG \) has wild representation type.

So even indecomposable representations of \( G = \mathbb{Z}/p \times \mathbb{Z}/p \) are thought to be impossible to classify ("Modules for \( G = \mathbb{Z}/p \times \mathbb{Z}/p \)" by Carlson, Friedlander, and Suslin, 2009).

7. \( \pi \)-points and Modules of Constant Jordan Type

The above theorem informs us that the representation theory of \( \mathbb{Z}/p\mathbb{Z} \) over a \( k \) of characteristic \( p \) is of finite type, so let’s consider the problem of classifying the indecomposable \( kE \)-modules where \( E = \mathbb{Z}/p\mathbb{Z} \).
If $E = \langle g \rangle$, then $kE = k[g]/(g^p - 1)$. It is often more convenient to consider the element $x := g - 1 \in kE. Notice that $x^p = (g - 1)^p = g^p - 1^p = 0$ so that $kE = k[x]/(x^p)$. Let $M$ be an indecomposable $kE$-module of dimension $n$. The action of $kE$ is completely determined by that of the element $x$. The action of $x$ on $M$ is given by an $n \times n$ nilpotent matrix whose $p^{th}$ power is $0$. Thus, the Jordan form of this matrix consists of Jordan blocks of sizes $1, 2, \ldots, p$. By indecomposability, there can only be one block, so we see that there are $p$ indecomposable $kE$-modules, one in each dimension $1, 2, \ldots, p$, where the action of $x$ is given by a matrix of $0$'s with $1$'s on the subdiagonal (or superdiagonal).

Notice that the regular representation (considering $kE$ as a module over itself) has dimension $p$ and corresponds to the partition of $p$ with just one part of size $p$ (one jordan block of size $p$).

This completely classifies all representations of $E$: they are direct sums of indecomposable representations. Thus representations of $E$ in dimension $n$ correspond to partitions of $n$ whose parts are less than or equal to $p$. Given a module $M$ of dimension $n$, form a partition of $n$ corresponding to the jordan block sizes of the action of $x$ (all of which are less than or equal to $p$). Given an appropriate partition of $n$, form a module $M$ of dimension $n$ as the direct sum of the indecomposable modules whose sizes correspond to the partition.

We will use this nice classification of indecomposable $kE$-modules to study $kG$-modules for an arbitrary finite group scheme $G$. To do this, we introduce the concept of the Jordan type of a module.

**Definition**

Let $M$ be a $k[x]/(x^p)$-module. Suppose the action of $x$ has $a_i$ blocks of size $i$ for $1 \leq i \leq p$. Define the Jordan type of $M$ to be the formal sum $a_1[1] + a_2[2] + \ldots + a_p[p]$. Notice that the blocks of size $p$ correspond to copies of the regular representation, which are projective. We ignore these summands when working in the stable module category, and this motivates the definition of the stable Jordan type of a $k[x]/(x^p)$-module $M$ to be the formal sum $a_1[1] + a_2[2] + \ldots + a_{p-1}[p-1]$.

Using this machinery to study $kG$-modules for arbitrary finite group schemes is based in the simple idea of considering modules restricted through maps. Thus we come to the definition of a $\pi$-point.

**Definition**

Let $G$ be a finite group scheme over $k$, and let $K/k$ be a field extension. A $\pi$-point for $G$ is a left flat map of $K$-algebras $\alpha_K : K[t]/t^p \to KG$ which factors through the group algebra $KC_K \subset KG_K$ of some unipotent abelian subgroup scheme $C_K \subset G_K$. If $K = k$ we call the map a $p$-point. For a given $kG$-module $M$, we say the Jordan type of the $\pi$-point $\alpha_K$ is the jordan type of the $K[t]/t^p$-module $\alpha_K^*(M_K)$ where $M_K = M \otimes K$ and $\alpha_K^*(M_K)$ is the $K[t]/t^p$-module obtained by restriction along $\alpha_K$.

The space of all $\pi$-points is too big, so we form an equivalence relation among them as follows:
8. Elementary Abelian p-groups

\[ k \text{- field, } \text{char } k = p \]
\[ G \text{- finite group, } p \parallel |G| \]
\[ E = (\mathbb{Z}/p)^r = \langle g_1, \ldots, g_r \rangle \leq G \]

Why are we interested in such subgroups?

**Chouinard’s Theorem** - 1975

Let \( G \) be a finite group, \( k \) a field. A \( kG \)-module \( M \) is projective (=free=injective=flat) iff \( M|_E \) is projective for all elementary abelian p-groups \( E \leq G \).

Other examples also show the importance of elementary abelian p-groups, including Quillen stratification, etc.

No, if \( r > 1 \), then \( kE \) has wild representation type by Drozd’s Theorem, so there are too many indecomposables to chew on. Thus, we restrict our attention to special modules I now define.

Let \( k = \bar{k} \). \( kE = \frac{k[g_1, \ldots, g_r]}{(g_1^p - 1, \ldots, g_r^p - 1)} = \frac{k[x_1, \ldots, x_r]}{(x_1^p, \ldots, x_r^p)} \), where \( x_i = g_i - 1 \). \( J(kE) = \langle x_1, \ldots, x_r \rangle \).

Let \( \alpha = (\lambda_1, \ldots, \lambda_r) \in A^r(\bar{k}) \), \( X_\alpha = \lambda_1 x_1 + \ldots + \lambda_r x_r \).

Since \( X_\alpha \) is nilpotent, its action on a finitely generated \( kE \)-module \( M \) splits into jordan blocks of lengths between 1 and \( p \). Write \( [p]^a p^{r-1} \) for the Jordan type of \( x_\alpha \). We say \( M \) has constant jordan type, or cJt, if \( (a_1, \ldots, a_p) \) is independent of \( \alpha \).

**Example**

The regular representation of \( kE \) has cJt \( [p]^p^{r-1} \).

**Theorem** (Dade) - 1978

\( M \) is projective iff \( M \) has cJt \( [p]^n \) for some \( n \). In particular \( p^{r-1}|n \).

Motivated by this fact, we call \( [p-1]^a p^{r-1} \) the stable constant jordan type, or scJt, of \( M \). So we see that certain constant jordan types are not possible, for example, if \( r = 3 \) there can be no cJt \( [p]^n \) for any squarefree positive integer \( n \). Here are some more restrictions on possible constant jordan types, and some conjectures along the same lines.

**Theorem** (Benson, MSRI, 2008)

If \( r \geq 2 \) and \( 2 \leq a \leq p - 2 \), then there are no modules of scJt \([a] \).

**Conjecture** (Suslin, 2008) - Appears in ”Modules of Constant Jordan Type,” a paper of Carlson, Friedlander, and Pevtsova.

Let \( r \geq 2 \). If \( 2 \leq a \leq p - 1 \) and \( M \) has cJt with blocks of length \([a]\), then it also has blocks of length \([a - 1]\) or \([a + 1]\)

**Conjecture** (Rickard, MSRI, 2008)

Let \( r \geq 2 \). If \( M \) has no blocks of length \( a \), then the total blocks of length greater than \( a \) is divisible by \( p \). (Some progress here due to Benson).

**Question** (Carlson, Friedlander, Suslin, 2009)
Do modules of cJt have wild representation type?

9. Vector Bundles on Projective Space

Vector bundles over schemes are locally free sheaves of modules, with a trivial vector bundle of rank $n$ corresponding to the sum of $n$ copies of the structure sheaf. Quillen-Suslin Theorem $\implies$ All algebraic vector bundles on $\mathbb{A}^n$ are trivial. The same is not true of $\mathbb{P}^n$.

**Conjecture** (Hartshorne)

All rank 2 vector bundles over $\mathbb{P}^n$, $n \geq 7$, are direct sums of line bundles. (Hartshorne admits he "not even sure it’s true," and doesn’t have sufficient evidence to make any conjectures for rank >3.)

In general, there are not a lot of known low rank vector bundles over proj. space. Modules of Constant Jordan type can actually be used to construct vector bundles on projective space. The goal of this section is to describe that process. Before doing so, we need to discuss a few preliminary points concerning bundles, sheaves, and modules.

**Sheaves of Modules on Spec A**

Suppose $X = \text{Spec } A$ is an affine scheme, and let $M$ be an $A$-module. To $M$ we associate a sheaf of modules on $X$, denoted $\widetilde{M}$, such that $\Gamma(X, \widetilde{M}) = M$. In other words, $M$ can be recovered from $\widetilde{M}$. $\widetilde{M}$ has the property that $\widetilde{M}|_{D(f)} = \widetilde{M}_f$ for all $f \in A$ where $D(f) = \{p \in \text{spec } A | f \notin p\} = \text{spec } A_f$, and $M_f$ is the localized $A_f$ module. We can also consider $D(f)$ to be the complement of $Z(f)$ in $X$ where $Z(f)$ are the zeroes of $f$ viewed as a "function" on $A$. Under this view, $f : A \to \prod_{p \in \text{spec } A} k(p)$ where $k(p) = A_p/pA_p$ is the residue field of $p$ and $f(p) = [f] \in k(p)$. With this definition, we see that $p \in Z(f) \iff f \notin p \iff p \notin D(f)$.

Since the $D(f)$ form an open cover of $X$, it follows that $\widetilde{M}$ is quasi-coherent.

The above paragraph establishes the following one-to-one correspondence:

Quasi-coherent sheaves of modules on spec $A \leftrightarrow$ A-modules

If we consider $A$ as a module over itself, what is the corresponding quasi-coherent sheaf $\mathcal{F}$? By our claims above, it should satisfy $\Gamma(X, \mathcal{F}) = A$ and $\Gamma(D(f), \mathcal{F}) = A_f$ and is thus precisely the structure sheaf $\mathcal{O}_X$. Under his correspondence, direct sums of modules correspond to direct sums of sheaves, and so free modules to free sheaves.

**Sheaves of Modules on Proj S**

In the previous section, we learned that quasi-coherent sheaves on spec $A$ and $A$-modules are in one-to-one correspondence. Here we establish a similar correspondence in Proj $S$ for some graded ring $S$.

Suppose $X = \text{Proj } S$, and let $M$ be a graded $S$-module. To $M$ we associate a sheaf of modules on $X$, denoted $\widetilde{M}$ in a way such that if we consider $S$ as a module over itself we obtain $\widetilde{S} = \mathcal{O}_X$ as before. One difference here is that we can’t completely recover $M$ from $\widetilde{M}$ as before, but we can up to a certain equivalence. First, let’s discuss the notion of a twisting sheaf.
If $M$ is a graded $S$-module, define $M(n)$ to be the graded $S$-module with $[M(n)]_d = M_{n+d}$. Hence, $M(n)$ is obtained from $M$ by shifting all degrees down by $n$. The multiplication from $S$ remains unchanged. Define $O(n) = S(n)$, and in general, for any sheaf of modules $\mathcal{F}$, define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} O(n)$.

Now we are ready to attempt to recover $M$ from $\tilde{M}$ by defining the graded $S$-module associated to any sheaf of modules on $X$.

**Definition**

Let $X = \text{Proj } S$ for some graded ring $S$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Define the graded $S$-module associated to $\mathcal{F}$, denoted $\Gamma_*(\mathcal{F})$, to be $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$ as an abelian group. (Need to understand the multiplication from $S$ better).

Now we can see how we might be able to go back and forth between graded $S$-modules and sheaves of $\mathcal{O}_X$-modules: to a module $M$ associate $\tilde{M}$ and to a sheaf of modules associate $\Gamma_*(\mathcal{F})$. Does it follow that $\Gamma_*(\tilde{M}) = M$? The answer is no, but close. (Hartshorne exercise 2.5.9, need to understand better).

Next in our quest to build vector bundles on projective space, we consider the Global $p$-Nilpotent Operator in the case of elementary abelian $p$-groups, a definition due to Eric Friedlander and Julia Pevtsova.

**Global $p$-Nilpotent Operator** (Friedlander and Pevtsova)

Here we revert our attention to elementary abelian $p$-groups of rank $r$, $E = (\mathbb{Z}/p\mathbb{Z})^r$ where $kE = k[x_1, ..., x_r]/(x_i^p)$ for $k$ a field of characteristic $p$. Let $A = k[x_1, ..., x_r]$ so that $\text{spec } A = \mathbb{A}_k^r$. Let $M$ be a $kE$-module of constant Jordan type, and define a map of $A$-modules as follows:

$$\Theta_M : A \otimes M \rightarrow A \otimes M$$

$$1 \otimes m \mapsto \sum_{i=1}^{r} X_i \otimes x_i(m)$$

Here the tensor product is over $k$, and we extend $A$-linearly. Notice that $A \otimes M$ is a free $A$-module of rank equal to the dimension of $M$, and so this map corresponds to a map of free sheaves on $\mathbb{A}_k^r$, which is a map of trivial vector bundles over $\mathbb{A}_k^r$. Let’s consider the motivation behind the definition of this map through specialization.

Abstractly, given two rings $A$ and $B$, a map $A \rightarrow B$, and an $A$-module $M$, we can give $B \otimes_A M$ the structure of a $B$ module. In our case, let $B = k$, and let the map $A \rightarrow k$ be a $k$-point of $X$. This amounts to choosing an $a_i \in k$ as an image for each $X_i \in A$. (Need to fill in some gaps here) The map above then specializes to:

$$\Theta_M : k \otimes_A A \otimes M \cong M \rightarrow k \otimes_A A \otimes M \cong M$$

$$m \mapsto \sum_{i=1}^{r} a_i x_i(m)$$
So the action of $\Theta_M$ on the fiber above a point $a = (a_1, a_2, ..., a_r)$ is given by the action of the $\pi$-point defined by $a$. Since $M$ has constant Jordan type, the kernel of this map is a locally free sheaf, i.e., a vector bundle over $A^r_k$. (Need to understand this better)

**Realisation Theorem** (Benson, Pevtsova)

Given a vector bundle $F$ of rank $s$ on $\mathbb{P}^{r-1}$, there exists a $kE$-module $M$ of scJt [1] s.t.

- if $p = 2$ then $F_1(M) \cong F$
- if $p$ is odd then $F_1(M) \cong F^*(F)$, where $F: \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius Map