1. The Nullcone and Restricted Nullcone

We will need the following preliminary definitions of subsets of restricted Lie algebras. Let $k$ be an algebraically closed field of characteristic $p$, and let $g$ be a restricted Lie algebra over $k$.

**Definition 1.1.** Define $N(g)$, the nullcone of $g$, to be all $[p]$-nilpotent elements $x \in g$, i.e., all elements such that $x^{[p]^n} = 0$ for some $n$. The restricted nullcone, $N_p(g)$ is the subset of $N(g)$ consisting of all $x$ such that $x^{[p]} = 0$.

Inside $g \times r$ we can consider the collection of all $r$-tuples with coordinates in the restricted nullcone, that is, let $N_p(g)^{\times r} = \{(x_1, \ldots, x_r) \mid x_i^{[p]} = 0\} \subset g^{\times r}$. Inside of $N_p(g)^{\times r}$ is the collection of all $r$-tuples with pairwise commuting coordinates in the restricted nullcone, that is, let $N_p(g)^{\times r} = \{(x_1, \ldots, x_r) \mid x_i^{[p]} = 0, [x_i, x_j] = 0\} \subset N_p(g)^{\times r}$. Finally, inside of $N_p(g)^{\times r}$ we have the collection of $r$-tuples with linearly independent, pairwise commuting coordinates in the restricted nullcone, that is, let $N_p(g)^{\times r} = \{(x_1, \ldots, x_r) \mid x_i^{[p]} = 0, [x_i, x_j] = 0, \dim \langle x_1, \ldots, x_r \rangle = r\} \subset N_p(g)^{\times r}$.

2. Elementary Subalgebras and $N_p(g\ell_n)$

We begin by defining the object of study.

**Definition 2.1.** Let $g$ be a restricted Lie algebra of dimension $n$. An elementary subalgebra, $\epsilon \subset g$, is an abelian Lie subalgebra with trivial $p$-restriction, i.e., $\epsilon$ is a subspace such that $[x, y] = 0$ and $x^{[p]} = 0$ for all $x, y \in \epsilon$. For $r \leq n$, let

$$E(r, g) = \{\epsilon \subset g \mid \epsilon \text{ elementary}, \dim(\epsilon) = r\}$$

There is only one elementary subalgebra of dimension 1, and we denote this subalgebra $g_a$.

Suppose $\{e_1, \ldots, e_r\}$ is a basis of $\epsilon$, and notice that the one dimensional subspace spanned by $e_i$ is an elementary subalgebra of dimension 1, and thus is isomorphic to $g_\epsilon$. It follows that $\epsilon \cong \oplus g_\epsilon$. Also, if we forget the Lie algebra structure we have an inclusion

$$E(r, g) \hookrightarrow \text{Grass}(r, g)$$

For the time being, we will work in the special case of $g = g\ell_n$, the restricted Lie algebra of $n \times n$ matrices with bracket given by commutator, and $p$-restriction given by $p$th powers.
of matrices. In this case, the subsets of $\mathcal{N}_p(\mathfrak{gl}_n)$, $\mathcal{N}_p^r(\mathfrak{gl}_n)$, and $\mathcal{N}_p^r(\mathfrak{gl}_n)^o$ have additional structure.

**Proposition 2.2.** If we identify $\mathfrak{gl}_n$ with $\mathbb{A}^{n^2}$, then $\mathcal{N}_p(\mathfrak{gl}_n) \subset \mathbb{A}^{n^2}$ is a closed variety, $\mathcal{N}_p^r(\mathfrak{gl}_n) \subset \mathbb{A}^{rn^2}$ is a closed variety, and $\mathcal{N}_p^r(\mathfrak{gl}_n)^o \subset \mathcal{N}_p^r(\mathfrak{gl}_n)$ is open in the Zariski topology.

**Proof.** Consider the 'general matrix of coordinate functions' whose $ij$th entry is the variable $x_{ij}$. Computing the $p$th power of this matrix we obtain

$$\left( \begin{array}{c} x_{ij} \\ \end{array} \right)^p = \left( \begin{array}{c} f_{ij} \\ \end{array} \right)$$

where the $f_{ij}$ are homogeneous polynomials of degree $p$ in the $x$'s. It follows that $\mathcal{N}_p(\mathfrak{gl}_n) = Z(f_{ij})$ is closed in $\mathbb{A}^{n^2}$.

Next, consider two 'general matrices', and compute their commutator:

$$\left( \begin{array}{c} x_{ij}^k \\ x_{ij}^l \\ \end{array} \right) - \left( \begin{array}{c} x_{ij}^l \\ x_{ij}^k \\ \end{array} \right) = \left( \begin{array}{c} g_{ij}^{kl} \\ \end{array} \right)$$

where the $g_{ij}^{kl}$ are homogeneous polynomials of degree 2 in the $x$'s. It follows that $\mathcal{N}_p^r(\mathfrak{gl}_n) = Z(f_{ij}^k, g_{ij}^{kl})$ is closed in $\mathbb{A}^{rn^2}$.

Finally, we have $\mathcal{N}_p^o(\mathfrak{gl}_n)^o = \{(A_1, \ldots, A_r) \mid A_i^{[p]} = 0, [A_i, A_j] = 0, \dim\langle A_1, \ldots, A_r \rangle = r\}$. View each coordinate matrix as a column of an $n^2 \times r$-matrix.

$$B = \left( \begin{array}{cccc} l & l & \cdots & l \\ x_{1i}^1 & x_{2i}^2 & \cdots & x_{ri}^r \\ l & l & \cdots & l \\ \end{array} \right)$$

Then the rank of this matrix is the dimension of $\langle A_1, \ldots, A_r \rangle$. This matrix has full rank if and only if all $r \times r$ minors are nonzero. For $\Sigma \subset (1, \ldots, n^2)$ with $|\Sigma| = r$, let $\Sigma B$ be the $r \times r$ submatrix of $B$ taking the rows corresponding to elements of $\Sigma$, and let $h_\Sigma = \det(\Sigma B)$. Notice that $h_\Sigma$ is a homogeneous polynomial of degree $r$, and that $\mathcal{N}_p^r(\mathfrak{gl}_n)^o = Z(f_{ij}^k, g_{ij}^{kl}) \cap Z(h_\Sigma)^c$ is open in $\mathcal{N}_p^o(\mathfrak{gl}_n)$.

**Remark 2.3.** I believe the proposition is true for all restricted Lie algebras $\mathfrak{g}$ and the associated sets $\mathcal{N}_p(\mathfrak{g})$, $\mathcal{N}_p^r(\mathfrak{g})$, and $\mathcal{N}_p^o(\mathfrak{g})^o$. I believe that any $\mathfrak{g}$ can be embedded in some $\mathfrak{gl}_n$. We will assume this is true, and continue on in complete generality, letting $\mathfrak{g}$ be any restricted Lie algebra. One may keep in mind $\mathfrak{gl}_n$ as a concrete example.

3. The Projective Variety $\mathbb{E}(r, \mathfrak{g})$

Here we wish to show that the inclusion (2.1) is closed in the sense that $\mathbb{E}(r, \mathfrak{g})$ embeds into Grass($r, \mathfrak{g}$) as a closed, projective subvariety, where Grass($r, \mathfrak{g}$) is viewed as a closed variety in $\mathbb{P}(\Lambda^r V)$ by the Plücker embedding. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathcal{N}_p^r(\mathfrak{g})^o & \longrightarrow & M_{n,r}(k)^o \\ \downarrow & & \downarrow \rho \\ \mathbb{E}(r, \mathfrak{g}) & \longrightarrow & \text{Grass}(r, \mathfrak{g}) \end{array}$$

(3.1)
The top horizontal map is the mapping of an $r$-tuple of elements of $\mathfrak{g}$ to an $n \times r$ matrix having chosen a basis for $\mathfrak{g}$. The bottom horizontal map is (2.1). The left vertical map takes an $r$-tuple of linearly independent, pairwise commuting elements of the restricted nullcone to the elementary subalgebra they define.

This commutative diagram is actually a Cartesian square, i.e., $N^r_p(\mathfrak{g})^\circ$ is the pullback in the category of sets of the diagram

\begin{equation}
\begin{array}{c}
\mathbb{M}_{n,r}(k)^\circ \\
\downarrow \quad p \\
\mathbb{E}(r, \mathfrak{g}) \ar@{^{(}->}[r] & \text{Grass}(r, \mathfrak{g})
\end{array}
\end{equation}

This can be seen by noticing that commutativity of the diagram below implies that $\text{im}\beta \subset N^r_p(\mathfrak{g})^\circ$, which allows us to define $\theta = \beta$.

\begin{equation}
\begin{array}{c}
X \\
\downarrow \quad \beta \\
N^r_p(\mathfrak{g})^\circ \ar@{^{(}->}[r] & \mathbb{M}_{n,r}(k)^\circ \\
\downarrow \quad \text{E}(r, \mathfrak{g}) \ar@{^{(}->}[r] & \text{Grass}(r, \mathfrak{g})
\end{array}
\end{equation}

Also, we can put $N^r_p(\mathfrak{g})^\circ$ into one-to-one correspondence with the set

$$\{(A, \epsilon) \in \mathbb{M}_{n,r}(k)^\circ \times \mathbb{E}(r, \mathfrak{g}) \mid p(A) = \epsilon\}$$

under the mapping

$$(u_1, \ldots, u_r) \mapsto \left(\begin{array}{cccc}
\ell & \ell & \ldots & \ell \\
\ell & u_1 & u_2 & \ldots & u_r \\
\ell & \ell & \ell & \ldots & \ell
\end{array}\right), \langle u_1, \ldots, u_r \rangle$$

We have already shown that $N^r_p(\mathfrak{g})$ is closed in $\mathbb{M}_{n,r}$, so that $N^r_p(\mathfrak{g})^\circ = N^r_p(\mathfrak{g}) \cap \mathbb{M}_{n,r}^\circ$ is closed in $\mathbb{M}_{n,r}^\circ$. Considering (3.1), it might follow that $\mathbb{E}(r, \mathfrak{g})$ is closed in $\text{Grass}(r, \mathfrak{g})$. However, $p$ is not necessarily a closed map. It is however, closed when restricted to $p^{-1}(U_\Sigma)$. To see this, consider the following proposition.

**Proposition 3.1.** Let $X$ and $Y$ be topological spaces with $Y$ quasi-compact. Then the projection $p : X \times Y \to X$ is a closed map where $X \times Y$ is given the product topology.

**Proof.** Let $E \subset X \times Y$ be closed, and take some $x \notin p(E)$. Hence, for each $y \in Y$ there are open sets $V_y \subset X$ and $W_y \subset Y$ such that $(x, y) \in V_y \times W_y$ and $(V_y \times W_y) \cap E = \emptyset$. By quasi-compactness, choose finitely many $W_y$ that cover $Y$. I claim that $\cap V_{y_i}$ is disjoint from $p(E)$, proving that $p(E)^c$ is open. If not, we have some $(x, y) \in E$ with $(x, y) \in V_{y_i} \times W_{y_i}$ for some $i$, a contradiction. \hfill \Box

Below is diagram 3.1 when restricted to the open set $U_\Sigma$. 

Since GL$_r$ is finite, it is quasi-compact (with the discrete topology). Hence the projection map is a closed map, and it follows that E($r, g$) ∩ $U_{Σ}$ is closed in $U_{Σ}$. The homeomorphism $p^{-1}(U_{Σ}) \cong U_{Σ} \times$ GL$_r$ follows from the fact that $p$ is a GL$_r$ torsor, which is trivial over $U_{Σ}$ (needs more explanation).

Finally, to show that E($r, g$) ↪ Grass($r, g$) is a closed embedding, we use (3.4) together with the following lemma from point-set topology.

**Lemma 3.2.** Let $X$ be a topological space and $Y \subset X$. Then $Y$ is closed if and only if there exists an open cover $U_i$ of $X$ with $Y \cap U_i$ closed in $U_i$ for all $i$.

**Proof.** If $Y$ is closed, then take the open cover to consist solely of the open set $X$. Next, suppose such an open cover exists. We will show that $Y^c$ is open. Since $Y$ is closed in $U_i$, there exist closed $Z_i \subset X$ with $Y \cap U_i = Z_i \cap U_i$.

$$Y^c = \bigcup_i Y^c \cap U_i = \bigcup_i Z_i^c \cap U_i$$

It follows that $Y^c$ is open. $\square$

Also, suppose $g$ is the Lie algebra associated to a linear algebraic group $G$. Then $G$ acts on $g$ by the adjoint action (conjugation). Since conjugation preserves linear independence, trivial bracket, and trivial $[p]$-restriction, it follows that E($r, g$) is $G$-stable. Typical examples are Lie(GL$_n$) = gl$_n$, Lie(GL$_n$) = sl$_n$, and Lie(U$_n$) = u$_n$, where U$_n$ is the set of unipotent, upper triangular matrices, and u$_n$ is the set of strictly upper triangular matrices.

4. **Examples**

To get a more concrete idea of how E($r, g$) embeds in Grass($r, g$), let’s look at a few examples.

**Example 4.1.** Let $g = u_3$, the set of all strictly upper triangular $3 \times 3$ matrices. u$_3$ is the Lie algebra of the Lie group $U_3$ of unipotent upper triangular $3 \times 3$ matrices, so that $U_3$ acts on $u_3$ by conjugation. It is also true that $B_3$, the set of upper triangular matrices $3 \times 3$ matrices, acts on u$_3$.

Consider E(2, u$_3$). Let $A = (a_{ij})$ and $B = (b_{ij})$ be two basis vectors for some $ε \in E(2, u_3)$, so $A$ and $B$ have the form:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

Since $A$ and $B$ commute, we know that $a_{12}b_{23} - a_{23}b_{12} = 0$. We will assume that $p > 2$ so that the requirement that $A$ and $B$ have trivial $[p]$-restriction is automatically satisfied, and leads to no new equations. We split into two cases:
(1) Suppose \( a_{12} = 0 \). Then it follows that either \( a_{23} = 0 \) or \( b_{12} = 0 \). If \( a_{23} = 0 \), we know that \( a_{13} \neq 0 \) so we can normalize \( A \) to assume it has the form:

\[
A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Then by subtracting \( b_{13}A \) from \( B \), we can assume \( B \) has the form:

\[
B = \begin{pmatrix}
0 & b_{12} & 0 \\
0 & 0 & b_{23} \\
0 & 0 & 0
\end{pmatrix}
\]

Any choice of \((b_{12}, b_{23}) \neq (0, 0)\) (up to scalar multiple) gives a unique elementary subalgebra.

In the case that \( b_{12} = 0 \), a dimension argument shows that the space spanned by \( A \) and \( B \) must be the space spanned by

\[
A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Notice this space is already covered in the first subcase if we choose \( b_{12} = 0 \) and \( b_{23} = 1 \).

(2) Suppose \( a_{12} \neq 0 \). Then as before, we can normalize both \( A \) and \( B \) and assume they are of the form:

\[
A = \begin{pmatrix}
0 & 1 & a_{13} \\
0 & 0 & a_{23} \\
0 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & b_{13} \\
0 & 0 & b_{23} \\
0 & 0 & 0
\end{pmatrix}
\]

Our commutation relation now requires that \( b_{23} = 0 \), so that after normalization we have the matrices:

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & a_{23} \\
0 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The subalgebra generated by these matrices is the same as that considered in the first case under the choice \( b_{12} = 1 \). Here \( A \) and \( B \) have switched roles.

In summary, the elementary abelian subalgebras of dimension 2 inside of \( u_3 \) are given up to scalar multiple by a choice of nonzero element in \( A^2 \), ie, as sets, \( \mathbb{E}(2, u_3) \cong \mathbb{P}^1 \).

To show that \( \mathbb{E}(2, u_3) \) embeds into \( \mathbb{P}^2 \) as the closed subvariety \( \mathbb{P}^1 \), let’s look at the Plücker coordinates. Let \( E_{12}, E_{13}, \) and \( E_{23} \) be an ordered basis for \( u_3 \), and write at the Plücker coordinates. Let \( E_{12}, E_{13}, \) and \( E_{23} \) be an ordered basis for \( u_3 \), and write \( A = (0, 1, 0) \) and \( B = (b_{12}, 0, b_{23}) \) as vectors. Form the \( 3 \times 2 \) matrix whose columns are \( A \) and \( B \), and embed into \( \mathbb{P}^2 \) using the Plücker coordinates:

\[
\begin{pmatrix}
b_{12} \\
1 & 0 \\
0 & b_{23}
\end{pmatrix} \mapsto [-b_{12} : 0 : b_{23}] \in \mathbb{P}^2
\]

The image of the Plücker embedding is thus the copy of \( \mathbb{P}^1 \) sitting in \( \mathbb{P}^2 \) whose second homogeneous coordinate vanishes.
Since $u_3$ is not commutative, it follows that $E(3, u_3) = \emptyset$. The proposition below computes $E(1, u_3)$.

**Proposition 4.2.** Let $p \geq n$. Then $E(1, u_n) \cong \text{Grass}(1, u_n) \cong \mathbb{P}(u_n)$. Also:

$$E\left(\frac{(n)(n-1)}{2}, u_n\right) = \begin{cases} \emptyset & n \geq 3 \\ \{(0 1 0 0)\} & n = 2 \end{cases}$$

**Proof.** Since $p \geq n$, all matrices in $u_n$ have trivial $[p]$-restriction. Also, since all one-dimensional subalgebras are abelian, it follows that any one-dimensional space in $u_n$ generates an elementary abelian subalgebra. This shows the first of the isomorphisms. The second isomorphism is by definition.

Now, as for the second claim, notice that $\dim(u_n) = n(n-1)/2$, and that $u_n$ is abelian if and only if $n = 2$. \qed

The proposition helps us deal with the limiting cases of $u_n$. Let us now consider $u_4$ for cases other than $r = 1, 6$. As before, we assume $p \geq 5$ so that the $[p]$-restriction condition adds no new requirements to our subalgebras.

In considering the $u_4$, it will be helpful to write down the general commutation relations. If $A$ and $B$ are strictly upper triangular, the condition that $AB = BA$ leads to the following three equations:

(4.1) $a_{12}b_{23} = b_{12}a_{23}$

(4.2) $a_{34}b_{23} = b_{34}a_{23}$

(4.3) $a_{12}b_{24} + a_{13}b_{34} = b_{12}a_{24} + b_{13}a_{34}$

Since $A$ and $B$ are arbitrary, all matrices we consider must pairwise satisfy the above three equations. Suppose that $a_{23} \neq 0$. Then equations (4.1) and (4.2) yield:

$$\frac{a_{12}b_{23}}{a_{23}} = b_{12} \quad \frac{a_{34}b_{23}}{a_{23}} = b_{34}$$

Notice that if $a_{12} = 0$ then $b_{12} = 0$, and also if $a_{12} \neq 0$ then $(b_{12}, b_{23}) = \lambda(a_{12}, b_{23})$. The same argument holds for $a_{34}$, so that in all cases we have $(b_{12}, b_{13}, b_{23}) = \lambda(a_{12}, a_{13}, a_{23})$ for some $\lambda \in k$. We have thus proven the following useful lemma for our calculations.

**Lemma 4.3.** Let $A_i, i = 1, \ldots, r$, be a collection of matrices in $u_4$ that pairwise commute. Suppose there is some $i$ with $A_i = A$, and $a_{23} \neq 0$. Then each superdiagonal of the $A_i$ is a scalar multiple of the superdiagonal of $A$.

In the case that there is some matrix whose $(2, 3)$ entry is non-zero, the lemma allows us to normalize the rest of the matrices and assume they have no superdiagonal. Let’s use this result to calculate $E(r, u_4)$ for $r = 3, 4,$ and $5$.

**Example 4.4.** $E(5, u_4)$. Let $A, B, C, D,$ and $E$ be linearly independent, pairwise commuting matrices in $u_4$. If all $(2, 3)$ entries are zero, a dimension argument shows that the space...
spanned by these matrices must be the five dimensional space:

\[
\left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid (a, b, c, d, e) \in \mathbb{A}^5 \right\}
\]

This space does not commute. It follows that there is some matrix, say \( A \), with \( a_{23} \neq 0 \). Normalizing, it follows that \( B, C, D, \) and \( E \) have no superdiagonal. This contradicts linear independence. In conclusion, we have \( \mathbb{E}(5, u_4) = \emptyset \).

**Example 4.5.** \( \mathbb{E}(4, u_4) \). Let \( A, B, C, \) and \( D \) be four linearly independent, pairwise commuting matrices in \( u_4 \). First assume there is some matrix, say \( A \), with \( a_{23} \neq 0 \). Normalizing we can assume that \( B = E_{13}, C = E_{14}, \) and \( D = E_{24} \). Equation (4.3) applied to \( A \) and \( B \), and \( A \) and \( D \) yields \( a_{34} = 0 \) and \( a_{12} = 0 \) respectively. Thus we may assume \( A = E_{23} \) so that we obtain the four dimensional elementary subalgebra \( e = \langle E_{13}, E_{14}, E_{23}, E_{24} \rangle \).

Next assume that all \( (2,3) \) entries are 0. If also all the \( (1,2) \) entries are zero, we obtain the nonabelian four dimensional subalgebra \( k\langle E_{13}, E_{14}, E_{24}, E_{34} \rangle \). Hence we may assume \( a_{12} \neq 0 \) and by normalizing, \( a_{12} = 1 \) and \( b_{12} = c_{12} = d_{12} = 0 \).

If all of the \( (3,4) \) entries are 0, then again we obtain a nonabelian subalgebra, so some \( (3,4) \) entry is nonzero. If the only such one is \( a_{34} \), then \( B, C, \) and \( D \) generate the three dimensional subalgebra \( k\langle E_{13}, E_{14}, E_{24} \rangle \), but this then does not commute with \( A \).

Hence, we may assume that \( b_{34} \neq 0 \). Normalizing all four matrices and checking commutation relations will show that \( C \) and \( D \) must both be \( E_{14} \), violating linear independence.

We conclude that there is only one elementary subalgebra of \( u_4 \) of dimension 4, so that \( \mathbb{E}(4, u_4) \) is just a point.

For the next example, we will also use the following Lemma.

**Lemma 4.6.** Let \( V \) be an \( n \) dimensional vector space. Then \( \text{Grass}(r, V) \cong \text{Grass}(n - r, V^*) \) for \( r = 0, \ldots, n \). Choosing an isomorphism of \( V \) with \( V^* \) gives \( \text{Grass}(r, V) \cong \text{Grass}(n - r, V) \) for \( r = 0, \ldots, n \).

**Proof.** Let \( U \subset V \) be a subspace of dimension \( r \). We thus have a map \( U \hookrightarrow V \) which induces a dual map \( V^* \twoheadrightarrow U^* \). The kernel of this map is an \( n - r \) dimensional subspace in \( V^* \). This is the desired isomorphism. \( \Box \)

Before moving onto our next computation, we use the previous lemma to prove a helpful and more general result.

**Proposition 4.7.** Let \( n = 2m \). Then \( \mathbb{E}(m^2 - 1, u_n) \) contains \( \mathbb{P}^{m^2 - 1} \) as a closed subvariety.

**Proof.** The upper-right \( m \times m \) block is a copy of \( \mathfrak{gl}_m \) which commutes. \( \Box \)

**Example 4.8.** \( \mathbb{E}(3, u_4) \). Let \( A, B, \) and \( C \) be three pairwise commuting, linearly independent matrices in \( u_4 \).

First assume all \( (1,2) \) entries and all \( (3,4) \) entries are 0. Then equations (4.1), (4.2), and (4.3) are automatically satisfied, and the matrices \( A, B, \) and \( C \) multiply like \( 2 \times 2 \) matrices (only the upper right \( 2 \times 2 \) block is nonzero). Thus by the lemma, the corresponding elementary subalgebras are \( \text{Grass}(3, \mathfrak{gl}_2) \cong \text{Grass}(1, \mathfrak{gl}_2) \cong \mathbb{P}^3 \).
One family of subalgebras in this copy of \( \mathbb{P}^3 \) is of the form \( k \langle aE_{13} + E_{14}, bE_{13} + E_{23}, cE_{13} + E_{24} \rangle \), where \((a,b,c) \in A^3\). The Plücker embedding gives:

\[
\begin{pmatrix}
a & b & c \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \mapsto [c : -b : a : 1] \in \mathbb{P}^3
\]

We see that this corresponds to the affine open set in \( \mathbb{P}^3 \) where \( x_3 \) doesn’t vanish. There are three other families like this, and they form the other three standard affine open sets of \( \mathbb{P}^3 \).

A tedious calculation of checking cases and normalizing matrices shows that minus the above copy of \( \mathbb{P}^3 \), all elementary subalgebras are in one of the following families:

\[
\mathcal{F}_1 = \left\{ k \langle \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rangle \mid (a,b,c) \in A^3 \right\}
\]

\[
\mathcal{F}_2 = \left\{ k \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rangle \mid (a,b,c) \in A^3 \right\}
\]

\[
\mathcal{F}_3 = \left\{ k \langle \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rangle \mid (a,b,c) \in A^3 \right\}
\]

Embedding each family into \( \mathbb{P}^{19} \), it can be shown that they all embed as \( \mathbb{A}^3 \). \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) intersect in \( \mathbb{A}^3 - \mathbb{A}^2 \) (this intersection is \( \mathcal{F}_1 - \{a=0\} = \mathcal{F}_2 - \{c=0\} \)). \( \mathcal{F}_3 \) is disjoint from \( \mathcal{F}_2 \) and \( \mathcal{F}_3 \).

In conclusion, we find that:

\[
\mathbb{E}(3, \mathfrak{u}_4) \cong \mathbb{P}^3 \sqcup (\mathcal{F}_1 \cup \mathcal{F}_2) \sqcup \mathcal{F}_3 \cong \mathbb{P}^3 \sqcup (\mathbb{A}^3 \sqcup \mathbb{A}^2) \sqcup \mathbb{A}^3
\]

Example 4.9. \( \mathbb{E}(2, \mathfrak{u}_4) \). Without giving the details of the computation, any elementary subalgebra of dimension 2 inside of \( \mathfrak{u}_4 \) is in one of the following disjoint families:

\[
\text{Grass}(2, 4) = \left\{ k \langle \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rangle \right\}
\]

\[
\mathcal{F}_1 = \left\{ k \langle \begin{pmatrix} 0 & 1 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & d & e \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rangle \mid (a,b,c,d,e) \in A^5 \right\}
\]
\[ F_2 = \left\{ k \left\langle \begin{pmatrix} 0 & 1 & a & 0 \\ 0 & b & c & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \mid (a, b, c, d) \in \mathbb{A}^4 \right\} \]

\[ F_3 = \left\{ k \left\langle \begin{pmatrix} 0 & 1 & 0 & b \\ 0 & 0 & a & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & e \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \mid (a, b, c, d, e) \in \mathbb{A}^5 \right\} \]

\[ F_4 = \left\{ k \left\langle \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \mid (a, b, c) \in \mathbb{A}^3 \right\} \]

\[ F_5 = \left\{ k \left\langle \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \mid (a, b, c, d) \in \mathbb{A}^4 \right\} \]

Above, there is a subcollection of Grass(2, 4) such that its union with \( F_2 \) and \( F_4 \) is \( P^4 \). Together these are represented by the family:

\[ \left\{ k \left\langle \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \mid [a : b : c : d : e] \in \mathbb{P}^4 \right\} \]

The last observation of the previous example leads us to the following general result.

**Proposition 4.10.** \( \mathbb{P}^{n/2 - 2} \hookrightarrow \mathbb{E}(2, \mathfrak{u}_n) \).

**Proof.** For every element of \( \mathbb{P}^{n/2 - 2} \) there is a unique one-dimensional subspace with no \((1, n)\) entry. Paired with the matrix \( E_{1,n} \), this subspace is an elementary subalgebra. \( \square \)

The above proposition follows from the observation that \( E_{1,n} \) spans the center of the lie algebra \( \mathfrak{u}_n \).

We now prove a generalization of proposition 4.7.

**Proposition 4.11.** Let \( m = \lfloor n/2 \rfloor \), and let \( 0 < r \leq r' \leq m^2 \). Then \( \mathbb{E}(r, \mathfrak{u}_n) \) contains at least \( \binom{m^2}{r} \) copies of Grass\((r, r')\), which are not necessarily disjoint.

**Proof.** Consider the \( m \times m \) square matrix in the upper right corner. If we only allow nonzero elements to appear in this matrix, then the commutation relations will necessarily be satisfied. Hence, choose \( r' \) entries in the \( m \times m \) matrix. Then any \( r \)-plane in the \( r' \)-dimensional subspace generated by the chosen entries is an elementary subalgebra. Hence Grass\((r, r') \hookrightarrow \mathbb{E}(r, \mathfrak{u}_n) \). Notice the same \( r \)-plane may be in multiple Grass\((r, r')\) for different choices of \( r' \) entries.

If \( r = r' \), then we have \( \binom{m^2}{r} \) points, or copies of \( \mathbb{P}^0 \). If \( r = r' - 1 \), then the each of the \( \binom{m^2}{r} \) copies of \( \mathbb{P}_1 \) intersects \( m^2 - r \) of the other copies in exactly one point. In general, for
fixed $r$, each Grass$(r,r')$ intersects exactly $r'(m^2-r')$ other Grass$(r,r')$ in a Grass$(r,r'-1)$. Thus as $r'$ grows, we are just describing embeddings of Grassmannians. When $r' = m^2$, we have 1 copy of Grass$(r,m^2)$ which contains all of the Grass$(r,r')$ glued together as described. Hence, the proposition simply states that $\mathbb{E}(r,u_n)$ contains a copy of Grass$(m^2)$, and we have described a sort of decomposition for it.

Example 4.12. For $r = 1, \ldots, n-1$, define $u_{r,n-r} \subset \mathfrak{gl}_n$ to be the set of $n \times n$ matrices with non-zero entries only in the upper right $r \times (n-r)$ block. Then $u_{r,n-r}$ is an elementary subalgebra of dimension $r(n-r)$.

If we consider the action of $GL_n$ on $\mathbb{E}(r(n-r),\mathfrak{gl}_n)$ by conjugation, we see that the stabilizer of $u_{r,n-r}$ is $P_r$, the subgroup of $GL_n$ with zero entries in the lower left $(n-r) \times r$ block, and with invertible matrices in the upper left $r \times r$ block and lower right $(n-r) \times (n-r)$ block. Hence there is a one-to-one correspondence between cosets of $P_r$ in $GL_n$ and elements of the orbit of $u_{r,n-r}$ in $\mathbb{E}(r(n-r),\mathfrak{gl}_n)$.

There is also a one-to-one correspondence between $GL_n/P_r$ and Grass$(r,n)$.

We will need the following Lemma for a proof of the next theorem.

Lemma 4.13. If $\epsilon$ is an elementary subalgebra of $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$, then the projection of $\epsilon$ onto the $i$th component is an elementary subalgebra of $\mathfrak{g}_i$.

Proof. We can take the direct sum of the other components to be one Lie algebra, so we may work in the case where $s = 2$. Let $\lambda \in k$, and suppose $x,y \in \mathfrak{g}_1$ such that there exists $x',y' \in \mathfrak{g}_2$ with $(x,x'),(y,y') \in \epsilon$. Then since $(x+y,x'+y') \in \epsilon$, $(\lambda x,\lambda x') \in \epsilon$, $0 = [(x,x'),(y,y')] = ([x,y],[x',y'])$, and $(x^{[p]},x'^{[p]}) = (x,x')^{[p]} \in \epsilon$, the lemma follows.

Corollary 4.14. If $\epsilon_i$ is the elementary subalgebra obtained by projecting $\epsilon$, then $\epsilon = \epsilon_1 \oplus \ldots \oplus \epsilon_s$.

Proof. This follows from the definition of direct sum.

Theorem 4.15. Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ be restricted Lie algebras over $k$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$. Then we have an injective map of algebraic varieties:

$$\mathbb{E}(r_1,\mathfrak{g}_1) \times \ldots \times \mathbb{E}(r_s,\mathfrak{g}_s) \to \mathbb{E}(\sum r_i,\mathfrak{g})$$

which sends $(\epsilon_1, \ldots, \epsilon_s)$ to $\epsilon = \epsilon_1 \oplus \ldots \oplus \epsilon_s$. The map is an isomorphism if each $r_i$ is the maximal dimension of an elementary subalgebra of $\mathfrak{g}_i$.

Proof. Suppose we have the maximum condition. Then for any elementary subalgebra $\epsilon$ of $\mathfrak{g}$ of dimension $r$, it follows by the corollary to Lemma 4.13 that $\epsilon = \epsilon_1 \oplus \ldots \oplus \epsilon_s$. The dimension of $\epsilon_i$ must be $r_i$ in order for the dimensions to add up. Hence the map is surjective.

5. Functoriality

Consider an injection of restricted Lie algebras $\varphi : \mathfrak{h} \hookrightarrow \mathfrak{g}$. This induces an injective map $\varphi^* : \mathbb{E}(r,\mathfrak{h}) \hookrightarrow \mathbb{E}(r,\mathfrak{g})$ of projective varieties, because if $\epsilon$ is an elementary subalgebra of $\mathfrak{h}$ of dimension $r$, then $\varphi(\epsilon)$ is an elementary subalgebra of $\mathfrak{g}$ of dimension $r$. Thus $\mathbb{E}(r,\cdot)$ is a functor from the category of restricted Lie algebras and injective morphisms to the category of projective varieties.
Recall that $\mathfrak{sl}_n$ is the restricted lie algebra of traceless $n \times n$ matrices. $\mathfrak{sp}_{2n}$ is the restricted lie algebra of all matrices of the form:

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where $A, B, C \in \mathfrak{gl}_n$ and $B$ and $C$ are symmetric. $\mathfrak{so}_{2l+1}$ is the lie algebra of all matrices of the form

$$\begin{pmatrix} 0 & u & v \\ -v^t & A & B \\ -u^t & C & -A^t \end{pmatrix}$$

where $A, B, C \in \mathfrak{gl}_l$, $B$ and $C$ are antisymmetric, and $u$ and $v$ are row vectors of length $l$. Finally, $\mathfrak{so}_{2l}$ is the lie algebra of all matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where $A, B, C \in \mathfrak{gl}_l$, and $B$ and $C$ are antisymmetric.

It may be useful to consider some standard injections of the above lie algebras into each other to discern pieces of the projective variety of elementary subalgebras.

6. $E(3, \mathfrak{gl}_4)$

Eric and Julia have attempted to write down a description of the projective variety $E(3, \mathfrak{gl}_4)$. In the opening of their computation, they for some reason neglect to make the following computation:

Let $e_{\text{min}} = E_{14}$, so that the jordan type of $e_{\text{min}}$ is $[2][1]^2$. Then the nilpotent part of the centralizer of $e_{\text{min}}$ is:

$$Z(e_{\text{min}}) \cap \mathcal{N} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & T & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | T \in \mathfrak{gl}_2 \text{ is nilpotent} \right\}$$