Problem 1: There are 16 socks (8 pairs) in a drawer. 4 are blue and long, 6 are blue and short, 2 are green and short, 4 are green and long. You pick two pairs at random.
   (1) What is the probability that you pick two socks of the same color?
   (2) What is the probability that you pick two socks of the same length?
   (3) You picked two short socks in the dark. What is the (conditional) probability that they are of the same color?

Problem 2: There are \( n \) independent components, each one working with probability \( \frac{1}{2} \). The system works if at least one of them is working.
   (1) What is the probability that the system is working?
   (2) You observe the system, and it is working. What is the (conditional) probability that component 1 is working?

Problem 3: There are 4 black balls, and 4 white ones in an urn. You pick 4 balls.
   (1) Compute the probability mass function of the number of black balls you get.
   (2) You win $2 per black ball you get, and an extra $20 if you get all 4 black balls. What is your expected gain?

Problem 4: You send a signal \( s \) (an integer number) to one of your friend. The signal received is \( s + X \), where \( X \) is a normal r.v. with mean 0 and Variance \( \frac{1}{2} \).
   (1) What is the probability that the signal received is in \([s - 1/2, s + 1/2]\), so that your friend won’t misunderstand it.
   (2) The friend sends you back the signal. What you receive is then \( s + X + Y \), where \( Y \) is another normal r.v. with mean 0 and Variance 1, independent of \( X \). What is the distribution of the signal that you receive?
   (3) What is the probability that the signal that you receive is not in the interval \([s - 1/2, s + 1/2]\) (so that you would misunderstand it)?

Problem 5: Let \( X \) have a Gamma distribution: \( f_X(x) = xe^{-x}, x \geq 0 \).
   (1) What is \( E[X] \)?
   (2) Compute the c.d.f of \( Y = \sqrt{X} \). Deduce its p.d.f.

Problem 6: 3 blue balls, 2 red, 1 black are placed in an urn.
   You pick 2 balls (without replacement). You win if you get at least one red ball, but picking the black ball make you lose (even if you got a red ball).
   (1) (5 points) What is the probability that you win?
   (2) (5 points) Your friend plays and win. What is the (conditional) probability that he got exactly 1 red ball?
Problem 7: There are two boxes in front of you, with money in it: the first box contains five $1 bills, the second box contains one $1 bill, one $5 bill, one $10 bill, one $20 bill and one $50 bill. You then pick a box at random, and pick ONE bill at random in the box you chose.

1. What is the probability of picking $50?
2. Find the expected value of the money you get.
3. You picked one bill: it’s a $1. What is the (conditional) probability that the box you picked the bill from is the one containing only $1 bills?
4. You’re allowed to pick another bill: you decide to take your chance in the other box. What is the probability that you get $50?

Problem 8: The quantity of gas consumed during your commute (directly converted in $), denoted \( X \), is a Normal random variable, with mean 5 and variance 1.

1. What is the probability that you consume more than $6 of gas in one commute?
2. What is the probability that, after 4 commutes, you spent less than $16 in gas?

Problem 9: You place \( m \) men and \( m \) women in a row, at random.

1. What is the probability that the last one in the line is a woman?
2. Let \( X \) be the number of men in the \( m \) first persons. Compute \( E[X] \). (Hint: use \( X_i \), the indicator function that the \( i \)th person is a man)
3. Compute \( Var(X) \). (Hint: you will need to compute \( Cov(X_i, X_j) \), make a difference when \( j = i \) and \( j \neq i \)).
4. Use Chebichev’s inequality to get a bound on the probability that there are more than 75% of men in the first \( m \) persons.

Problem 10: You throw a fair coin many times. Each time you get Head, you gain $1, and each time you get Tail, you lose $1. We denote \( X_i \) the amount you gain on the \( i \)th flip (+1 if Head, −1 if Tail), and \( X = \sum_{i=1}^{N} X_i \) the total gain after \( N \) independent flips.

1. Compute \( E[X] \) and \( Var(X) \).
2. Use the Central Limit Theorem to approximate the probability that, after 900 turns, you have won more than $45.

Problem 11: You have a reserve of 16 light bulbs. The \( i \)th light bulb has a lifetime \( X_i \) (in years), an exponential random variable, with parameter \( \lambda = 1/2 \). You do not need to buy any more light bulbs for a period of time which is \( X = \sum_{i=1}^{16} X_i \), where the \( X_i \)’s are independent.

1. Compute the expectation and variance of \( X \).
2. Use the Central Limit Theorem to approximate the probability that you last less than 20 years with your reserve?