MATH 126 NOTES

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Notation

$\mathbb{R}$ will denote the set of all real numbers. If $A$ and $B$ are sets, we denote that $A$ is a subset of $B$ by $A \subset B$. $f: A \rightarrow B$ denotes a function $f$ with domain $A$ and codomain $B$. At times, for convenience, we write $f: \mathbb{R} \rightarrow \mathbb{R}$ even when the domain of $f$ is some subset of the real numbers. If $x \in A$, we denote by $f(x)$ the value in $B$ assigned to $x$ by the function $f$.

5. Inverse Functions

5.1. Inverse Functions. Suppose two players are to play a game with a given function $f: A \rightarrow \mathbb{R}$, where $A \subset \mathbb{R}$. The first player thinks of an $x$-value, and then speaks aloud the value of $f(x)$. The second player wins if she can state the first player’s chosen $x$-value. Can the second player guarantee victory? The answer to this question depends on the initial choice of the function $f$.

First, let us consider the game played with the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x$. Notice that whatever value is spoken by the first player, say $y$, the second player can correctly state that the first player’s chosen $x$-value is $y/2$. In other words, she can define another function $g(y) = y/2$ that undoes what the function $f$ did to $x$. Even more concretely, if the first player thought of the number 7, he would speak the number $f(7) = 2 \cdot 7 = 14$. The second player would then use her function and the spoken value to recover the number 7: $g(14) = 14/2 = 7$.

Let’s see what would happen if the game was played with the function $f: \mathbb{R} \rightarrow [-1, 1]$ defined by $f(x) = \cos x$. Suppose the first player speaks aloud the value 1. Notice that $1 = \cos(0) = \cos(\pm 2\pi) = \cos(\pm 4\pi) = \ldots$, so that the second player cannot determine the first player’s $x$-value. We see that with the function $f$, there is no function $g$ that undoes what $f$ did to $x$. The second player may have some idea of possible values of the original $x$-value, but she has no way of knowing exactly which one to guess.

Let’s give a rigorous definition of the property that a function $f: A \rightarrow B$ must possess so that the second player can guarantee victory.

**Definition 5.1.** A function $f: A \rightarrow B$ is **one-to-one** if

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

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Example 5.2. $f(x) = 2x$ is one-to-one because the condition $2x_1 = 2x_2$ implies that $x_1 = x_2$. However, the function $f(x) = \cos x$ is not one-to-one because although $\cos(0) = \cos(2\pi)$, we have the inequality $0 \neq 2\pi$.

Question 5.3. Is the function $f(x) = x^3$ one-to-one? What about the function $g(x) = x^2$?

How can we tell if a given function is one-to-one? There is a nice graphical method used to determine if a function is one-to-one, often referred to as the Horizontal Line Test. Notice that if a function $f$ is not one-to-one if there are distinct $x$-values $x_1$ and $x_2$ such that $c = f(x_1) = f(x_2)$. Graphically, this means that the horizontal line $y = c$ intersects the graph $y = f(x)$ in two places. This line of reasoning leads to the following theorem.

Theorem 5.4 (Horizontal Line Test). A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if and only if no horizontal line intersects its graph more than once.

Challenge 5.5. For which values of $a$ is the function $f(x) = ax + \cos x$ one-to-one.

In the two player game described at the beginning of this section, if the given function $f$ was one-to-one, the second player was able to define a special function $g$ that undid $f$. This special function has a name.

Definition 5.6. Given a one-to-one function $f : A \to B$, the inverse function $f^{-1} : B \to A$ is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Remark 5.7. The one-to-one condition is required here. Suppose $x_1$ and $x_2$ are two distinct values in the domain $A$ that are mapped to the same value $y$ in the range $B$, ie, $f(x_1) = f(x_2) = y$. Then it is not clear how to define $f^{-1}(y)$. If we start with a one-to-one function, we won’t have this ambiguity.

We can always restrict the domain of a given function to make it one-to-one.

Example 5.8 (Restricting the Domain). The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is one-to-one, because we only consider positive square roots. Similarly, the function $g : [0, \pi] \to [-1, 1]$ defined by $g(x) = \cos x$ is also one-to-one, as we only consider angles between 0 and $\pi$. 
Three Steps to Finding Inverse Functions

Step 1: Write \( y = f(x) \)
Step 2: Switch \( x \) and \( y \)
Step 3: Solve for \( y \)

**Exercise 5.9.** Let \( f(x) = 2x^2 - 8x, x \geq 2 \). Find \( f^{-1}(x) \) and graph \( f \) and \( f^{-1} \) on the same axes.

**Remark 5.10.** Since a point \((x, y)\) on the graph of \( f(x) \) corresponds to a point \((y, x)\) on the graph of \( f^{-1}(x) \), the graph of \( f^{-1} \) is the reflection of the graph of \( f \) across \( y = x \).

Intuitively, a function is continuous if its graph is "connected," ie, it can be drawn without having to lift your pencil from the paper. Also, we can understand differentiability as "smoothness" of the graph of a function, ie, the absence of any corners or cusps. Since reflecting a graph across the line \( y = x \) preserves "connectedness" and "smoothness," part of the following theorem is expected.

**Theorem 5.11.**

1. If \( f : A \to B \) is continuous, then \( f^{-1} : B \to A \) is continuous.
2. If \( f : A \to B \) is differentiable, then \( f^{-1} : B \to A \) is differentiable and

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

**Remark 5.12.** We now have two ways to find the derivative of an inverse function. We can first find a formula for the inverse using the method described above, and then differentiate, or we can use the theorem.

**Exercise 5.13.**

1. Let \( f(x) = \frac{1}{x - 1}, x > 1 \).
   (a) Show that \( f \) is one-to-one.
   (b) Calculate \( f^{-1} \) and find its domain and range.
   (c) Calculate \((f^{-1})'(2)\) directly and using the theorem.
   (d) Sketch the graphs of \( f \) and \( f^{-1} \) on the same axes.

2. If \( f(x) = \sqrt{x^3 + x^2 + x + 1} \), find \((f^{-1})'(2)\).

3. If \( f^{-1} \) is differentiable, \( G(x) = \frac{1}{f^{-1}(x)}, f(3) = 2 \), and \( f'(3) = \frac{1}{9} \), find \( G'(2) \).
5.6. **Inverse Trigonometric Functions.** The standard trigonometric functions are not one-to-one. To define the inverse trigonometric functions, we must restrict domains.

**Example 5.14.** The function \( f : \mathbb{R} \to [-1, 1] \) defined by \( f(x) = \cos x \) is not one-to-one, so \( f^{-1} \) does not exist. However, the function \( g : [0, \pi] \to [-1, 1] \) defined by \( g(x) = \cos x \) is one-to-one. Hence, the function \( g^{-1} : [-1, 1] \to [0, \pi] \) exists, and \( g^{-1}(x) \) is equal to “the angle in between 0 and \( \pi \) whose cosine is \( x \).”

<table>
<thead>
<tr>
<th>Inverse Trigonometric Functions</th>
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<tbody>
<tr>
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<tr>
<td>( \cos^{-1} x )</td>
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<td>( \sin^{-1} x )</td>
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<td>( \tan^{-1} x )</td>
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**Exercise 5.15.**

1. Find \( \arcsin(1) \).
2. Find \( \tan^{-1} \left( \tan \left( \frac{4\pi}{3} \right) \right) \).
3. Simplify \( \tan(\sin^{-1}(x)) \).
4. Find the derivative of \( y = \tan^{-1}(\cos \theta) \).
5. Let \( f(x) = \arcsin(e^x) \). Find \( f'(x) \), and the domains of \( f \) and \( f' \).
6. Evaluate \( \lim_{x \to 0^+} \tan^{-1}(\ln x) \).

**Challenge 5.16.** A lighthouse is on an island 3 km offshore, and its light makes 4 revolutions per minute. How fast is the light moving along the shore when it is 1 km from the point onshore nearest the lighthouse?

5.7. **Hyperbolic Functions.** Hyperbolic functions are very helpful in modeling natural phenomena. They are defined using the exponential function. They are simply shorthand notation for their more complicated definitions.
Definition 5.17.

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\end{align*}
\]

The other three hyperbolic functions are defined analogously to the trigonometric functions. The reason for the analogy with trigonometric functions will be seen shortly.

5.7.1. Graphs.

5.7.2. Identities.

1. \(\sinh(-x) = -\sinh(x)\)
2. \(\cosh(-x) = \cosh(x)\)
3. \(\cosh^2 x - \sinh^2 x = 1\)

More identities on pg. 290 of the text.

5.7.3. Derivatives.

Example 5.18. Using the definitions of \(\sinh x\), \(\cosh x\), and \(\tanh x\) find their derivatives.

More derivatives on pg. 291 of the text.

5.7.4. Inverses.

Exercise 5.19. Derive a formula for

\[
\frac{d}{dx} (\cosh^{-1} x)
\]

Challenge 5.20. Derive a formula for the inverse of \(f(x) = \cosh x\).

5.8. Indeterminate Forms and L’Hospital’s Rule. Sometimes in evaluating limits, there are two competing “forces” at work, and it is not immediately clear which “force” wins, or what value the two “forces” agree on. Such limits are called indeterminate forms. L’Hospital’s Rule is a helpful method for evaluating indeterminate limits.

Theorem 5.21 (L’Hospital’s Rule). If \(\lim_{x \to a} \frac{f(x)}{g(x)}\) is of the form \(\frac{0}{0}\) or \(\frac{\infty}{\infty}\), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

Remark 5.22. The rule involves differentiating the numerator and denominator as separate functions. We don’t apply a quotient rule.
Example 5.23. Evaluate \( \lim_{x \to 0} \frac{\sin 5x}{\tan 3x} \).

When dealing with other indeterminate forms, try to write them in one of the forms

\[
\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}
\]

to apply L’Hospital’s Rule.

**Other Indeterminate Forms**

(1) \(0 \cdot \infty\): rewrite as \(\frac{0}{\frac{1}{\infty}} = \frac{0}{0}\) or \(\frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}\).

Example 5.24. Evaluate \( \lim_{x \to \infty} x (e^{5/x} - 1) \).

(2) \(\infty - \infty\): rewrite by factoring or using common denominators.

Example 5.25. Find \( \lim_{x \to \infty} (x \ln(x + 2) - x \ln x) \).

(3) \(0^0, \infty^0, \text{ and } 1^\infty\): rewrite using ln.

Example 5.26. Evaluate \( \lim_{x \to 0} (1 - 3x)^{1/x} \).

Exercise 5.27.

\[
(1) \lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \quad (3) \lim_{x \to 0} (\sin(x))^x \\
(2) \lim_{x \to \pi/4} (1 - \tan x) \sec 2x \quad (4) \lim_{x \to 1} \left( \frac{x}{x - 1} - \frac{1}{\ln(x)} \right)
\]
6. Techniques of Integration

Chapter 6 in Stewarts equips us with many different integration techniques. This will expand the types of functions for which we can find an antiderivative. Each technique is best suited for certain types of functions, so besides learning each technique, it will also be important to learn to recognize when to use which technique. We will try to discuss general rules for when to use the techniques, but there are many exceptions to these rules. The best way to develop an eye for how to integrate certain functions is to do many example problems.

6.1. Integration by Parts. First we recall the method of substitution. A good substitution can significantly simplify an integral.

Recall u-substitution: often useful for inside functions (a chain rule for antidifferentiation).

Example 6.1. Evaluate \( \int \frac{x^3}{\sqrt{1 - x^2}} \, dx \).

Integration by Parts Formula

\[
\int udv = uv - \int vdu
\]

Integration by Parts: often useful for products of functions (a product rule for antidifferentiation), and for functions involving ln and inverse trigonometric functions.

Remark 6.2. Some people find the acronym ”LIATE” a helpful mnemonic to decide which part of the integral should be set equal to u. Here the letters stand for logarithm, inverse trigonometric, algebraic (polynomials), trigonometric, and exponential, respectively.

Exercise 6.3.

1. \( \int \frac{\arcsin(\sqrt{x})}{\sqrt{x}} \, dx \)
2. \( \int x \sin^2(2x) \, dx \)
3. \( \int e^{2x} \cos(e^x) \, dx \)
4. \( \int x \tan^{-1} x \, dx \)
5. \( \int_1^2 (\ln(x))^2 \, dx \)
6. \( \int \frac{x \, dx}{\cos^2 x} \)

6.2. Trigonometric Integrals and Substitution.

Trigonometric Integration: often useful for integrating products of trigonometric functions using u-substitution and trigonometric identities.

Example 6.4. Evaluate \( \int \sin^3 7x \, dx \).
Evaluating integrals of the form $\int \cos^m x \sin^n x \, dx$.

(1) If $m$ is odd, let $u = \sin x$ and use the identity $\cos^2 x = 1 - \sin^2 x$.
(2) If $n$ is odd, let $u = \cos x$ and use the identity $\sin^2 x = 1 - \cos^2 x$.
(3) If both $n$ and $m$ are even, use the identities

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

**Example 6.5.** Evaluate $\int \tan^3(x) \sec^4(x) \, dx$.

Evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$.

(1) If $m$ is odd, let $u = \sec x$ and use the identity $\tan^2 x = \sec^2 x - 1$.
(2) If $n$ is even, let $u = \tan x$ and use the identity $\sec^2 x = 1 + \tan^2 x$.
(3) If $n$ is odd and $m$ is even, express everything in terms of $\sec x$ and use integration by parts on odd powers.

### Helpful Formulas

$$\int \tan x \, dx = \ln |\sec x| + C \quad \text{and} \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

**Exercise 6.6.**

\[
\begin{align*}
(1) & \quad \int \frac{\sin^4 x}{\cos^6 x} \, dx & (3) & \quad \int x \tan^2(x) \, dx \\
(2) & \quad \int \tan^3(x) \, dx & (4) & \quad \int \frac{(1 - \sin(x))^2}{\cos^2(x)} \, dx
\end{align*}
\]

Trigonometric Substitution: often useful for integrating functions that contain an expression of the form $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$.

**List of Trigonometric Substitutions**

(1) For $\sqrt{a^2 - x^2}$, use $x = a \sin \theta$.
(2) $\sqrt{a^2 + x^2}$, use $x = a \tan \theta$.
(3) For $\sqrt{x^2 - a^2}$, use $x = a \sec \theta$.

**Remark 6.7.** Sometimes both $u$-substitution and trigonometric substitution can be done to solve the same integral. Since $u$-substitutions are faster and easier, be careful not to waste time and effort by doing an unnecessary trigonometric substitution.
Exercise 6.8.

(1) \( \int \sqrt{1-x^2} \, dx \)  
(3) \( \int \frac{1}{x^2 \sqrt{1+x^2}} \, dx \)  
(5) \( \int \frac{x^3}{\sqrt{1+x^2}} \, dx \)

(2) \( \int \frac{t^3}{\sqrt{4-t^2}} \, dt \)  
(4) \( \int \frac{x}{\sqrt{3-x^4}} \, dx \)  
(6) \( \int \frac{dx}{x^4 \sqrt{x^2-5}} \)

6.3. Partial Fractions. Partial Fraction Decomposition allows us to rewrite complicated rational functions as sums of functions which are much easier to integrate.

Partial Fractions: often useful for integrating rational functions

\[ R(x) = \frac{P(x)}{Q(x)} \]

where \( P(x) \) and \( Q(x) \) are polynomials. If \( \text{deg}(P(x)) \geq \text{deg}(Q(x)) \), first apply polynomial long division. We will consider four cases determined by how \( Q(x) \) factors. The cases will be dealt with in increasing levels of complexity, but by the end, a clear pattern should emerge.

6.3.1. \( Q(x) \) is a product of distinct linear factors. If \( Q(x) = (a_1x+b_1) \cdots (a_nx+b_n) \), then write

\[ R(x) = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n} \]

and solve for the \( A_i \).

Example 6.9. \( \int \frac{2x+1}{x^2-7x+12} \, dx \)

6.3.2. \( Q(x) \) has repeated linear factors. Apply the previous method, using an extra term for each repeated factor.

Example 6.10. \( \int \frac{4}{x^2(x+2)} \, dx \)

6.3.3. \( Q(x) \) contains distinct irreducible quadratic factors. Use a general linear term \( Ax+B \) in the numerators.

Example 6.11. \( \int \frac{x-1}{(x+2)(x^2+1)} \, dx \)

6.3.4. \( Q(x) \) contains repeated irreducible quadratic factors. Apply the previous method, using an extra term for each repeated factor.

Remark 6.12. After finding the partial fraction decomposition of a rational function, integrating each separate term may involve a technique of its own (u-substitution, splitting fractions, completing the square).

(1) \[ \int \frac{2x^2 + x + 2}{x(x^2 + 1)} \, dx \]  
(2) \[ \int \frac{2}{x^4 + x^2} \, dx \]  
(3) \[ \int \frac{dx}{x^4 - 16} \, dx \]  
(4) \[ \int \frac{x + 1}{x^2 - 4x + 6} \, dx \]  
(5) \[ \int \frac{1 + x}{x + x^3} \, dx \]  
(6) \[ \int \frac{8x - 11}{x^2 - 3x + 2} \, dx \]

6.4. Approximate Integration. Some functions have no elementary antiderivative, so we cannot use the Fundamental Theorem of Calculus to evaluate their definite integrals. However, we may still be interested in the area under their graphs. Approximate integration gives us a method for finding definite integrals of these ”unintegrable” functions. Each of the following rules is derived from geometric considerations.

Midpoint Rule

\[
\int_a^b f(x) \, dx \approx \Delta x \left[ f(\bar{x}_1) + \ldots + f(\bar{x}_n) \right] \text{ where } \bar{x}_i = \frac{x_{i-1} + x_i}{2} \text{ and } \Delta x = \frac{b - a}{n}.
\]

Trapezoidal Rule

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n) \right] \text{ where } x_i = a + i\Delta x
\]

When approximating anything, it is often helpful to know how accurate the approximation is. This means we would like to find a range of values in which the actual value lies. This can be achieved by finding an upper bound on the error of the approximation, ie, the largest possible difference between the actual value of the integral and the approximation. The following formulas give us such a bound.

Error Bounds

\[
|E_T| \leq \frac{K(b - a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b - a)^3}{24n^2} \text{ where } |f''(x)| \leq K \text{ on } [a, b]
\]

Example 6.14. If the Trapezoidal Rule is used to approximate \[ \int_1^2 e^{x^2} \, dx \] how large should \( n \) be to guarantee that the error is at most \( 10^{-6} \)?

Example 6.15. If the Midpoint Rule is used to approximate \[ \int_1^3 \sqrt{1 + x^3} \, dx \] how large should \( n \) be to guarantee that the error is at most \( 0.0002 \)?
**Simpson’s Rule**

\[ \int_{a}^{b} f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right] \]

where \( n \) is even.

**Error Bound**

\[ |E_S| \leq \frac{K(b-a)^5}{180n^4} \text{ where } |f^{(4)}(x)| \leq K \text{ on } [a, b]. \]

**Challenge 6.16.** Without using the error bounds, verify that the midpoint and trapezoid approximations are exact for linear functions, and that Simpson’s rule is exact for cubic functions.

6.5. **Improper Integrals.** Improper integrals involve finding areas of unbounded regions of the plane, that is, regions that in some sense go on forever. If the part of the region that goes on forever gets small enough fast enough, the area still might be a finite number.

Improper integrals combine techniques of integration and finding limits.

6.5.1. **Type 1 - Infinite Intervals.**

\[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \]

\[ \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]

\[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{0} f(x) \, dx + \lim_{b \to \infty} \int_{0}^{b} f(x) \, dx \]

**Definition 6.17.** If the limit defining the integral is finite, the integral **converges**, else it **diverges**.

**Example 6.18** (Important!). For what values of \( p \) does the integral

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx \]

**converge?**
6.5.2. Type 2 - Discontinuous Integrands.

\[
\int_a^b f(x) \, dx = \lim_{s \to a^+} \int_s^b f(x) \, dx
\]

\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
\]

\[
\int_a^b f(x) \, dx = \lim_{t \to c^-} \int_a^t f(x) \, dx + \lim_{s \to c^+} \int_s^b f(x) \, dx
\]

Exercise 6.19.

(1) \(\int_2^\infty \frac{x^2 - 4x - 1}{(x^2 - 1)(x^2 + 1)} \, dx\)

(2) \(\int_0^\infty \frac{dx}{x^2 + 3x + 2}\)

(3) \(\int_1^5 \frac{y}{y - 1} \, dy\)

(4) \(\int_2^3 \frac{1}{\sqrt{3 - x}} \, dx\)

(5) \(\int_{-2}^{14} \frac{dx}{\sqrt{x + 2}} \, dx\)

(6) \(\int_0^{\pi/6} \frac{\cos x}{\sin^2 x} \, dx\)

Sometimes improper integrals are very complicated, or even impossible, to evaluate, so it is not practical to evaluate them to determine convergence or divergence. If we are only interested in whether an integral converges or not, and we are not interested in the actual finite value to which it converges, we can use the Comparison Theorem. The Comparison Theorem is based on two simple ideas: if \(a < b\) and \(b\) is finite, then \(a\) is also finite, and a number larger than infinity is also infinity.

**Theorem 6.20** (Comparison Theorem). Suppose \(f\) and \(g\) are continuous and \(f(x) \geq g(x) \geq 0\) for all \(x \geq a\). Then

i) If \(\int_a^\infty f(x) \, dx\) converges, then \(\int_a^\infty g(x) \, dx\) also converges.

ii) If \(\int_a^\infty g(x) \, dx\) diverges, then \(\int_a^\infty f(x) \, dx\) also diverges.

Exercise 6.21. Determine whether the following integrals are convergent or divergent.

(1) \(\int_0^\pi \frac{e^{-x^2}}{x} \, dx\)

(2) \(\int_1^\infty \frac{2 + \sin x}{\sqrt{1 + x^2} + x^4} \, dx\)

(3) \(\int_1^\infty \frac{\sin^2(x)}{x^3 + 1} \, dx\)

(4) \(\int_{-\infty}^\infty e^{-x^2} \, dx\)

(5) \(\int_0^1 \frac{\cos(2x) - \cos(3x)}{x^2} \, dx\)

(6) \(\int_0^\infty \frac{\arctan x}{x^2} \, dx\)
7. APPLICATIONS OF INTEGRATION

Chapter 7 in Stewarts explores different uses for integration. The bulk of the battle here is setting up an integral that represents a desired physical quantity such as area, volume, length, work, or force. The integrals themselves are much easier to evaluate than most we encountered in Chapter 6, but setting them up correctly requires practice. The main idea is the standard one of integration: split the desired physical quantity into small pieces, and integrate over the pieces. All throughout this chapter, drawing accurate pictures (or at least visualizing the picture) should prove to be very helpful.

7.1. Area Between Curves.

\[ A = \int_{a}^{b} [f(x) - g(x)] \, dx \]

Sometimes it is much easier to integrate in the \( y \)-direction (Draw a picture first).

Remark 7.1. If you ever find that your "upper" and "lower" function are the same, try integrating in the other variable.

Exercise 7.2. (1) Let \( R \) be the region of the plane bounded by the line \( y = x - 1 \) and the curve \( x = 1 + y^2 \). Sketch the region and find its area.

(2) Let \( R \) be the region in the first quadrant bounded by the curves \( y = x^3 \) and \( y = 2x - x^2 \). Calculate the area of \( R \).

(3) Set up, but do not evaluate, the integral to find the area of the "sunglasses" formed by the graphs of \( y = 2x - x^3 \) and \( y = x^3 \).

7.2. Volumes. Here we introduce a 3-step method for setting up integrals that represent desired physical quantities. The method is general enough to apply to volumes, work, and force. It is only one of many ways to think about such problems. Through practice, find a method that works for you.

Volumes by Cross Sections

\[ V = \int_{a}^{b} dV(x) = \int_{a}^{b} A(x) \, dx \]

Step 1: Draw the solid in a coordinate system. Include a representative cross section, and label the variable that determines the cross section.

Step 2: Find the area of the cross section as a function of the determining variable.

Step 3: Integrate the area function along all possible cross sections.

Example 7.3. Let \( R \) be the bounded region of the plane enclosed by the \( x \)-axis and the graph of \( y = 3x - x^2 - 2 \). Find an integral that gives the volume of the solid obtained by rotating \( R \) about the \( x \)-axis.
Exercise 7.4.

(1) Consider the region \( R \) bounded by the line \( y = 3x \) and the parabola \( y = 2 + x^2 \).
   i) Find the volume of the solid obtained by rotating the region \( R \) about the \( y \)-axis.
   ii) Set up but do not evaluate an integral giving the volume of the solid obtained
       by rotating the region \( R \) about the line \( y = 1 \).

(2) The base of a solid \( S \) is a disk of radius one. Parallel cross sections perpendicular to
    the base are equilateral triangles. Set up an integral for the volume of the solid, but
    do not evaluate it.

7.3. Volumes by Cylindrical Shells. Some solids of revolution have cross sections whose
inner and outer radii are represented by the same function. We saw before in finding areas
between curves that this represents a problem. In this case, we can use cylindrical shells
instead of cross sections to find volume.

Volumes by Cylindrical Shells
\[
V = \int_a^b dV(x) = \int_a^b 2\pi r(x)h(x) \, dx
\]
Step 1: Draw the solid in a coordinate system. Include a representative cylindrical shell,
and label the variable that determines the shell.
Step 2: Find the volume of the shell as a function of the determining variable.
Step 3: Integrate the volume function along all possible shells.

Example 7.5. Let \( R \) be the bounded region of the plane enclosed by the \( x \)-axis and the
graph of \( y = 3x - x^2 - 2 \). Find an integral that gives the volume of the solid obtained
by rotating \( R \) about the line \( x = -3 \).

Exercise 7.6.

(1) A region enclosed by \( y = \sin x \) and \( y = 0 \) for \( 0 \leq x \leq \pi \) is rotated about \( x = -\pi/2 \).
   Find the volume of the resulting solid.

(2) Let \( \mathcal{R} \) be the region of the plane bounded by the line \( y = x - 1 \) and the curve
   \( x = 1 + y^2 \). Write an explicit integral for the volume \( V \) of the solid obtained by
   rotating \( \mathcal{R} \) about the \( y \)-axis.

Remark 7.7. Although some volumes are more easily determined by one method or the
other, in some cases neither method holds a distinct advantage. In these cases, you are free
to choose whichever method you prefer.

7.4. Arc Length. Setting up an integral representing the length of some function is fairly
straightforward. Here we will find that the integration can be tricky.
The arc length formula gives the length of a function between defined starting and stopping points. The argument of the arc length function is the stopping point, so the arc length function is the length of the given function from a defined starting point to a variable stopping point.

**Challenge 7.9.** Use the arc length formula to find the length of the perimeter of the ellipse defined by the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

### 7.5. Applications to Physics and Engineering

Here the 3-step method used to find volumes returns to find work and force. When applied to volumes, the second step of the method required geometric formulas to determine areas of shapes. In the cases of work and force, the second step will require formulas from physics.

#### 7.5.1. Work

Formulas necessary for calculating work:

1. \( W = Fd \)
2. \( F = \rho V g \) (liquid)
3. \( F = \mu l \) (rope)

where \( W \) = work, \( F \) = force, \( d \) = distance, \( \rho \) = density, \( V \) = volume, \( g \) = gravity, \( \mu \) = linear density, and \( l \) = length.

**Calculating Work**

Step 1: Draw a picture and define a coordinate system. Include a representative cross section, and label the variable that determines the cross section.

Step 2: Find the work done on the cross section as a function of the determining variable.

Step 3: Integrate the work function along all possible cross sections.
Example 7.10. A tank has the shape obtained by rotating about the $y$-axis the region bounded by $y = x^2, x = 0, y = 1,$ and $y = 9,$ where $x$ and $y$ are measured in meters. If the tank is filled up to the level $y = 7$ with a liquid of density $\rho,$ find an integral that expresses the work done in pumping all of the liquid out of the top of the tank.

Exercise 7.11.

(1) A tank has a (truncated) conical shape obtained by rotating $y = 5(x - 3), 0 \leq y \leq 5$ about the $y$-axis, where $x$ and $y$ are measured in meters. Water is filled in the tank to a height of $y = 4.$ Set up the integral that expresses the work needed to pump all the water to the top of the tank.

(2) A well is 100 ft deep, and a water bucket is hauled from the bottom to the top, using a rope which weighs 0.1 pounds per foot. The water bucket weighs 40 pounds. How much work is needed to bring the bucket and the rope to the top of the well?

(3) A monkey is 10 ft off the ground in a tree with a long rope. She spots a bunch of bananas directly below on the ground and manages to "lasso" the bananas with the rope. If the bunch of bananas weighs 15 lbs, and the rope weighs 2 lbs per foot, how much work is done by the monkey in pulling up the bunch of bananas?

7.5.2. Hydrostatic Force. Formulas necessary for calculating hydrostatic force:

(1) $F = PA$

(2) $P = \rho gd$ (for m, kg)

(3) $P = \delta d$ (for ft, lb)

where $P =$ pressure, $A =$ area, $d =$ depth, and $\delta =$ weight density.

Calculating Hydrostatic Force

Step 1: Draw a picture and define a coordinate system. Include a representative cross section, and label the variable that determines the cross section.

Step 2: Find the hydrostatic force on the cross section as a function of the determining variable.

Step 3: Integrate the force function along all possible cross sections.

Example 7.12. A plate in the shape of a symmetric trapezoid three meters wide at the bottom, five meters wide at the top, and two meters high is submerged vertically with its top at the surface of a liquid of density $\rho.$ Find an integral that gives the hydrostatic force on one side of the plate.

Exercise 7.13. (1) A plate in the shape of an equilateral triangle with side length 2 feet is suspended vertically in a liquid of weight density $\delta.$ The side closest to the surface is parallel to the surface and 2 feet below it. Find an integral which expresses the hydrostatic force of the liquid on the side of the plate.
(2) The end of an open tank containing water is vertical and is a semicircle with diameter 16 meters at the top. The surface of the water is 4 meters from the top. Set up an integral for the hydrostatic force exerted against the end of the tank.

8. Differential Equations

Definition 8.1.

(1) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

Example: \( \frac{dy}{dx} = xye^{x^2} \)

(2) A separable equation can be written in the form

\[ \frac{dy}{dx} = f(x)g(y) \]

Example: \( \frac{dy}{dx} = xe^{x^2}y \)

To solve a separable equation, separate variables and integrate. If an initial condition is given, use it to solve for the integration constant. (Suppose in our example \( y(0) = 1 \))

Exercise 8.2.

(1) The time rate of change of a population of bacteria grown in a controlled laboratory environment is modelled by the differential equation

\[ \frac{dy}{dt} = \frac{1}{2}t(170 - y) \]

where \( y \) is the population of the bacteria and \( t \) is the elapsed time in hours. If the initial population of bacteria is \( y = 10 \), find the population at time \( t \), and the equilibrium state, that is, the value \( \lim_{t \to \infty} y(t) \)

(2) \( \frac{dy}{dx} = 3x^2\sqrt{1 - y^2} \) and \( y(0) = \frac{1}{2} \)

(3) \( \frac{dy}{dx} = \frac{1 + x}{xy}, x > 0, y(1) = -4 \)
9. Series

Chapter 8 deals with infinite series, that is, the sum of infinitely many terms. To even make sense of this, we will ask similar questions of convergence and divergence that we asked when dealing with improper integrals. We will learn many different techniques to help determine convergence, and as when we learned integrating techniques, it will be important to recognize when to use which technique. We will also find that most functions can be represented by an infinite series, and this representation will allow us to integrate functions that cannot be integrated with techniques we’ve learned so far.

9.1. Sequences. Although the main topic of this chapter is series, we begin with sequences. A series is a sum of infinitely many terms, while a sequence is simply an infinite list of numbers. The similarities often cause confusion, and it is important to be careful when distinguishing between convergence of a sequence, and convergence of a series.

**Definition 9.1.** A sequence is a list of numbers written in a certain order: \{1, 2, 3, 4, \ldots\} or \{2, 4, 6, 8, \ldots\}.

**Describing a Sequence**

1. Using a formula: \(a_n = \frac{n}{n + 1}\); \(a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, a_3 = \frac{3}{4}, \ldots \Rightarrow \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}\)
2. Recursion: \(a_1 = 1, a_n = 3a_{n-1}\); \(a_1 = 1, a_2 = 3, a_3 = 9, \ldots \Rightarrow \{1, 3, 9, \ldots \}\)
3. Description: \(a_n = n^{th}\) digit in the decimal expansion of \(\pi \Rightarrow \{3, 1, 4, 1, 5, 9, \ldots \}\)

**Limit of a Sequence**

Consider the sequence \(a_n = (-1)^n \frac{1}{n} \Rightarrow \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots \}\)

**Definition 9.2** (Intuitive). A sequence \(\{a_n\}\) has the limit \(L\) if for all ”large enough” \(n\), \(a_n\) is ”very close” to \(L\). If a sequence has a limit, it converges; else it diverges.

**Finding the Limit of a Sequence**

1. If the sequence is defined by a formula \(a_n = f(n)\), then
   \[\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)\]
   so the limit of the sequence is the limit of the defining function.

**Example 9.3.** Find the limit of the sequence

\[a_n = \left( \frac{4n - 3}{n} - \frac{3n}{n + 1} \right)^{n^2 + n}\]
(2) The Squeeze Theorem: If \( a_n \leq b_n \leq c_n \) for all \( n \), and
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L
\]
then
\[
\lim_{n \to \infty} b_n = L
\]

**Example 9.4.** Find the limit of the sequence \( b_n = (3 + \sin(n))^{1/n} \)

(3) Monotonic Sequence Theorem: If a sequence is bounded (ie, there is some number \( M \) such that \(-M \leq a_n \leq M\) for all \( n \)) and monotonic (ie, always increasing or always decreasing) then
\[
\lim_{n \to \infty} a_n
\]
exists.

**Remark 9.5.** Using the Monotonic Sequence Theorem to find the limit of a sequence involves a mathematical tool called induction. There are examples worked out in the text.

**Exercise 9.6.** Determine whether the following sequences converge or diverge. If the sequence converges, find its limit.

1. \( c_n = (-1)^n(1 - (1/n)) \)
2. \( \frac{n + (-1)^n \sqrt{n}}{n + \sqrt{n}} \)
3. \( a_n = \frac{(3 \cos(n))^n}{5^n + 2^n} \)
4. \( a_n = (-1)^{n+1} \frac{4n^2 + 1}{5n - n^2} \)
5. \( a_n = \frac{(\ln n)^2}{n} \)

### 9.2. Series
Any sequence defines a series simply by adding all of the terms. A series in turn defines a sequence: its sequence of partial sums, where the nth partial sum is the finite sum of the first \( n \) terms. Convergence of the series is then defined by convergence of its sequence of partial sums.

**Definition 9.7.**

1. An infinite series is obtained by summing a sequence:
\[
a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n
\]
(2) The \( n^{th} \) partial sum is the finite sum

\[
s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i
\]

(3) The partial sums of a series form a sequence: \( s_1, s_2, s_3, \ldots \). If \( \{s_n\} \) converges to \( s \), we say

\[
\sum_{n=1}^{\infty} a_n
\]
converges, and write

\[
\sum_{n=1}^{\infty} a_n = s
\]

If \( \{s_n\} \) diverges, then

\[
\sum_{n=1}^{\infty} a_n
\]
diverges.

**Telescoping Series**

Use partial fractions for series with factorable denominators.

**Example 9.8.**

(1) Determine if

\[
\sum_{n=3}^{\infty} \frac{1}{n(n-1)}
\]
converges or diverges. If it converges find the sum.

(2) Determine if

\[
\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}
\]
converges or diverges. If it converges find the sum.

**Geometric Series**

**Definition 9.9.** A geometric series has the form

\[
a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1}
\]
a is called the initial term and \( r \) is called the common ratio.
Theorem 9.10. The geometric series
\[ \sum_{n=1}^{\infty} ar^{n-1} \]
is convergent if \( |r| < 1 \), and
\[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \]
If \( |r| \geq 1 \), the series diverges.

Remark 9.11. This theorem makes dealing with geometric series very straightforward. All of the desired information is encoded in the initial term and the common ratio.

Example 9.12.
(1) Determine if
\[ \sum_{n=1}^{\infty} \left( \frac{3}{5} \right)^{n+1} \left( \frac{7}{5} \right)^{n} \]
converges or diverges. If it converges find the sum.
(2) Determine if
\[ \sum_{n=0}^{\infty} \frac{5^n - 2}{7^n} \]
converges or diverges. If it converges find the sum.

Theorem 9.13 (Test for Divergence). If \( \lim_{n \to \infty} a_n \) doesn’t exist, or if \( \lim_{n \to \infty} a_n \neq 0 \), then
\[ \sum_{n=1}^{\infty} a_n \]
diverges.

It makes sense that a series can only converge if its terms approach zero. This is because any other number added to itself infinitely many times is positive or negative infinity.

(1) Determine if
\[ \sum_{n=1}^{\infty} \frac{\sin(1/n)}{\tan(1/n)} \]
converges or diverges. If it converges find the sum.
(2) Determine if
\[ \sum_{n=1}^{\infty} \frac{\ln \frac{n}{3n + 1}}{21} \]
converges or diverges. If it converges find the sum.

Remark 9.15 (Warning!). If \( \lim_{n \to \infty} a_n = 0 \), then

\[
\sum_{n=1}^{\infty} a_n
\]

may or may not converge. The harmonic series

\[
\sum_{n=1}^{\infty} \frac{1}{n}
\]

is an example of a divergent series whose terms converge to 0.

9.3. The Integral and Comparison Tests. The previous section provided three methods of determining convergence (Telescoping Series, Geometric Series, and the Test for Divergence). Here we learn four more. The integral test says that the convergence of certain series are determined by their integrals.

9.3.1. Integral Test. Suppose \( f(x) \) is positive, continuous, and decreasing on \([1, \infty)\), and let \( a_n = f(n) \). Since

\[
\sum_{n=1}^{\infty} a_n \approx \int_{1}^{\infty} f(x) \, dx
\]

we can integrate to determine if

\[
\sum_{n=1}^{\infty} a_n
\]

converges or diverges.

Theorem 9.16 (Integral Test). If \( f \) is positive, continuous, and decreasing on \([1, \infty)\), and \( a_n = f(n) \) then

\[
\sum_{n=1}^{\infty} a_n
\]

converges if and only if \( \int_{1}^{\infty} f(x) \, dx \) converges.

Example 9.17 (\( p \)-series). For what values of \( p \) is the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) convergent?

Remark 9.18. This is a very important example, often referred to as the "\( p \)-series test." It provides us with a benchmark to use in any sort of comparison we might employ. We will refer back to this example many times.

Exercise 9.19.
(1) Determine if
\[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n)^2} \]
converges or diverges.

(2) Determine if
\[ \sum_{n=0}^{\infty} \frac{1}{\sqrt{n} e^{\sqrt{n}}} \]
converges or diverges.

9.3.2. Comparison Test. We’ve already seen the Comparison Theorem for integrals. Here is the version for series. The intuition remains the same.

**Theorem 9.20** (Comparison Test). Suppose \( \sum a_n \) and \( \sum b_n \) are series with positive terms, and \( a_n \leq b_n \) for all \( n \).

i) If \( \sum b_n \) converges, then \( \sum a_n \) converges.

ii) If \( \sum a_n \) diverges, then \( \sum b_n \) diverges.

Example 9.21.

(1) Determine if \( \sum_{n=1}^{\infty} \frac{\sqrt{n} + 3 \sqrt{n^3}}{(2n)^2} \) converges or diverges.

(2) Determine if \( \sum_{n=1}^{\infty} \frac{1}{n + \ln n} \) converges or diverges.

9.3.3. Limit Comparison Test. The Comparison Test has the disadvantage of requiring carefully constructed inequalities. The Limit Comparison Test is a bit less discerning in which series can be compared, and for this reason is often easier to use.

**Theorem 9.22** (Limit Comparison Test). Suppose \( \sum a_n \) and \( \sum b_n \) are series with positive terms. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) where \( 0 < c < \infty \), then either both series converge or both diverge.

Example 9.23.

(1) Determine if \( \sum_{n=1}^{\infty} \frac{1}{e^n - n^e} \) converges or diverges.

(2) Determine if \( \sum_{n=1}^{\infty} \frac{n^2 + 2}{(n + 1)^4} \) converges or diverges.
9.4. Other Convergence Tests. We now have seven different methods for determining convergence of a series. Here we learn four more tests before finally examining some applications of infinite series.

9.4.1. Alternating Series Test.

**Theorem 9.24** (Alternating Series Test). If the alternating series \( \sum_{n=1}^{\infty} (-1)^n b_n \) satisfies:

i) \( b_{n+1} \leq b_n \)

ii) \( \lim_{n \to \infty} b_n = 0 \)

then the series is convergent.

**Example 9.25.**

(1) Determine if \( \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n} \) converges or diverges.

(2) Determine if \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + 3} \) converges or diverges.

**Error Bound for Alternating Series**

This theorem will become more useful in the following sections. We just state it here with no example.

**Theorem 9.26.** If \( s = \sum_{n=1}^{\infty} (-1)^n b_n \), then \( |R_n| = |s - s_n| \leq |b_{n+1}| \)

9.4.2. Absolute Convergence.

**Definition 9.27.** A series \( \sum a_n \) is absolutely convergent if \( \sum |a_n| \) is convergent. A series \( \sum a_n \) is conditionally convergent if \( \sum a_n \) converges but \( \sum |a_n| \) diverges.

**Theorem 9.28.** Absolutely convergent series are convergent.

**Example 9.29.**

(1) Determine if \( \sum_{n=1}^{\infty} \frac{\sin(100n)}{(1.01)^n} \) is conditionally convergent, absolutely convergent, or divergent.

(2) Determine if \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n} + 9} \) is conditionally convergent, absolutely convergent, or divergent.
9.4.3. **Ratio Test.** Use to determine absolute convergence of series with powers and/or factorials.

**Theorem 9.30** (Ratio Test). Let \( L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \).

i) If \( L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

ii) If \( L > 1 \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.

iii) If \( L = 1 \), then the ratio test is inconclusive.

**Example 9.31.**

(1) Determine if \( \sum_{n=1}^{\infty} \frac{n}{2^n} \) is conditionally convergent, absolutely convergent, or divergent.

(2) Determine if \( \sum_{n=2}^{\infty} (-1)^n \frac{n2^n}{(n-1)!} \) is conditionally convergent, absolutely convergent, or divergent.

9.4.4. **Root Test.** Use to determine absolute convergence of series whose terms are \( n \)th powers.

**Theorem 9.32.** Let \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \).

i) If \( L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

ii) If \( L > 1 \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.

iii) If \( L = 1 \), then the root test is inconclusive.

**Example 9.33.**

(1) Determine if \( \sum_{n=0}^{\infty} \left( \frac{n^2 + 5}{2n^2 + 1} \right)^n \) is conditionally convergent, absolutely convergent, or divergent.

(2) Determine if \( \sum_{n=0}^{\infty} \left( \frac{n}{3n + 2} \right)^n \) is conditionally convergent, absolutely convergent, or divergent.
9.5. **Power Series.** Power series begin our exploration into using infinite series to integrate functions without elementary antiderivatives. By adding an \( n \)th power of \( x \) to the \( n \)th term of an infinite series, we turn the series into a function of \( x \). What results is an “infinite polynomial” called a power series.

**Definition 9.34.**

(1) A power series has the form \( \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \)

(2) A power series centered at \( a \) has the form \( \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \ldots \)

**Theorem 9.35.** For a power series \( \sum_{n=0}^{\infty} c_n (x-a)^n \) there are 3 possibilities.

i) The series converges only at \( x = a \).

ii) The series converges for all \( x \).

iii) The series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \) for some \( R \).

**Definition 9.36.** \( R \) is called the radius of convergence, and the set of \( x \) for which the series converges is called the interval of convergence.

**Remark 9.37.** Use the Ratio Test to determine intervals of convergence.

**Example 9.38.** Find the radius and interval of convergence of the following power series.

1. \( \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n5^n} \)
2. \( \sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n3^n} \)

9.6. **Representing Functions as Power Series.** Power series are functions, and different coefficients give different functions. Over the next few sections, we develop the theory of choosing the coefficients so as to make the power series equal to some given function in its radius of convergence. Using the geometric series, we can already represent many functions as power series.

Using the Geometric Series we have: \( \sum_{n=0}^{\infty} = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \) for \( |x| < 1 \).
Example 9.39. Find a power series expansion for the following functions, and determine the interval of convergence.

1) \( g(x) = \frac{1}{x^2 - 7x + 12} \)

2) \( f(x) = \frac{x}{1 + x^4} \)

## Differentiating and Integrating Power Series

We mention the following theorem because we will use it later when integrating power series. It states that we can differentiate and integrate power series term by term to obtain new power series whose radius of convergence is equal to the original.

**Theorem 9.40.** If \( \sum_{n=0}^{\infty} c_n(x-a)^n \) is a power series with radius of convergence \( R \), then:

i) \( \frac{d}{dx} \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} \)

ii) \( \int \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} \)

where the power series on the right each have radius of convergence \( R \).

9.7. Taylor and Maclaurin Series. In the last section, our only tool for finding power series representations of given functions was the geometric series. Here we vastly expand the functions for which we can find power series representations. The main tool is Taylor’s Theorem, which relates the \( n \)th coefficient of a power series to the \( n \)th derivative of the function it represents.

**Theorem 9.41.** Suppose \( f(x) \) has a power series representation at \( a \), ie, \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \). Then \( c_n = \frac{f^{(n)}(a)}{n!} \).

**Definition 9.42.** This representation is the Taylor Series of \( f \) centered at \( a \). If \( a = 0 \), it is also called the Maclaurin Series.

This theorem allows us to compute the coefficients of power series representing certain functions. If we apply this to some of the most basic functions, we obtain the following results, found on page 468 of the text, but reproduced here because of their importance.
Useful Maclaurin Series

<table>
<thead>
<tr>
<th>Function</th>
<th>Maclaurin Series</th>
<th>Radius of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1-x}$</td>
<td>$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$</td>
<td>$R = 1$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$(1+x)^k$</td>
<td>$1 + kx + \frac{k(k-1)x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{k^n}{n!} x^n$</td>
<td>$R = 1$</td>
</tr>
</tbody>
</table>

Remark 9.43. The last series is called the Binomial Series, and the binomial coefficient is given by the following formula:

$$\left(\begin{array}{c}k \\ n\end{array}\right) = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$$

We can use these standard Maclaurin Series to find others.

Example 9.44.

1. Find the Maclaurin series for $g(x) = xe^{-x^2}$, then find $g^{(17)}(0)$ and $g^{(20)}(0)$.

2. Write down the Maclaurin series for the function $f(x) = \sqrt{1+x}$ and use this to obtain a series expansion for $\int_{0}^{1/2} x \sqrt{1+x^3}$. 

3. Find the Maclaurin series of $f(x) = x^2 e^{-x^2}$. Using this, find $f^{(20)}(0)$, and evaluate $\int_{0}^{1} f(x)dx$ correct to within 0.01.

4. Find the Maclaurin series of $g(x) = x^4 e^{-x^2}$ and its radius of convergence. Using this, find $g^{(84)}(0)$, and evaluate $\int_{0}^{1} g(x)dx$ correct to within $10^{-3}$.

9.7.1. Taylor Polynomials. The above functions are exactly equal to their Taylor series within each radius of convergence. However, sometimes it is not practical to keep infinitely many terms in the series. Taylor polynomials only contain finitely many terms of the series, so they are only approximations, but they have the advantage of being easier to work with.
Definition 9.45. In the Taylor series \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \), if we consider only terms with degree \( \leq n \), we obtain \( T_n(x) \), the \( n \)th degree Taylor polynomial of \( f \) at \( a \).

Example 9.46. (1) If \( T_n(x) \) is the \( n \)th Taylor polynomial for \( \sin(x) \) around \( x = 0 \), what \( n \) is sufficient for \( T_n(1) \) to approximate \( \sin(1) \) with an error less than 0.001?

(2) Write the first 6 terms of the Taylor approximation of \( \cos x \) at \( a = \frac{\pi}{4} \).

(3) Let \( f(x) = \int_{0}^{x} \sin(t^2)dt \). Find an upper bound on the error \( |f(1) - T_9(1)| \), where \( T_9 \) is the ninth degree Taylor polynomial centered at 0.

(4) Consider the Taylor series expansion of the function \( f(x) = x \ln x \) about \( x = 1 \). Give the first 4 non-zero terms.

9.8. Applications of Taylor Polynomials. In this final section we learn how to determine the accuracy of a Taylor polynomial.

Taylor polynomials help us approximate functions. We can determine the error \( |R_n(x)| = |f(x) - T_n(x)| \) of our approximation in one of two ways:

(1) If the Taylor series is alternating, then \( |R_n(x)| \leq |\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}| \).

(2) In all cases Taylor’s Formula gives \( R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \) for some \( z \) in between \( x \) and \( a \).

Remark 9.47. If the Taylor series is alternating, it is fairly straightforward to use the alternating series error bound to determine the accuracy of our Taylor polynomial. We’ve already seen some examples in the last section. Only use Taylor’s formula if you must. Here are some examples.

Example 9.48.

(1) Find \( T_3(x) \), the third Taylor polynomial of \( g(x) = x^{4/3} \) about 8. Use \( T_3(x) \) to approximate \( 7^{4/3} \) as a sum of fractions, and determine the accuracy.

(2) Approximate \( f(x) = x^{1/3} \) by its Taylor polynomial of degree 3 at \( a = 1 \) and estimate the accuracy of this approximation at \( x = 0.5 \).
The standard way of describing points in a plane is to use Cartesian coordinates, i.e., \( x \) and \( y \) coordinates. However, some curves in the plane are more easily described using polar coordinates. We first familiarize ourselves with graphs in polar coordinates, and then apply methods of calculus to answer questions about tangents, areas, and lengths.

### 10.3. Polar Coordinates

Polar Coordinates \((r, \theta)\) are an alternate method of describing points in the coordinate plane.

The following relations help us convert between Cartesian and polar equations:

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2 \\
\tan \theta &= \frac{y}{x}
\end{align*}
\]

**Example 10.1.**

1. Find the polar equation for the circle centered at \((1, 0)\) of radius 1.
2. Find the cartesian equations for the circles \(r = 2 \sin \theta\) and \(r = \sin \theta + \cos \theta\).

### 10.3.1. Graphs

To draw graphs of polar equations, it is often helpful to draw the graph in the \((r, \theta)\) plane.

**Example 10.2.** Sketch the graphs of the following polar equations.

\[
\begin{align*}
(1) r &= 2 \cos \theta \\
(2) r &= \cos^2(\theta) \\
(3) r &= 1 \\
(4) r &= 1 + \cos \theta \\
(5) r &= \sin(4\theta) \\
(6) r &= 2 + \cos 2\theta
\end{align*}
\]

### 10.3.2. Tangent Lines

If \(r = f(\theta)\), then

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
\]

**Example 10.3.** Find the equation of the line tangent to \(r = 1 + \cos \theta\) at \(\theta = \frac{\pi}{2}\).

### 10.4. Areas and Lengths in Polar Coordinates

#### 10.4.1. Area

\[
A = \int_{a}^{b} \frac{1}{2} r^2 d\theta = \int_{a}^{b} \frac{1}{2} [f(\theta)]^2 d\theta
\]
Example 10.4.

(1) Let \( D \) be the region inside the curve \( r = 2 \cos \theta \) and outside the curve \( r = 1 \). Sketch the region \( D \) and find its area.

(2) Sketch the circles \( r = 2 \sin \theta \) and \( r = \sin \theta + \cos \theta \) and find the area that lies inside both circles.

(3) Find the area of the region enclosed by one loop of the curve \( r = \sin(4\theta) \).

(4) Sketch the curve \( r = \cos^2(\theta) \) and then find the area enclosed by the curve.

10.4.2. Length.

\[
L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]

Example 10.5.

(1) What integral gives the length of the graph of \( r = 1/\theta \) from \( \theta = \pi \) to \( \theta = 2\pi \)?

(2) Draw \( r = e^{-\theta} \) for \( 0 \leq \theta \leq 4\pi \), and find its length, and the length for \([\theta, \infty)\).

(3) Draw \( r = \theta \) for \( 0 \leq \theta \leq 2\pi \) and find an integral representing its length.

(4) Sketch \( r = \sin^3\left(\frac{\theta}{3}\right) \) for \( 0 \leq \theta \leq 2\pi \), and find its length.