CANTOR'S SERIES FOR VECTORS

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Decimal expansions have been generalized in several directions. First, there are the well-known expansions with arbitrary integer bases. For a sequence of integers \(q_1, q_2, \ldots (q_i \geq 2)\) Cantor [1] obtained the expansion
\[
x = \sum_{m=1}^{\infty} \frac{a_m}{q_1 q_2 \cdots q_m},
\]
where \(x \in [0, 1)\) and \(a_m \in \{0, 1, \ldots, q_m - 1\}\). See [5] for a survey on expansions with references to Cantor's series. There are also expansions with non-integer bases [4, 5, 8, 9], negative bases [7], and similar expansions exist for complex numbers [2].

In \(n\)-dimensional Euclidean space, matrix expansions and their associated transformations have been studied extensively [3, 8]. The purpose of this note is to generalize Cantor series (1) to matrix expansions and to give a list of examples of such expansions.

Let \(n\) be a fixed positive integer and \(Q_1, Q_2, \ldots\) be a sequence of nonsingular \(n \times n\) matrices. We take \(|\cdot|\) to be a norm on \(n\)-dimensional Euclidean space and take \(||\cdot||\) to be a compatible matrix norm (i.e., \(|Qx| \leq ||Q|| \cdot |x|\)). (See Lancaster [6] for material on matrix norms.) Let \(I = \times Q_n [0, 1)\) and assume \(|x| \leq B\) for \(x \in I\). Next we make some fundamental definitions.

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\[
T^{(0)}(x) = Q_0x - [Q_0x] \\
T_m = T^{(m)} \circ T^{(m-1)} \circ \cdots \circ T^{(1)} \\
a_m(x) = [Q_m x], \\
a_n(x) = [Q_m(T_{m-1}(x))]
\]

where \([y_1, y_2, \ldots, y_n]^T = ([y_1], [y_2], \ldots, [y_n])^T\) and \([\ ]\) is the usual greatest integer function in this last expression.

**Theorem.** Assume \(\sup_i \|Q_i^{-1}\| < 1\). Then
\[
x = \sum_{i=1}^{\infty} (Q_i Q_{i-1} \cdots Q_0)^{-1} a(x).
\]

**Proof.** \(x = Q_0^{-1} Q_1 x = Q_0^{-1} (a_0(x) + T_0(x)) = Q_0^{-1} a_0(x) - Q_0^{-1} T_0(x).\) For our induction, assume for a positive integer \(m\) that
\[
x = \sum_{i=1}^{m} Q_i^{-1} Q_{i-1} \cdots Q_1^{-1} a(x) + Q_i^{-1} Q_{i-1} \cdots Q_1^{-1} T_{m}(x).
\]
Then $Q_1^{-1}Q_2^{-1} \cdots Q_m^{-1}T_m(x) = Q_1^{-1}Q_2^{-1} \cdots Q_m^{-1}(a_{m+1}(x) + T_{m+1}(x))$ and (3) holds for all $m$. Therefore, since $T_m(x) \leq B$,

$$\lim_{m \to \infty} \left| x - \sum_{i=1}^{m} Q_i^{-1}Q_2^{-1} \cdots Q_i^{-1}a_i(x) \right| = \lim_{m \to \infty} \left| Q_1^{-1}Q_2^{-1} \cdots Q_m^{-1}T_m(x) \right|$$

$$\leq \lim_{m \to \infty} B \left\| Q_1^{-1} \right\| \left\| Q_2^{-1} \right\| \cdots \left\| Q_m^{-1} \right\| \leq B \lim_{m \to \infty} \left( \sup_{i} \left\| Q_i^{-1} \right\|^m \right) = 0.$$  

Thus (2) holds and the theorem is proved.

Clearly the theorem holds when $\left\| (Q_mQ_{m-1} \cdots Q_2)^{-1} \right\| \to 0$ as $m \to \infty$. Under the assumptions of the theorem, the convergence is geometric with rate $B(\sup_i \left\| Q_i^{-1} \right\|)^m$.

Next we give some examples, both general and numerical, of the theorem. As noted below, examples 1, 2, and 3 are known. In the case $n > 1$ and the $Q_i$ are not all identical, the result was not known. Example 4 gives a numerical example of this situation.

**Example 1.** If $n = 1$ and $\left\| \cdot \right\|$ are both the usual absolute value on the real line, we can obtain the results mentioned above for any $x \in [0,1)$. If $Q_i = q$ is a positive integer then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{q^m}, \quad a_m(x) \in \{0,1,\cdots,q-1\}.$$  

This is the usual $q$-adic expansion. If $Q_i = q$ where $\beta > 1$ and $\beta$ is not an integer, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\beta^m}, \quad a_m(x) \in \{0,1,\cdots,[\beta]\}.$$  

These $\beta$-expansions have been extensively studied [3, 4, 5, 9, 10]. If we let $Q_i = \gamma_i, \gamma_i \in R, \inf_i |\gamma_i| > 1$, then

$$x = \sum_{m=1}^{\infty} \frac{a_m(x)}{\gamma_1\gamma_2 \cdots \gamma_m}.$$  

This last formulation allows expansions with negative bases (e.g., $-10$) as well as mixtures of integral and non-integral positive and negative numbers. For some material on expansions with negative radices, see [7].

**Example 2.** Let $n = 2$. If $Q_i \equiv \begin{bmatrix} r-q & q \\ q & r \end{bmatrix} = Q$, then $Qx = (r_1x_1 - q_1x_2, q_1x_1 + r_2x_2)^T$ which is equivalent to $(x_1 + ix_2)(r + iq) = (r_1x_1 - q_1x_2) + i(q_1x_1 + r_2x_2)$. Therefore if we take

$$|x| = \sqrt{x_1^2 + x_2^2} \text{ and } \|Q\| = \sqrt{r^2 + q^2},$$
we have \(|Qx| \leq \|Q\| \cdot |x|\) by the theory of complex numbers. Of course, \(\|Q^{-1}\| = (r^2 + q^2)^{-1/2}\). Thus

\[
x = \sum_{m=1}^{\infty} Q^{-m} a_m(x),
\]

and the expansion is valid for complex numbers \(x = x_1 + x_2i\) (\(|x_j| < 1\)) with complex base satisfying \(r^2 + q^2 > 1\). This transformation and the expansion have been studied by Fischer [2, 3].

**Example 3.** If \(|x|^2 = \sum_{i=1}^{n} x_i^2\), then \(\|Q\|^2 = \sum_{i,j} q_{ij}^2\) is a compatible matrix norm.

As a specific numerical example with \(n = 2\), we take \(x^T = (1/2, 3/4)\) and \(Q = \begin{bmatrix} 2 & 0 \\ 2/3 & 4/3 \end{bmatrix}\). Then \(Q^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/4 & 3/4 \end{bmatrix}\) and \(\|Q^{-1}\|^2 = 7/8 < 1\). We compute

\[
\begin{align*}
a_1(x) &= (1, 1)^T \\
a_2(x) &= a_3(x) = a_4(x) = (0, 0)^T \\
a_5(x) &= (0, 1)^T.
\end{align*}
\]

The fifth order approximation to \(x\) is \(Q^{-1}a_1(x) - Q^{-5}a_5(x) = (1/2, 75511024)^T\).

Finally we check the rate of convergence.

\[
|x - Q^{-1}a_1(x) - Q^{-5}a_5(x)| = |(0, 13/1024)^T| = (13/1024) < 2(7/8)^{5/2}.
\]

The last inequality is by the guaranteed rate of convergence given in the proof of the theorem where \(B = \sup x \in \mathbb{R} \|x\| = 2\).

**Example 4.** If \(|x| = \max \{|x_i| : 1 \leq i \leq n\}\), then \(\|Q\| = n \max \{|q_{ij}| : 1 \leq i, j \leq n\}\) is a compatible matrix norm. For the matrices given in Table 1 below, we have

| Table 1. Expansion of \((1/2, 1/2)^T\) |
|---|---|---|---|
| \(m\) | 1 | 2 | 3 | 4 |
| \(Q_m\) | \(\begin{bmatrix} 3/2 & 3/2 \\ -3/2 & 3/2 \end{bmatrix}\) | \(\begin{bmatrix} 4 & 0 \\ 4/3 & 8/3 \end{bmatrix}\) | \(\begin{bmatrix} 8 & -4 \\ -8 & 8 \end{bmatrix}\) | \(\begin{bmatrix} 3/2 & -3/2 \\ 3/2 & 3/2 \end{bmatrix}\) |
| \(Q_m^{-1}\) | \(\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}\) | \(\begin{bmatrix} 1/4 & 0 \\ -1/8 & 3/8 \end{bmatrix}\) | \(\begin{bmatrix} 1/4 & 1/8 \\ 1/4 & 1/4 \end{bmatrix}\) | \(\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}\) |
| \(\|Q_m^{-1}\|\) | 2/3 | 3/4 | 1/2 | 2/3 |
| \(a_m(x)\) | (0, 1) | (2, 0) | (−3, 5) | (0, 1) |
| \(A_m(x)\) | (1/3, 1/3) | (7/12, 5/12) | (97/192, 91/192) | (1/2, 1/2) |


\[(1/2,1/2)^T = Q_1^{-1}(1,0)^T + (Q_2Q_1)^{-1}(2,0)^T + (Q_3Q_2Q_1)^{-1}(-3,5)^T + (Q_4Q_3Q_2Q_1)^{-1}(0,1)^T.\]

Let \(A_m(x) = \sum_{s=1}^{m} Q_s \cdots Q_1 a_s(x)\) in Table 1.

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References


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