1. Consider the affine coordinate algebra $A$ of the plane curve $Z(y^2 - x^3) \subset \mathbb{A}^2$, and let $K$ be the field of fractions of $A$. Find an element $\alpha \in K$ which is integral over $A$ but not in $A$.

Proof. We will calculate the integral closure of $A$ inside $K$. We begin with a more general discussion.

Suppose $C$ is an irreducible, affine curve, that is, $C$ is some irreducible affine variety of dimension 1. Let $A$ be the ring of regular functions on $C$, so that $A$ is a noetherian domain of Krull dimension 1. Localization preserves the noetherian condition, as well as the domain condition. Thus, the local ring $\mathcal{O}_x$ at any point $x \in C$ is a noetherian domain. It also has Krull dimension 1 as the dimension of any local ring of an irreducible variety is equal to the dimension of the variety.

If $C$ has a singular point $x$, then $\mathcal{O}_x$ is not regular. We know that for noetherian local domains of dimension 1, regularity is equivalent to integrally closed, so that $\mathcal{O}_x$ is not integrally closed. Since the localization of an integrally closed domain is integrally closed, if $C$ has singular points, we can conclude that $A$ is not integrally closed. On the other hand, if $C$ is smooth, then $A$ must be integrally closed, and thus a Dedekind domain.

Now suppose we resolve the singularities of $C$ to obtain some smooth curve $C'$ and a map $C' \to C$ of varieties. Suppose in addition that $C'$ happens to be affine, with coordinate ring $A'$, which by the above discussion is a Dedekind domain. This induces an injection $A \hookrightarrow A'$. I will show that in the case of this problem, the above injection makes $A'$ a finitely generated $A$-module, which is enough to show $A'$ is integral over $A$.

To see this, suppose $A'$ is finitely generated as an $A$-module. Let $f \in A'$. Then if $\{b_1, \ldots, b_n\}$ are generators, let $b = (b_1, \ldots, b_n)^t \in A^n$, and notice that $fb = Mb$ for some $M \in M_n(A)$. It follows that $(fI_n - M)b = 0 \in A^n$. Multiplying on the left by the adjoint matrix gives $\det(fI_n - M)b = 0$. By definition, of $b$, this implies that $\det(fI_n - M)$ is in the annihilator of $A'$, but $A'$ is a faithful $A$ module, so $\det(fI_n - M) = 0$. Expanding the determinant gives the desired monic polynomial with coefficients in $A$ that $f$ satisfies.

Now we specialize to the case of the plane curve $Z(y^2 - x^3)$.

Let $C = Z(y^2 - x^3)$. By problem 2 (below), we can resolve the singularity of $C$ to obtain the affine line, $\mathbb{A}^1 = C'$. Specifically, we have the blowup

$$
\varphi: C' \to C
$$

$$
t \mapsto (t^2, t^3)
$$

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Let $A = k[x, y]/(y^2 - x^3)$ and $A' = k[t]$ be as above, so that this corresponds to a map of rings given by

$$
\varphi^* : A \hookrightarrow A'
\begin{align*}
x & \mapsto t^2 \\
y & \mapsto t^3
\end{align*}
$$

The image of $A$ inside of $A'$ consists of all polynomials with no linear term. Hence, $A'$ is generated as an $A$ module by the finite set $\{1, t\}$. This shows that $A'$ is isomorphic to the integral closure of $A$ inside of $K$. The isomorphism sends $t$ to $yx^{-1}$, so we can describe the integral closure of $A$ in $K$ as follows:

$$
\overline{A}^K = \left\{ \frac{f(x, y)}{1} + ay \left\lfloor a \in k \right. \right\} \subset K
$$

noticing that $A \subset \overline{A}^K$ is the collection of all elements with $a = 0$. Hence, let $\alpha \in \overline{A}^K$ be any element with $a \neq 0$.

As a specific example, take $\alpha = yx^{-1} \in K$. First, if $\alpha$ were in $A$, then there would exist some $f \in k[x, y]$ such that $y - fx \in (y^2 - x^3)$. This is impossible because no element in $(y^2 - x^3)$ has a linear term in $y$. Next, notice that $\alpha$ satisfies the polynomial $t^2 - x \in A[t]$. Indeed, we have:

$$
\left( \frac{y}{x} \right)^2 - x = \frac{y^2 - x^3}{x^2} = 0 \in K
$$

So $\alpha$ is integral over $A$, but $\alpha \not\in A$.

\[\square\]

2. Use the blow-up construction to resolve the singularity of $Z(y^2 - x^3)$. You should get a map of algebraic varieties $B \rightarrow Z(y^2 - x^3)$. Show that this blow-up is smooth.

\textbf{Proof.} Let $Y = Z(y^2 - x^3) \subset \mathbb{A}^2$, and let $[t : u]$ be homogenous coordinates of $\mathbb{P}^1$. If $\varphi : X \rightarrow \mathbb{A}^2$ is the blow-up of $\mathbb{A}^2$ at the origin, then the blow-up of $Y$ at the origin is by definition $B := \varphi^{-1}(Y - O) \subset X$. Let’s compute $\varphi^{-1}(Y - O)$.

First, suppose $t \neq 0$. Then, since $t$ is a homogeneous coordinate, we may assume $t = 1$ and treat $u$ as an affine coordinate. Then we have:

$$
\varphi^{-1}(Y) \cap \{t \neq 0\} = \{(x, y) \times [1, u] \mid xu = y, y^2 = x^3\}
$$

Solving these two equations gives the solutions $(0, 0) \times [1, u]$, which is in the fiber over $O$, and $(u^{-2}, u^{-3}) \times [1 : u]$, $u \neq 0$, which is in $B$.

Next, suppose $u \neq 0$ so we assume $u = 1$. Then we have

$$
\varphi^{-1}(Y) \cap \{u \neq 0\} = \{(x, y) \times [t, 1] \mid x = yt, y^2 = x^3\}
$$

Solving these two equations gives the solutions $(0, 0) \times [t, 1]$, which is in the fiber over $O$, and $(t^2, t^3) \times [t : 1]$, which is in $B$.

Any element of the form $(u^{-2}, u^{-3}) \times [1 : u]$, $u \neq 0$ is also of the form $(t^2, t^3) \times [t : 1]$, so that $B = \{(t^2, t^3) \times [t : 1] \mid t \in \mathbb{A}^1\} \subset X$. It follows that $B \cap E = \{(0, 0) \times [0 : 1]\}$ where $E$ is the exceptional curve, i.e., the fiber over $O$.

To show that $B$ is smooth, we will show that $B \cong \mathbb{A}^1$. $B$ is isomorphic to the affine variety $X = Z(x^3 - z, x^2 - y) \subset \mathbb{A}^3$ by mapping $(t^2, t^3) \times [t : 1]$ to the point $(t, t^2, t^3)$, and $X$ is isomorphic to $\mathbb{A}^1$ because $k[X]$ is isomorphic to $k[s]$ by mapping $x$ to $s$. Affine varieties are isomorphic if and only if there coordinate algebras are isomorphic, so $B \cong X \cong \mathbb{A}^1$. \[\square\]
3. Embed $Gr(2, 5)$ in a projective space. Show that $Gr(2, 5)$ is a union of open subvarieties $A_i$ each of which is isomorphic to $\mathbb{A}^n$. Find the coordinate algebra of the intersection $A_i \cap A_j$, $i \neq j$.

**Proof.** We use Plücker coordinates. It is no more difficult to work in the more general case of 2-planes in and $n$ dimensional vector space, so we will. Any 2-plane $U$ in $k^n$ is spanned by 2 linearly independent vectors $u$ and $v$. Expanding $v$ and $u$ in the standard basis of $k^n$ gives a $n \times 2$ matrix.

$$
\begin{pmatrix}
    u_1 & v_1 \\
    u_2 & v_2 \\
    u_3 & v_3 \\
    \vdots & \vdots \\
    u_n & v_n 
\end{pmatrix}
$$

We map $U$ into $\mathbb{P}^{n \choose 2}$ with homogeneous coordinates $X_{ij}$ where $1 \leq i < j \leq n$ by $U \mapsto (u_i v_j - u_j v_i)$. Since the spanning vectors are linearly independent, one of the homogeneous coordinates must be nonzero. Also, choosing different spanning vectors amounts to multiplying the above matrix on the right by an invertible $2 \times 2$ matrix, so that the resulting coordinates are multiplied by the determinant of the invertible matrix, which leaves the corresponding element of $\mathbb{P}^{n \choose 2}$ unchanged. Therefore, this is a well-defined map.

To see that the map is an embedding, suppose that two 2-planes $U$ and $U'$ define the same homogeneous coordinates. Let $A$ and $B$ be any $n \times 2$ matrices which define $U$ and $U'$ respectively. By reordering, we may assume that the first two rows in each matrix have nonzero determinant, and so multiplying $A$ and $B$ by appropriate $2 \times 2$ matrices, we can assume that the first two rows of each form the $2 \times 2$ identity matrix. It then follows that $A = B$ by equating determinants of the $2 \times 2$ minors involving the first two rows of $A$ and $B$. Thus $U = U'$ and the map is injective.

Next, we wish to see that under this embedding, $Gr(2, n)$ is equal to the solutions of some set of homogeneous polynomial equations. In fact, we will show that $Gr(2, n)$ is the projective variety defined by the vanishing of the $n \choose 2$ homogeneous polynomial equations of degree 2 given by $f_{ijkl} = X_{ij} X_{kl} - X_{ik} X_{jl} + X_{il} X_{jk}$, $1 \leq i < j < k < l \leq n$.

It is a routine and unenlightening computation to show that if $X_{ij} = u_i v_j - u_j v_i$, then each $f_{ijkl}$ vanishes. This shows that $Gr(2, 5)$ injects into the zero set of the $f_{ijkl}$.

Next, to see that this injection is onto, suppose $c_{st} \neq 0$, $1 \leq s < t \leq n$, satisfy each $f_{ijkl}$. Since some $c_{st} \neq 0$, by relabeling we may assume $c_{12} \neq 0$. Then we may assume $c_{12} = 1$ and multiplying by the appropriate $2 \times 2$ invertible matrix, that $u_1 = v_2 = 1$ and $u_2 = v_1 = 0$. Then for $m = 3, \ldots, n$, let $u_m = -c_{2m}$ and $v_m = c_{1m}$. With these definitions, we must show that $u_s v_t - u_t v_s = c_{st}$. If either $\{s, t\} \cap \{1, 2\} \neq \emptyset$, then the result holds by definition of $u_1, u_2, v_1,$ and $v_2$. Else, using the fact that the $c_{st}$ satisfy each $f_{ijkl}$ we have:

$$
u_s v_t - u_t v_s = c_{2t}c_{1s} - c_{2s}c_{1t} = c_{12} c_{st} = c_{st}$$

because we have assumed $c_{12} = 1$. Thus the map of $Gr(2, 5)$ is onto the zero set of the $f_{ijkl}$.

Now, consider the open set $U_{ij} \subset \mathbb{P}^{n \choose 2}$ where $X_{ij} \neq 0$. Then, let $A_{ij} = U_{ij} \cap Gr(2, n)$. Notice that $A_{ij}$ is an open subvariety of $Gr(2, n)$, and that the $A_{ij}$ cover $Gr(2, n)$ because the $U_{ij}$ cover $\mathbb{P}^{n \choose 2}$. Next, since $X_{ij} \neq 0$, we can normalize so that $X_{ij} = 1$. Each 2-plane
$U$ in $A_{ij}$ has a unique representing matrix whose minor defined by rows $i$ and $j$ is the $2 \times 2$ identity matrix. The coordinate ring of $A_{ij}$ is thus:

$$k[A_{ij}] \cong k[y_{kl} \mid 1 \leq k \leq n, 1 \leq l \leq 2]/(y_{i1} - 1, y_{i2}, y_{j1}, y_{j2} - 1)$$

which is isomorphic to a polynomial ring in $2(n-2)$ variables. Thus $A_{ij} \cong \mathbb{A}^{2(n-2)}$ as affine varieties.

Finally, let’s consider the intersection $A_{ij} \cap A_{kl}$, where $A_{ij} \neq A_{kl}$. This is an open set in $A_{ij} \cong \mathbb{A}^{2(n-2)}$ defined as the complement of the hypersurface $y_{k1}y_{l2} - y_{i1}y_{k2} = 0$, and is therefore affine itself. The coordinate ring $k[A_{ij} \cap A_{kl}]$ is:

$$k[y_{mn}, t \mid 1 \leq m \leq n, 1 \leq n \leq 2]/(y_{i1} - 1, y_{i2}, y_{j1}, y_{j2} - 1, t(y_{k1}y_{l2} - y_{i1}y_{k2}) - 1)$$

This is the simplest general form covering all such intersections $A_{ij} \cap A_{kl}$. In all specific cases however, the above expression simplifies to a polynomial ring in $2n - 3$ variables with one relation generating the ideal.

As an explicit example, let $n = 5$, and let’s consider $A_{12} \cap A_{13}$. The above expression simplifies to

$$k[[y_{31}, y_{41}, y_{51}, y_{32}, y_{42}, y_{52}], t]/(ty_{32} - 1)$$

So that $A_{12} \cap A_{13}$ is the complement of the hypersurface $y_{32} = 0$ in $\mathbb{A}^6$.

4. Take as the definition of the $p$-adic integers $\mathbb{Z}_p$ the completion of $\mathbb{Z}$ with respect to the absolute value $\|\cdot\|_p$. Show that $\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n$.

**Proof.** We begin by making precise what structure the isomorphism sign in the problem respects.

Given a ring $R$ and a metric $d : R \times R \to \mathbb{R}_{\geq 0}$, we can complete $(R, d)$ to $(\hat{R}, \hat{d})$ by considering equivalence classes of Cauchy sequences. That is, if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $R$, we define the equivalence relation

$$\{x_n\} \sim \{y_n\} \iff \lim_{n \to \infty} d(x_n, y_n) = 0$$

Then if $[x_n]$ and $[y_n]$ are equivalence classes of Cauchy sequences, define

$$\hat{d}([x_n], [y_n]) := \lim_{n \to \infty} d(x_n, y_n)$$

This is all well-defined (the equivalence relation and $\hat{d}$) by the standard properties of the metric $d$. As a metric space, $R$ embeds as a dense subspace into $\hat{R}$ as equivalence classes represented by constant sequences, and $\hat{d}$ restricts to $d$ on $R$.

Place a ring structure on $\hat{R}$ by componentwise multiplication and addition of Cauchy sequences, noting that the sum and product of Cauchy sequences are Cauchy sequences. The operations are well-defined by the triangle inequality and the fact that Cauchy sequences are bounded.

If the original ring is a topological ring with respect to the metric $d$, i.e., if addition and multiplication are continuous maps from $R \times R$ to $R$, then $\hat{R}$ is also a topological ring with respect to $\hat{d}$. The proof of this statement is too analytical for this class. In the case of this problem, we begin with $R = \mathbb{Z}$ and $d(x, y) = \|x - y\|_p$.

Considering the other side of the isomorphism sign in the problem, let $R_\ell$ be a collection of topological rings. Then if we give the projective limit the initial topology with respect to the canonical morphisms, and if we define addition and multiplication componentwise, then
the projective limit is also a topological ring. In the case of this problem, we give the finite rings \( \mathbb{Z}/p^n \) the discrete topology.

I claim then that \( \mathbb{Z}_p \simeq \lim_{n \to \infty} \mathbb{Z}/p^n \) in the category of topological rings.

To see this, define a map \( \varphi : \lim_{n \to \infty} \mathbb{Z}/p^n \to \mathbb{Z}_p \) by sending \( (\overline{a_n}) \) to \([a_n]\). Here, \( \overline{a_n} \in \mathbb{Z}/p^n \) and for all \( n > m \), \( a_n - a_m \in (p^m) \). This just ensures that \( (\overline{a_n}) \) is an element of the projective limit. Also, \([a_n]\) is the equivalence class of the Cauchy sequence \( \{a_n\} \). Notice \([a_n]\) is indeed Cauchy by the property above, ie, \( d(a_n, a_m) \leq p^{-m} \) for all \( n > m \).

Since we’ve defined addition and multiplication componentwise in both the completion and the limit, \( \varphi \) is a ring map.

Now, suppose that \( [a_n] \sim [0] \), the constant Cauchy sequence whose terms are 0. By definition, this means \( \lim d(a_n, 0) = 0 \). Thus, for any \( m > 0 \), there is some \( N > 0 \) such that \( a_n \in (p^m) \) for all \( n > N \). Thus \( \overline{a_n} = \overline{0} \) for \( n \) large enough, and then since \( a_n \equiv a_m \) mod \( (p^m) \) for all \( n > m \), it follows that \( \overline{a_n} = \overline{0} \) for all \( n \), and thus \( \varphi \) is injective.

Next, I claim that each equivalence class of Cauchy sequences has a representative of the form \( \{a_n\} \) where \( a_n \equiv a_m \) mod \( (p^m) \) for all \( n > m \). This will prove surjectivity of \( \varphi \).

To see this, notice that the Cauchy condition implies that any Cauchy sequence defines a residue class of \( \mathbb{Z}/p^n \) for all \( n \), and these classes are compatible as above. Indeed, let \( \{b_n\} \) be a Cauchy sequence, and define \( \{a_n\} \) as follows. Let \( \varepsilon = p^{-n} \). The Cauchy condition guarantees that for large enough \( k > l \), \( b_k - b_l \in (p^l) \), ie, that each member of the tail of the Cauchy sequence belongs to the same residue class of \( \mathbb{Z}/p^n \). Let \( a_n \) be any representative of this class. Notice then that for large enough \( k \) and for \( n > m \), \( a_n - b_k \in (p^l) \subset (p^m) \) so that \( a_n - a_m \in (p^m) \).

We now must show that \( \varphi \) is a homeomorphism, ie, that it is both continuous and open.

For continuity, fix \( \varepsilon > 0 \), and let \( B \) be a ball of radius \( \varepsilon \) centered at \( \varphi((\overline{a_n})) = [a_n] \in \mathbb{Z}_p \). We want to find an open \( U \subset \lim_{n \to \infty} \mathbb{Z}/p^n \) such that \( (\overline{a_n}) \in U \) and \( \varphi(U) \subset B \). The generators for the topology on \( \lim_{n \to \infty} \mathbb{Z}/p^n \) are sets of the form \( \pi_m^{-1}(V) \) where \( V \subset \mathbb{Z}/p^m \) for some \( m \). Here \( V \) is an arbitrary subset because \( \mathbb{Z}/p^m \) has the discrete topology.

Take \( m \) to be such that \( p^{-m} < \varepsilon \), and let \( V \) be the singleton \( \overline{a_m} \) in \( \mathbb{Z}/p^m \). Then \( U = \pi_m^{-1}(V) \) consists of all elements \( \overline{b_n} \in \lim_{n \to \infty} \mathbb{Z}/p^n \) such that \( \overline{b_n} = \overline{a_m} \). Then

\[
\overline{d}([a_n], [b_n]) = \lim_{n \to \infty} d(a_n, b_n) \leq d(a_n, b_m) \leq p^{-m} < \varepsilon
\]

The first inequality here follows from the fact that \( d(a_n, b_m) \) is a nonincreasing sequence. Indeed, if \( a_n - b_n \in (p^k) \), then \( a_{n+1} - b_{n+1} \in (p^k) \) also by the compatibility of the \( a \)'s and \( b \)'s. Thus, \( \varphi \) is continuous.

Next, we show that \( \varphi \) is an open map. As above, let \( \pi_m^{-1}(V) \) be open in \( \lim_{n \to \infty} \mathbb{Z}/p^n \). Then

\[
\varphi(\pi_m^{-1}(V)) = \{[a_n] \mid [\overline{a_n}] \in V\}
\]

Take any such \( [a_n] \). By the proof of surjectivity, we may assume that \( a_n - a_m \in (p^m) \) for all \( n > m \). Let \( B \) be the ball around \([a_n] \) of radius \( p^{-m} \), and take any \([b_n]\) inside of this ball. This means that

\[
\overline{d}([a_n], [b_n]) = \lim_{n \to \infty} d(a_n, b_n) \leq p^{-m}
\]

So for \( n \) large enough, we have \( a_n - b_n \in (p^m) \), but by our assumed form, this implies that \( a_m - b_m \in (p^m) \) so that \( \overline{b_m} = \overline{a_m} \) and thus \( [b_n] \in \varphi(\pi_m^{-1}(V)) \). This shows that \( \varphi \) is an open map, completing the proof.

\( \square \)