Math 125 Notes

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CHAPTER 1

Functions and Limits

1. Functions and Their Representations

**Definition 1.1.** A function \( f \) is a rule that assigns to each element \( x \) in a set \( D \) exactly one element, called \( f(x) \), in a set \( E \). The domain of \( f \) is the set \( D \), and the range of \( f \) is the set of all possible values of \( f(x) \) as \( x \) varies throughout \( D \).

**Example 1.2.** State the domain of the function \( f(x) = \frac{x}{x^2 - 4} \).

**Example 1.3.** Define \( f(x) = \lfloor x \rfloor \) to be the greatest integer less than or equal to \( x \). Draw the graph of \( f \).

**1.1. Piecewise Defined Functions.** Piecewise defined function are functions defined by different formulas in different parts of their domains.

**Example 1.4.** Draw the graph of

\[
  f(x) = \begin{cases} 
  x + 1 & x < 1 \\
  3 & x = 1 \\
  5x^2 - 4 & x > 1 
  \end{cases}
\]

**Example 1.5.** Write the absolute value function \( f(x) = |x| \) as a piecewise defined function. Draw the graph of \( f \).

**Example 1.6.** Let

\[
  f(x) = \frac{|x - 2|}{x^2 - 4}
\]

Find the domain of \( f \) and write \( f \) as a piecewise defined function.

**1.2. Symmetry.**

**Definition 1.7.** Let \( f \) be a function with domain \( D \). Then \( f \) is an even function if \( f(-x) = f(x) \) for every \( x \) in \( D \), and \( f \) is an odd function if \( f(-x) = -f(x) \) for every \( x \) in \( D \).

**Example 1.8.** Determine if the following functions are even, odd, or neither.

a) \( f(x) = 2 + x^2 \)  
b) \( g(x) = x + x^3 \)  
c) \( f(x) = x + x^2 \)

**Question 1.9.** Can you define a function that is both even and odd?

**Question 1.10.** What type of symmetry do the graphs of even and odd functions possess?
1.3. Increasing and Decreasing Functions.

**Definition 1.11.** A function $f$ is *increasing* on an interval $I$ if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

A function $f$ is *decreasing* on an interval $I$ if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I$$

**Question 1.12.** If an odd (even) function is increasing on $I$, what can we say about its behavior on $-I$?

2. A Catalog of Essential Functions

2.1. Linear Functions. A *linear function* has the form $f(x) = mx + b$.

**Question 2.1.** What is the domain of a linear function? What is the range of a linear function?

2.2. Polynomials. A *polynomial* is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

with $a_n \neq 0$. The *degree* of $P(x)$ is $n$.

**Question 2.2.** What is the domain of a polynomial? What is the range of a polynomial?

2.3. Power Functions. A *power function* has the form $f(x) = x^a$, for some constant $a$.

- **2.3.1.** $a = n$, where $n$ is a positive integer. $f(x) = x^n$
- **2.3.2.** $a = 1/n$, where $n$ is a positive integer. $f(x) = \sqrt[n]{x}$
- **2.3.3.** $a = -1$. $f(x) = 1/x$

**Question 2.3.** What is the domain of a power function? What is the range of a power function?

2.4. Rational Functions. A *rational function* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

**Question 2.4.** What is the domain of a rational function?

2.5. Trigonometric Functions. The two fundamental trigonometric functions are

$$f(x) = \sin x \quad \text{and} \quad g(x) = \cos x$$

**Question 2.5.** What is the domain of $\sin x$? What is the range of $\cos x$?

2.6. Transformations of Functions. Given the graph of a function $f(x)$, and a constant $c$, we can obtain the graph of the functions

$$f(x) + c, \quad f(x + c), \quad cf(x), \quad \text{and} \quad f(cx)$$

by translating, stretching, and reflecting the graph of $f$ appropriately. See page 17 of the textbook.
2.7. Combinations of Functions. Given two functions \( f \) and \( g \) with domains \( A \) and \( B \), respectively, we can define the functions 
\[
(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x) \quad (fg)(x) = f(x)g(x)
\] 
with domain \( A \cap B \). The function 
\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}
\] 
has domain \( \{x \in A \cap B \mid g(x) \neq 0\} \).

Definition 2.6. Given two functions \( f \) and \( g \), the composite function, \( f \circ g \), is defined by 
\[
(f \circ g)(x) = f(g(x))
\]

Example 2.7. Let \( f(x) = \sqrt{2 - x} \) and \( g(x) = \sqrt{1 + x} \). Find the domain of 
\[
\frac{f}{g} \quad \text{and} \quad f \circ g
\]

3. The Limit of a Function

Consider the following two player game. The parameters of the game are a function \( f \), with domain \( D \), a value \( a \in D \) and a number \( L \). The game is played as follows: the first player sets up a 'target' for the second player to hit by stating a positive number \( \epsilon \). This target is given by the interval \((L - \epsilon, L + \epsilon)\). The second player will then use the given function to 'shoot' values to try to hit the target. However, the second player cannot specify exactly which values in the domain she will shoot, she can only specify a shooting range with a positive number \( \delta \). The shooting range is then given by \((a - \delta, a + \delta) \setminus \{a\}\). The actual shot will be taken from somewhere in this range. If the shot hits the target, the second player wins, otherwise, the first player wins.

No matter how the first player defines the target, can the second player make her shooting range small enough to ensure that any shot taken from that range will hit the target? The answer to this question is 'yes' if and only if the limit of the function \( f(x) \) at \( a \) is \( L \).

Example 3.1. If \( f(x) = x^2 \), \( a = 0 \), and \( L = 0 \), show that the second player can always win the game.

Proof. The target for the second player is \((-\epsilon, \epsilon)\). So she would like to find a shooting range \((-\delta, \delta) \setminus \{0\}\) such that for any \( x \) in the range, we have \( f(x) = x^2 \in (-\epsilon, \epsilon) \). This means 
\[
|x^2 - 0| < \epsilon \iff |x| = |x - 0| < \sqrt{\epsilon}
\] 
It follows that if we define our shooting range by \( \delta = \sqrt{\epsilon} \), notice that for any \( x \in (-\sqrt{\epsilon}, \sqrt{\epsilon}) \setminus 0 \), we have \( x^2 \in (-\epsilon, \epsilon) \). \( \square \)

Example 3.2. If \( f(x) = \lfloor x \rfloor \), \( a = 1 \), and \( L = 1 \), show that the first player can always win the game.

Definition 3.3. Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly \( a \) itself. Then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write 
\[
\lim_{x \to a} f(x) = L
\]
if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

**Remark 3.4.** The notion of a one-sided limit is nearly identical to the notion of a limit as we’ve just defined, except that our shooting range only extends to one side of $a$.

**Definition 3.5 (Left-hand Limit).** Write
$$
\lim_{x \to a^-} f(x) = L
$$
if for any $\varepsilon > 0$ there is some $\delta > 0$ such that
if $a - \varepsilon < x < a$ then $|f(x) - L| < \varepsilon$

**Question 3.6.** How should this definition be modified to describe a right-hand limit?

4. Calculating Limits

Here we explore different methods of calculating limits that avoid the tedious precise definition of a limit.

To evaluate one-sided limits, draw a picture of the graph.

**Example 4.1.** Let
$$
f(x) = \begin{cases} 
  x + 1 & x < 1 \\
  3 & x = 1 \\
  5x^2 - 4 & x > 1 
\end{cases}
$$
and calculate
$$
\lim_{x \to 1^-} f(x) \text{ and } \lim_{x \to 1^+} f(x)
$$

Other methods include algebraic manipulation and the Squeeze theorem. The following theorem should also prove helpful.

**Theorem 4.2.**
$$
\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)
$$

**Example 4.3.** Let
$$
f(x) = \begin{cases} 
  3x & x < 1 \\
  x^2 + x & x \geq 1 
\end{cases}
$$
and calculate
$$
\lim_{x \to 1^-} f(x) \text{ and } \lim_{x \to 1^+} f(x)
$$

**Example 4.4.** Evaluate the following limits.

a) $\lim_{x \to 2} \frac{|x - 2|}{x^2 - 4}$

b) $\lim_{x \to 0} \left[ \sin \left( \frac{2}{x^2} \right) + 1 \right] |x|

c) $\lim_{x \to 3^+} \frac{x^2 - 9}{x + 3}$

d) $\lim_{x \to 16} \frac{4 - \sqrt{x}}{16 - x}$
5. Continuity

**Definition 5.1.** A function $f$ is **continuous at** $a$ if

$$\lim_{x \to a} f(x) = f(a)$$

**Remark 5.2.** Notice that the definition of continuity implies that $f(a)$ exists, that $\lim_{x \to a} f(x)$ exists, and that the two are equal. Intuitively, a function is continuous if it can be drawn with a pencil without having to lift the pencil from the paper. Any point where the pencil must be lifted is a point of discontinuity.

**Question 5.3.** What is a function that is continuous everywhere? What types of functions are continuous everywhere on their domain? What is a function that is discontinuous at infinitely many points? What is a function that is continuous nowhere?

**Example 5.4.** Determine whether the following function is continuous at 2.

$$f(x) = \begin{cases} \frac{|x - 2|}{x^2 - 4}, & x \neq 2 \\ \frac{1}{4}, & x = 2. \end{cases}$$

**Example 5.5.** Consider the function $f$ that is given by

$$f(x) = \begin{cases} x + 1, & x < 1 \\ c, & x = 1 \\ 5x^2 - 4, & x > 1 \end{cases}$$

For what value(s) of $c$ is the function continuous at $x = 1$?

**Example 5.6.** Consider the function $f$ that is given by

$$f(x) = \begin{cases} 3x, & x < 1 \\ x^2 + x, & x \geq 1 \end{cases}$$

Is $f$ continuous at $x = 1$?

**Example 5.7.** For what values of $c$ is the function $f(x)$ continuous everywhere?

$$f(x) = \begin{cases} cx + 1, & x \leq 3 \\ cx^2 - 1, & x > 3 \end{cases}$$
Example 5.8. Let
\[ f(x) = \begin{cases} 
  x^2 \cot x & x \neq 0 \\
  0 & x = 0 
\end{cases} \]
Show that \( f \) is continuous at \( x = 0 \).

Theorem 5.9 (The Intermediate Value Theorem). Suppose that \( f \) is continuous on the closed interval \([a, b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

Remark 5.10. The theorem agrees with our intuition concerning continuous functions in the sense that they cannot jump across values (the pencil cannot be lifted from the paper).

Example 5.11. Consider the equation \( x^5 + 2x^3 + 4x - 10 = 0 \). Use the Intermediate Value Theorem to show that the equation has at least one solution.

Example 5.12. Show that the equation \( \sin x - \cos x - 3x = 0 \) has a solution.

Example 5.13. Show that the equation \( x^4 + 4x - 3 = 0 \) has at least two real solutions.

6. Limits Involving Infinity

Having done enough work with \( \varepsilon, \delta \) proofs in section 1.3, here we abandon our desire to give a precise definition of a limit involving infinity, instead only appealing to intuition.

Definition 6.1. The notation
\[ \lim_{x \to a} f(x) = \infty \]
means that the values of \( f(x) \) can be made arbitrarily large by taking \( x \) sufficiently close to \( a \).

Similar definitions can be made for the notations involving \(-\infty\) and one-sided limits. In all such cases, the graph of \( f(x) \) has a vertical asymptote at \( x = a \).

Example 6.2. Evaluate the following limits.
\[ a) \lim_{x \to 1^+} \frac{x^2 - 9}{x^2 + 2x - 3} \quad \quad \quad b) \lim_{x \to 2^+} e^{\frac{1}{x-2}} \]

Definition 6.3. The notation
\[ \lim_{x \to \infty} f(x) = L \]
means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large.

Similar definitions can be made for the notation involving \(-\infty\). In both cases, the graph of \( f(x) \) has a horizontal asymptote at \( y = L \).

Example 6.4. Evaluate the following limits.
a) \( \lim_\limits_{x \to \infty} \left( \sqrt{x^4 + \pi x^2} - x^2 \right) \)

b) \( \lim_\limits_{x \to -\infty} \left( x + \sqrt{x^2 + 2x + 3} \right) \)

c) \( \lim_\limits_{x \to \infty} \left( \sqrt{x^2 + x + 1} - x \right) \)

d) \( \lim_\limits_{x \to \infty} \left( \sqrt{x^2 + ax} = \sqrt{x^2 + bx} \right) \)

e) \( \lim_\limits_{t \to -\infty} \frac{\sin(-t)}{t} \)

f) \( \lim_\limits_{n \to \infty} \frac{5n(n + 1)(2n + 1)}{6n^3} \)

g) \( \lim_\limits_{x \to -\infty} \frac{\sin^2 x}{x\sqrt{x^2 + 1}} \)

h) \( \lim_\limits_{x \to \infty} \left( \ln(1 + 2x) - \ln(1 + x) \right) \)
CHAPTER 2

Derivatives

2. The Derivative as a Function

DEFINITION 2.1. The derivative of a function $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

REMARK 2.2. The derivative of a function $f$ evaluated at a point $a$ is the slope of the tangent line of $f$ at $a$.

EXAMPLE 2.3. Use the definition of the derivative to find $f'(x)$ if

$$f(x) = \frac{1}{x}$$

EXAMPLE 2.4. Use the definition of the derivative to find $f'(x)$ if

$$f(x) = \sqrt{x}$$

THEOREM 2.5. If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

REMARK 2.6. Intuitively, a function is differentiable if it’s graph is smooth, ie, it has no holes, corners, or kinks.

QUESTION 2.7. Where is the function $f(x) = |x|$ differentiable?

3. Basic Differentiation Formulas

REMARK 3.1. In practice, we don’t compute derivatives using the definition. Instead, we collect certain rules and formulas that allow us to make computations quickly.

All of the following formulas can be shown using the definition, but should be used freely. Here $c$ is an arbitrary constant, $n$ is any real number, $f$ and $g$ are differentiable functions, and $s(t)$, $v(t)$, and $a(t)$ are the position, velocity, and acceleration, respectively, of a particle.
at time $t$.

$$\frac{d}{dx}(c) = 0 \quad \text{(Derivative of a constant)}$$

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{(Power Rule)}$$

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \quad \text{(Constant Multiple Rule)}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad \text{(Sum Rule)}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dt}s(t) = v(t)$$

$$\frac{d^2}{dt^2}s(t) = \frac{d}{dt}v(t) = a(t)$$

**Example 3.2.** Find the derivatives of the following functions.

a) $f(x) = 1 + x^2$

b) $f(x) = \sqrt{x}$

c) $f(x) = x(x^2 + 1)$

d) $f(x) = 2 + \cos x$

**4. The Product and Quotient Rules**

Here are two new rules that allow us to find derivatives of products and quotients. Again, these can be proven using the definition, but you are free to use them without proof.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \quad \text{(Product Rule)}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \text{(Quotient Rule)}$$

**Remark 4.1.** The quotient rule can be used in conjunction with what we know about the derivatives of sin and cos to show the following formulas, which you may also use without
proof.

\[ \frac{d}{dx}(\tan x) = \sec^2 x \]

\[ \frac{d}{dx}(\csc x) = -\csc x \cot x \]

\[ \frac{d}{dx}(\sec x) = \sec x \tan x \]

\[ \frac{d}{dx}(\cot x) = -\csc^2 x \]

**Example 4.2.** Find the derivatives of the following functions.

- a) \( f(x) = \frac{x}{1 + x} \)
- b) \( h(t) = \frac{3t - 2}{\sqrt{2t + 1}} \)
- c) \( f(x) = x(x^2 + 1) \)
- d) \( f(x) = \frac{x^2 - 3}{x^3} \)

**5. The Chain Rule**

As in the previous section, here we learn a rule that expands the number of functions for which we can find a derivative without relying on the definition.

\[ \frac{df(g(x))}{dx} = f'(g(x))g'(x) \quad \text{(Chain Rule)} \]

**Example 5.1.** Find the equation of the line tangent to

\[ \frac{(1 - x)^2}{1 + (1 + x)^2} \]

at \( x = 0 \).

**Example 5.2.** Find the derivatives of the following functions.

- a) \( f(x) = \tan \sqrt{1 + x^2} \)
- b) \( f(x) = \tan^2(x^3 + 1) \)
- c) \( f(x) = \frac{\sin(2x)}{3x^3 - x} \)
- d) \( f(x) = \tan^3(2x + 5) \)
- e) \( w(\theta) = \frac{1}{\sqrt{1 + \cos(2\theta)}} \)
- f) \( f(x) = x^2 \sin(2x) \)

**6. Implicit Differentiation**

Curves in the plane are not always described by functions \( y = f(x) \). For example, consider the circle \( x^2 + y^2 = 1 \). However, curves not described by functions still have tangent lines, and thus it makes sense to determine the slope of such curves at points \( (x, y) \). Note that we expect the slope to depend on both \( x \) and \( y \) because for a single \( x \) value, the curve may have many corresponding \( y \) values, each with their own unique slope.
Example 6.1. Consider the curve that is given by the equation

\[ 2(x^2 + y^2)^2 = 25xy. \]

Use implicit differentiation to find the equation of the tangent line at the point (1, 2).

Example 6.2. Find \( \frac{dy}{dx} \) if \( x^2 \cos y + \sin 2y = xy. \)

Example 6.3. Consider the curve that is given by \( x^3 + xy^3 = 9(y - 1) \). Find an equation of the tangent line to the curve at the point (1, 2).

Example 6.4. Find the equation of the line tangent to the curve

\[ (x^2 + y^2)^{3/2} = 2xy \]

at the point \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

Example 6.5. Consider the curve

\[ xy + y^2 = 1 \]

(1) Find the equation of the line tangent to the curve at the point (0, -1).

(2) Find \( \frac{d^2y}{dx^2} \) at the point (0, -1).

7. Related Rates

Related rates problems ask one to find the relationship between two changing quantities. Here

Step 1: Draw a picture, labeling fixed quantities with numbers and changing quantities with variables. Identity the desired rate.

Step 2: Use the picture and a geometric identity to write an equation relating the quantities in the picture.

Step 3: Take the derivative of both sides of the equation with respect to time, remembering that all variables are functions of time.

Step 4: Rearrange for the desired rate and plug in known quantities.

Example 7.1. A tank is in the shape of an inverted right circular cone with radius 2 m at the top and depth \( 6\pi \) m. Suppose water is pumped in at a rate of 1 m\(^3\) per hour. How fast is the water level in the tank rising when the water is 2 m deep?

Example 7.2. Water is poured into a tank shaped as an inverted cone of radius 15 m and of height 10 m. The water flows at the rate of 2 m\(^3\) per minute when the water is 6 m deep. How fast is the water level rising at that time?

Example 7.3. Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop’s radius increases at a constant rate.

Example 7.4. A tank in the shape of an inverted circular cone is standing on its tip. It has a diameter of 4 m, and a height of 9 m. If the water leaks out at a rate of \( 2m^3/min \) how fast is the water level falling when the water level is 3 m?
Example 7.5. A cube is increasing in volume at a rate of 10 cm³/sec. Find the rate of change of the surface area of the cube when one edge has length 2 cm.

8. Linear Approximations and Differentials

A function that is differentiable at a point resembles its tangent line near the point. In some sense, the tangent line is the best line approximating the curve, and can be used to give good approximations of curves.

Definition 8.1. Let \( f \) be a function differentiable at \( x = a \). The linear approximation of \( f \) at \( a \) is the function

\[
L(x) = f(a) + f'(a)(x - a)
\]

The preceding paragraph says that \( L(x) \approx f(x) \) as long as \( x \) is near \( a \). The approximation is exact at \( x = 1 \), and becomes less accurate as \( x \) gets further from \( a \).

Example 8.2. Write down the linear approximation to

\[ f(x) = \frac{(1-x)^2}{1+(1+x)^2} \]

at the value \( a = 0 \). Use the linear approximation to estimate \( \frac{0.99^2}{1+1.01^2} \).

Example 8.3. Find the linear approximation of the function \( y = (1 + x)^{1/2} \) at \( a = 0 \), and use it to approximate \( (0.98)^{1/2} \).

Example 8.4. Consider the curve that is given by \( x^3 + xy^3 = 9(y - 1) \). Find an equation of the tangent line to the curve at the point \((1, 2)\). Estimate the \( y \)-coordinate of that point on the curve that is close to \((1, 2)\) and whose \( x \)-coordinate is equal to 0.9.

8.1. Differentials. Suppose \( y = f(x) \) represents \( y \) as a function of \( x \). The differential of \( f \) helps us estimate how much the \( y \)-value changes given a small change in the \( x \)-value.

\[
\text{Change in } x = dx = \Delta x \\
\text{Change in } y = \Delta y = f(x + dx) - f(x) \\
\text{Differential } dy = f'(x)dx \approx \Delta y
\]

Example 8.5. The pressure \( P \) and the volume \( V \) inside a spherical soap bubble satisfy the equation \( PV = K \), where \( K \) is constant. Initially, the radius of the bubble is 6 cm. If the pressure increases of 0.5% (in other words, if \( \Delta P = 0.005P \)), estimate the change in the radius of the bubble.
CHAPTER 3

Applications of Differentiation

1. Maximum and Minimum Values

Derivatives help us locate maximums and minimums of differentiable functions.

**Definition 1.1.** Let \( f \) be a function with domain \( D \).

1. The **absolute maximum** (minimum) of \( f \) is the largest (smallest) value \( f \) obtains on \( D \).

2. A **local maximum** (minimum) is a value \( f(c) \) that is larger (smaller) than all other values \( f(x) \) for values of \( x \) near \( c \).

**Definition 1.2.** A **critical number** of a function \( f \) is a number \( c \) such that \( f'(c) = 0 \) or \( f'(c) \) does not exist.

**Theorem 1.3.** If \( f \) has a local extrema at \( c \), then \( c \) is a critical number of \( f \).

**Example 1.4.** Consider the function \( f \) that is given by
\[
f(x) = \frac{6}{x^2 + 3}
\]
Of all lines tangent to the graph of \( f \), find the line of maximum slope.

To find the absolute extrema of a function \( f \) on a closed interval \([a, b] \):

1. Find the values of \( f \) at critical points in \([a, b]\).

2. Find the values of \( f \) at the endpoints of \([a, b]\).

3. Compare the values from (1) and (2).

**Example 1.5.** Find the absolute maximum and absolute minimum values of the function
\[
f(x) = 2x^2 - x^4
\]
on the closed interval \([-\frac{1}{2}, 2]\).

**Example 1.6.** Sketch the graph of a function that is defined on the interval \([1, 5]\), has local minima at \( x = 2 \) and \( x = 4 \) and no other local minima, and has no local or global maxima.

2. The Mean Value Theorem

**Theorem 2.1** (Mean Value Theorem). Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there is some \( c \in (a, b) \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]
Theorem 2.2. If \( f'(x) = g'(x) \) for all \( x \) in \((a,b)\), then \( f(x) = g(x) + c \) where \( c \) is a constant.

Proof. Let \( F(x) = f(x) - g(x) \). Then \( F'(x) = f'(x) - g'(x) = 0 \) on \((a,b)\) so that \( F(x) = c \) on \((a,b)\). It follows that \( f(x) - g(x) = c \) on \((a,b)\). \( \square \)

Example 2.3. Consider the equation \( x^4 + 4x - 3 = 0 \). Show that the equation has at most two real solutions.

Example 2.4. Consider the equation \( x^4 + 6x^2 - 5 = 0 \). Show that the equation has at most two real solutions.

Example 2.5. Consider the equation \( x^5 + 2x^3 + 4x - 10 = 0 \). Show that the equation has at most one solution.

Example 2.6. Consider the equation \( x^5 + 3x = 1 \). Show that the equation has only one solution between 0 and 1.

3. Derivatives and the Shapes of Graphs

4. Curve Sketching

Example 4.1. Let \( f(x) = \frac{x}{x^2 - 4} \).
(a) State the domain of \( f \).
(b) Find any asymptotes of the graph of \( f \).
(c) Find the intervals of increase and decrease of \( f \).
(d) Locate all local maxima and minima.
(e) Find the intervals of concavity of \( f \).
(f) Locate all inflection points.
(g) Sketch the graph of \( f \).

Example 4.2. Let \( f(x) = \left( \frac{x}{1 + x} \right)^2 \), and follow \((a) - (g)\) of the previous example.

Example 4.3. Let \( f(x) = \frac{x - 1}{x^2} \), and follow \((a) - (g)\) of the previous example.

Example 4.4. Let \( f(x) = \frac{1}{x} - \frac{1}{x - 1} \), and follow \((a) - (g)\) of the previous example.

Example 4.5. Let \( f(x) = x^{2/3}(x - 5) \), and follow \((a) - (g)\) of the previous example.

Example 4.6. Let \( f(x) = \frac{x^2 - 3}{x^3} \), and follow \((a) - (g)\) of the previous example.

Example 4.7. Let \( f(x) = \frac{x}{x^2 + 4} \), and follow \((a) - (g)\) of the previous example.

Example 4.8. Let \( f(x) = \frac{x^2 + 3x + 1}{x^2 + 1} \), and follow \((a) - (g)\) of the previous example.
5. Optimization Problems

Example 5.1. The perimeter of a rectangular copper sheet is 12 inches. The two opposite edges are welded together to form a cylindrical pipe. What dimensions of the rectangle will result in a cylinder of maximum volume?

Example 5.2. Find the volume of the largest circular cone that can be inscribed in a sphere of radius \( r \).

Example 5.3. A rectangular box is designed to have a square base and an open top. The material for the bottom costs 2 dollar per square foot, while the material for the sides costs 3 dollars per square foot.

a) If the box must have a volume of 9 ft\(^3\) find the dimensions of the box that would minimize the cost of materials.

b) If the cost of the box is to be 216 dollars find the dimensions that maximize the volume of the box.

Example 5.4. Find the points on the ellipse \( 4x^2 + y^2 = 4 \) that are farthest away from the point \((1, 0)\).

Example 5.5. A cylindrical can is to be constructed so that its volume will be 40 cm\(^3\). If the material used to make the top and bottom is twice as expensive as the material used to make the side, find the dimensions of the least expensive can.

Example 5.6. Find the dimension of the right circular cylinder of greatest volume that can be inscribed in a given right circular cone of height 1 and radius 1 meter.

7. Antiderivatives

Example 7.1. Find the most general antiderivative of the function \( f(x) = \frac{x^3 - x^2 + x^{3/2}}{\sqrt{x}} \).

Example 7.2. Find the most general antiderivative of the function \( f(x) = \frac{x^2 - 3\sqrt{x}}{x} \).

Example 7.3. Find the most general antiderivative of the function \( f(x) = \frac{x^2 + x + 1}{x^2} \).

Example 7.4. Find the most general antiderivative of the function \( f(x) = \frac{1 - \sqrt{x}}{x} \).

Example 7.5. Suppose that the acceleration of a particle (in m/sec\(^2\)) at time \( t \) (in sec) is given by the function

\[ a(t) = 6t - 6 \]

for \( t \geq 0 \).

a) Find the velocity \( v(t) \) of the particle as a function of time if \( v(0) = -9 \) m/sec.

b) Find the net displacement of the particle from \( t = 0 \) to \( t = 4 \).
CHAPTER 4

Integrals

1. Areas and Distances

Example 1.1. Express the following limit as a definite integral.
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)}
\]

Example 1.2. Write the limit
\[
\lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^{3/2}
\]
as an integral.

Example 1.3. Write the limit
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^{5}
\]
as an integral.

Example 1.4. Write the limit
\[
\lim_{n \to \infty} \left(\frac{1}{n}\right) \left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]
\]
as an integral.

Example 1.5. Write the limit
\[
\lim_{n \to \infty} \frac{1}{n} \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)
\]
as an integral.

2. The Definite Integral

Example 2.1. Write the following limit as an integral.
\[
\lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \sqrt{4 - \left(\frac{2i}{n}\right)^2}
\]
Compute the limit by interpreting the integral as an area.

3. Evaluating Definite Integrals

Example 3.1. Evaluate:
Example 3.2. Find the area bound by the graph $y = x^3 - x$ and the $x$-axis.

4. The Fundamental Theorem of Calculus

Example 4.1. Differentiate each of the following:

(a) $f(x) = \int_{\cos x}^{\sin x} \sqrt{t^4 + 1} \, dt$

(b) $f(x) = \int_{x}^{x^2011} e^{-t^2} \, dt$

(c) $f(x) = \int_{1}^{x} e^t \sin t \, dt$

(d) $y = \int_{\cos x}^{x^2} \sqrt{1 + t^4} \, dt$

(e) $y = \int_{0}^{\tan x} \sqrt{t^2 + 4} \, dt$

5. The Substitution Rule

Example 5.1. Evaluate the integral:

(a) $\int_{0}^{\pi} x \sin(x^2 + \pi) \, dx$

(b) $\int_{x}^{x^5} \sqrt{1 + x^3} \, dx$

(c) $\int_{0}^{\sin x} \frac{x}{1 + \cos x} \, dx$

(d) $\int_{x}^{x^3} \sqrt{x^2 + 1} \, dx$

(e) $\int_{0}^{\pi} \tan \left( \frac{x}{3} \right) \, dx$

(f) $\int_{0}^{3 \tan x} \frac{x}{\cos^2 x} \, dx$

(g) $\int_{x}^{x^5} \sqrt{x^3 + 4} \, dx$

(h) $\int_{x}^{x^3} \frac{x}{x^2 + 1} \, dx$

(i) $\int_{0}^{x} \frac{dx}{\sqrt{2 - 3x}}$

(j) $\int_{0}^{1} \sin \pi x \, dx$

(k) $\int_{0}^{\sin t} \frac{\sin t}{\cos t + 12} \, dt$

(l) $\int_{0}^{2x^3} \sqrt{x^2 + 1} \, dx$

(m) $\int_{0}^{1} t 10^{-t^2} \, dt$

(n) $\int_{0}^{x} \frac{x}{\sqrt{1 + 2x}} \, dx$

(o) $\int_{0}^{3 \cos \theta} \frac{\theta}{5} \, d\theta$

(p) $\int_{0}^{\sin \sqrt{x}} \frac{dx}{\sqrt{x}}$
CHAPTER 5

Inverse Functions

1. Inverse Functions

2. The Natural Logarithm Function

Example 2.1. Differentiate each of the following:

(a) \( y = (4 + x^2)^x \)
(b) \( f(x) = (2 + \cos x)^x \)
(c) \( f(x) = \ln[(x^2 + a^2)^5(x + b)^6(x + c)^7] \)
(d) \( f(x) = x^x \)
(e) \( y = (\cos x)^x \)
(f) \( f(x) = \int_2^{x^2} \ln(\cos t) \, dt \)
(g) \( f(x) = (\ln(5x))^3 \)
(h) \( f(x) = \int_0^{\ln x} \sqrt{t^5 + 1} \, dt \)
(i) \( f(x) = \ln \frac{\sqrt{x + 1}}{x - 1} \)
(j) \( f(t) = \sin(\ln t) \)
(k) \( f(x) = \frac{1 + \ln x}{1 - \ln x} \)
(l) \( f(x) = \int_1^{\ln x} e^{x^2} \, du \)
(m) \( g(x) = \int_1^{\cos x} \ln t \, dt \)
(n) \( f(x) = x^{x^2} \)
(o) \( y = \ln(2x^2 - 3x) \)
(p) \( y = x^{\sin x} \)
(q) \( y = \frac{(x - 1)(x - 4)}{(x - 2)(x - 3)} \)

Example 2.2. Evaluate each of the following:

(a) \( \int \frac{\ln(x^2)}{x} \, dx \)
(b) \( \int_0^1 \frac{x + 1}{x^2 + 2x + 3} \, dx \)
(c) \( \int \frac{\ln x}{x} \, dx \)
(d) \( \int_1^e \frac{\sqrt{\ln x}}{x} \, dx \)
(e) \( \int_e^{e^4} \frac{dt}{t\sqrt{\ln t}} \)

Example 2.3. Consider the curve given by the equation

\[ x \ln y + y = 1 - \ln x \]

a) Find the equation of the tangent line to the curve at the point \((1, 1)\).

b) Estimate the \(y\)-coordinate of that point on the curve that is close to \((1, 1)\) and whose \(x\)-coordinate is equal to 1.2.
Example 2.4. Evaluate the limit
\[
\lim_{x \to \infty} (\ln(1 + 2x) - \ln(1 + x))
\]

Example 2.5. Consider the function \( f(x) = \ln x - (x - 4)^2 \). Show that the \( f(x) \) has exactly one root in \([1, 3]\).

Example 2.6. Show that the equation \( \ln x - x + 2 \) has exactly two solutions.

3. The Natural Exponential Function

Example 3.1. The following limit represents the derivative of some function \( f \) at some number \( a \):
\[
\lim_{x \to 0} \frac{e^x - 1}{x}
\]
Write down one possible function \( f \) and the corresponding value of \( a \). What is the limit?

Example 3.2. Differentiate each of the following:

(a) \( y = e^{x(x^2 + 1)} \)  
(b) \( f(x) = e^{\left(\frac{x}{x^2 + 1}\right)} \)  
(c) \( f(x) = x \cdot e^{\sin x} \)  
(d) \( f(x) = xe^{\sqrt{x}} \)  
(e) \( f(x) = \ln(xe^x) \)  
(f) \( f(t) = t^2 e^{-t} \)  
(g) \( y = \frac{2x - 3}{e^{x+1}} \)  
(h) \( xe^{-y} + ye^{-x} = 3 \)

Example 3.3. Evaluate each of the following:

(a) \( \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} \, dx \)  
(b) \( \int e^{3\cos(x)} \sin(x) \, dx \)  
(c) \( \int_0^1 \frac{e^x}{5e^x - 3} \, dx \)  
(d) \( \int e^{2x} \sqrt{1 + e^x} \, dx \)

Example 3.4. Consider the set of all rectangles that lie in the first quadrant, have one edge along the positive \( x \)-axis, one corner at the origin, and the diagonally opposite corner on the curve \( y = e^{-x} \). Find the dimensions of the rectangle in the set that has the largest area.

Example 3.5. Consider the curve that is given by the equation
\[
e^x + \ln y = x^2 - y^2 + 2
\]
Find the equation of the tangent line at the point \((0, 1)\).

Example 3.6. Show that for any \( x \geq 0 \), \( e^x \geq 1 + x \). (Hint: apply the mean value theorem to \( f(x) = e^x \)). Then show that \( g(x) = e^x - x^2/2 - 2 \) for \( x \geq 0 \) has exactly one real root.

Example 3.7. Let \( f(x) = \frac{x^e}{xe}, x > 0 \).

a) Find the critical numbers of \( f \), and determine the intervals where \( f \) is increasing, and the intervals where \( f \) is decreasing.
b) Which is larger, $e^\pi$ or $\pi^e$?

**EXAMPLE 3.8.** Evaluate the following limits

\[
\begin{align*}
\text{a) } & \lim_{x \to 0^-} e^{1/x} \cos x \\
\text{b) } & \lim_{x \to 1} \frac{e^x - e}{x - 1} \\
\text{c) } & \lim_{x \to 2^+} e^{\pi/x}
\end{align*}
\]

**EXAMPLE 3.9.** In the $xy$-plane, consider the curve given by the equation $y = e^{xy}$. Find the equation of the line tangent to the curve at the point $(-e, \frac{1}{e})$.

**EXAMPLE 3.10.** Find the area of the region bounded by the curves $y = 0$, $y = xe^{x^2}$, $x = 0$ and $x = 1$.

**EXAMPLE 3.11.** Find the absolute maximum and minimum of $f(x) = xe^{-3x}$ on the interval $[-1, 1]$.

4. General Logarithmic and Exponential Functions

5. Exponential Growth and Decay

**EXAMPLE 5.1.** A 500 gm. sample of radioactive substance decays so that after 2 years only 350 gms. remain.

\begin{enumerate}
\item[a)] Find a formula for the amount of the substance present at time $t$.
\item[b)] What is the half-life of the substance?
\item[c)] How much of the substance is left after five years?
\end{enumerate}

**EXAMPLE 5.2.** A sample of radon-222 decayed to 58% of its original amount in 3 days.

\begin{enumerate}
\item[a)] What is the half-life of radon-222?
\item[b)] How long does it take a sample of radon-222 to decay to 10% of its original amount?
\end{enumerate}