Math 108: Pre-calculus
10/2/2012 - Midterm 1 (50 minutes)

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USC ID:

Signature:

1. Write your name and ID number in the spaces above.
2. Show all your work (answers need explanations) and circle your final answer. Simplify as much as you can your answer.
3. No books, no cell phone or other notes are permitted.
4. Calculators are NOT permitted.
5. There are 5 Problems, total grade is on 100 points.
6. Open your exam only when you are instructed to do so.

1. Problem 1: 14 points
2. Problem 2: 14 points
3. Problem 3: 21 points
4. Problem 4: 25 points
5. Problem 5: 26 points

Total: /100 points
Problem 1. (14 points)

Equation 1: \( x^2 - 2x = 2 \)

1. (6 points) Using the graph of the function \( f(x) = x^2 - 2x \) given below, give the number of solutions Equation 1 has, and then give an approximate values of the solutions (if they exist).

   There are 2 intersections, so there are 2 solutions seem to be
   \[ x = 2.48 \]
   \[ x = -0.48 \]

2. (8 points) Solve Equation 1 algebraically: give the exact values of the solutions (if they exist).

   Equation is
   \[ x^2 - 2x - 2 = 0 \]
   The solutions are
   \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
   with \( a = 1 \), \( b = -2 \), \( c = -2 \)

   \[ x = \frac{2 + \sqrt{4 - 4(-2)(-2)}}{2} = 1 + \frac{\sqrt{12}}{2} = 1 + \sqrt{3} \]
   \[ x = \frac{2 - \sqrt{4 - 4(-2)(-2)}}{2} = 1 - \frac{\sqrt{12}}{2} = 1 - \sqrt{3} \]

   (Note that \( \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3} = 2\sqrt{3} \))
Problem 2. (14 points)

Equation 2: \( \sqrt{2x^2 + 1} = 2x - 1 \)

3. (7 points) Using the graph of the function \( f(x) = \sqrt{2x^2 + 1} \) given below, give the number of solutions Equation 2 has, and then give an approximate values of the solutions (if they exist).

There is one intersection, so there is only one solution, which seems to be \( x = 2 \)

4. (7 points) Solve Equation 2 algebraically: give the exact values of the solutions (if they exist).

\[
\sqrt{2x^2 + 1} = 2x - 1 \quad \iff \quad 2x^2 + 1 = (2x - 1)^2
\]

AND \( 2x - 1 \geq 0 \) (otherwise it cannot be a square)

*From the graph, we only have 1 solution, close to \( x = 2 \)

\[
\begin{align*}
2x^2 + 1 &= 4x^2 - 4x + 1 \\
2x^2 - 4x &= 0 \\
2x(x - 2) &= 0
\end{align*}
\]

\[
\begin{cases}
x = 0 \\
x = 2
\end{cases}
\]

but \( x = 0 \) is not a solution, since \( 2x - 1 < 0 \) for \( x = 0 \)

Solution is therefore \( \{x = 2\} \)
Problem 3. (21 points)
You have 3 points $A = (-3, -1)$, $B = (1, 2)$ and $C = (\frac{5}{2}, 0)$.

1. (3 points) Place the 3 points, and draw the triangle $ABC$.

2. (6 points) Compute the distances $AB$, $BC$ and $AC$.

   \[
   AB = \sqrt{(1 - (-3))^2 + (2 - (-1))^2} = \sqrt{16 + 9} = \sqrt{25} = 5
   \]

   \[
   BC = \sqrt{(\frac{5}{2} - 1)^2 + (0 - 2)^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{9 + 16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}
   \]

   \[
   AC = \sqrt{(\frac{5}{2} - (-3))^2 + (0 - (-1))^2} = \sqrt{(\frac{11}{2})^2 + 1} = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}
   \]

3. (2 points) What are the coordinates of the midpoint of $[AC]$?

   \[
   \text{Midpoint has coordinates } \left( \frac{-1}{4}; -\frac{1}{2} \right) = \left( \frac{5 + (3)}{2}; \frac{0 + (-1)}{2} \right)
   \]

4. (5 points) Give the equation of the circle $C$ whose diameter is $[AC]$. Does $B$ lie on that circle?

   \[
   \text{Circle has center the midpoint of } [AC] \text{, diameter } \frac{5\sqrt{5}}{4}
   \]

   \[
   \text{radius } \frac{5\sqrt{5}}{8} \text{ equation is } \left( x + \frac{1}{4} \right)^2 + \left( y - \frac{1}{2} \right)^2 = \left( \frac{5\sqrt{5}}{8} \right)^2 \Rightarrow \text{Yes, } B \text{ is on the circle.}
   \]

5. (5 points) Place the point $D = (-3/2, -3)$. What is the midpoint of $[BD]$? Does $D$ lie on the circle $C$?

   \[
   \text{The midpoint is also the midpoint of } AC
   \]

   \[
   \left( \frac{-1}{4}, -\frac{1}{2} \right) = \left( \frac{-3/2 + 1}{2}; \frac{-3 + 2}{2} \right)
   \]

   \[
   \rightarrow BD \text{ is also a diameter of the circle }
   \]

   \[
   \text{Its midpoint is the center? } D \text{ is on the circle.}
   \]
Problem 4. (25 points)
You are given the following function

\[ f(x) = \begin{cases} 
-1 & \text{if } x \leq 0 \\
2\sqrt{x} & \text{if } 0 < x < 1 \\
3 - x & \text{if } x \geq 1 
\end{cases} \]

1. (3 points) Compute \( f(0), f(1/4), f(2) \).

\[ f(0) = -1, \quad f\left(\frac{1}{4}\right) = 2\sqrt{\frac{1}{4}} = 1, \quad f(2) = 3 - 2 = 1 \]

2. (14 points) Draw the graph of the function \( f(x) \).

3. (4 points) What are the \( x \)-intercepts? What are the \( y \)-intercepts?

The \( x \)-intercept is \( x = 3 \) \( (x = 0 \text{ is NOT on the graph}) \)

The \( y \)-intercept is \( y = 0 \) \( (y = 0 \text{ is NOT on the graph}) \)

4. (4 points) For which value of \( x \) is the function maximal?

The function is maximal for \( x = 1 \)
Problem 5. (26 points)
You want to build a rectangular garden. You already have a wall, and you put fences on the other sides. You have $2,000. Fence costs $1 per ft, but you also want to decorate the wall with plants, which you estimate will cost $3 per ft. We denote by $x$ the length of the wall that you will cover with plants, and by $y$ the width of your garden (see the illustration).

1. (3 points) How are $x$ and $y$ related (because of your budget constraint)?

\[
\frac{3x}{2} + \frac{2y+x}{x} = 2,000 \quad \text{cost on the wall} \quad \text{cost of the fence}
\]

It simplifies to $2x + y = 1,000$.

2. (5 points) Express the Area of your garden only in terms of $x$.

\[
\text{Area} = x \cdot y = x \cdot (1,000 - 2x) = -2x^2 + 1,000x
\]

3. (10 points) Draw the graph of the function $A(x) = -2x^2 + 1000x$ (don't forget to put the scale on the axis). Label the $x$-intercepts (write down their value).

The graph of $A(x)$ is shown with the $x$-intercepts marked.

4. (8 points) What are the dimensions of your garden ($x$ and $y$) corresponding to that maximal area? What is the maximal Area you can obtain?

The maximum is attained for $x = 250$

\[
y = 1,000 - 2x = 500
\]

The area is then $250 \times 500 = 125,000 \text{ sq ft.}$