Polya urn

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1 Intro

In probability theory, statistics and combinatorics Urn Models have a long and colorful history. Classical mathematicians Laplace and Bernoulis, amongst others, have made notable contributions to this class of problems. For example, Jacob Bernoulli in his Ars Conjectandi (1731) considered the problem of determining the proportions of different-colored pebbles inside an urn after a number of pebbles are drawn from the urn. This problem also attracted the attention of Abraham de Moivre and Thomas Bayes.

Urn models can be used to represent many naturally occurring phenomena, such as atoms in a system, biological organisms, populations, economic processes, etc. In fact, in 1923 Eggenberger and Polya devised an urn scheme (commonly known as Polya Urn) to model such (infinite) processes as the spread of contagious diseases. In one of its simplest forms, the Polya Urn originally contains x white balls and y black balls. A ball is randomly drawn from the urn and replaced back to the urn along with additional (fixed) h balls of the same color as the drawn ball. This process continues forever. The fraction of white balls in the urn converges almost surely to a random limit, which has a beta distribution with parameters x/h and y/h.

There are a number of generalizations of the basic Polya Urn model. For example, when a ball is drawn, not only it is returned to the urn, but if it is white, then A additional white balls and B additional black balls are returned to the urn, and if the drawn ball is black, then C additional white balls and D additional black balls are returned to the urn. This set-up involves the matrix \([A \ B; C \ D]\), known as the replacement matrix; a rule used for effecting change in the composition and number of balls after each drawing.

Or, instead of only two differently colored balls, there may be several differently colored balls present in the urn. Or, instead of the additional fixed h balls noted above, after every draw a different number, that may or may not be deterministic, may be added to the urn.

In the rest of the paper, Polya urn model is carefully studied in section 2. As an application of polya urn, edge-reinforced random walk is studied in section 4. Finally, numerical simulation is presented in section 5.

2 Polya Urn study

2.1 Definition and Notations

A Polya urn is an urn containing balls of up to K (K \(\geq 2\)) different colors. The urn evolves in discrete time steps - at each step, one ball is sampled uniformly at random; The color of the withdrawn ball is observed, and two balls with that color are returned to the urn.

2.2 Count vector as a markov chain

We start with the markov property of Polya urn, which is not only of importance in itself, but also serves as an important building block in later analysis.
Lemma 2.1. Let $X_{jn}$ denote the number of balls with $j$th color in the urn at time step $n$, then the vector \{(X_{1n}, \ldots, X_{Kn})\}_{n \geq 1}$ is a homogeneous Markov Chain.

To show it is a Markov Chain, observed that at each step one ball is sampled uniformly at random. Picking a ball of a particular color only relies on balls in the last step. Since the transition matrix does not depend on time index $n$, it is homogeneous.

Despite the simple proof, it is not clear how this result can be used since the state space of the vector is infinite and it never goes back to the same state.

2.3 Ratio as a Martingale

Another quantity of interests is the ratio of number of balls with different colors. Statistically it is very important and sometimes good enough to know this ratio.

Note it is not a Markov chain. For example, at one time step this ratio is one half. Then the next step we don’t know what the ratio value would be since the ratio may come from the situation with 1 white ball plus 1 black ball or the situation with 2 white balls plus 2 black balls, which, however, give different next positions.

But it is a martingale as stated in the following lemma.

Lemma 2.2. Let $\rho_i = \frac{X_i}{\sum_j X_{jn}}$, $n \geq 0$ be the ratio of $i$th color ball in the urn at the conclusion of step $n$. Define $\mathcal{F}_n = \sigma(c_i)$, where $c_i$ is the color of $i$th ball. Then $(\rho_n)_{n \geq 0}$ is a Martingale w.r.t. $\mathcal{F}_{n-1}$.

Proof.

$$E[\rho_{n+1} | \mathcal{F}_n] = E\left[\frac{X_{i,n+1}}{\sum_j X_{jn,n+1}} | \mathcal{F}_n\right] = E\left[\frac{X_{i,n+1}}{\sum_j X_{j,n+1}} | \mathcal{F}_n\right] = \frac{1}{\sum_j X_{j,n+1}} E[X_{i,n+1} | \mathcal{F}_n]$$

$$= \frac{1}{\sum_j X_{j,n} + n + 1} E[X_{i,n} + 1(c_{n+1} = i) | \mathcal{F}_n] = \frac{1}{\sum_j X_{j,n} + n + 1} (X_{i,n} + \frac{X_{i,n}}{\sum_j X_{j,n} + n}) = \rho_n$$

Which implies that as the urn evolves the expected ratio of a particular color stays the same. Furthermore, when $n \to \infty$, we have:

Lemma 2.3. Starting from 1 ball of each color, $\lim \rho_n$ exists almost surely. $y_{\infty}$ is uniform in $[0, 1]$.

Proof. Since $\rho_i$ is a martingale and upper bounded by 1, by convergence theorem, we know the limit exists almost surely. The distribution of $y_{\infty}$ is uniform is proved later in more general cases.

As time goes on sufficiently long, the proportion of picking balls of different colors stabilizes. So, what is the proportion’s distribution? In the next subsection we are going address this problem.

2.4 Proportion Limiting Distribution

Seeing that Polya urn is a Markov Chain, it is natural to seek its limiting discrete distributions underlying the process in closed form. Particularly, we are interested in how many balls of a color would be picked when the process goes on for a sufficiently long time. As the first step, we give the closed form probability of picking a given number of balls starting from a particular configuration.

Lemma 2.4. Suppose a Polya urn starts with $X_{j0}$ balls with $j$th color ($j = 1, \ldots, K$) and $\tau_0 = \sum_j X_{j0}$. Let $X_{jn}$ be the number of $j$th ($j = 1, 2, \ldots, K$) color balls drawn in the Polya urn after $n$ draws, then

$$P(X_{jn} = y_j, j = 1, \ldots, m) = \frac{\prod_{j=1}^{K} [X_{j0}(X_{j0} + 1) \cdot \ldots \cdot (X_{j0} + y_j - 1)]}{\tau_0(\tau_0 + 1) \cdot \ldots \cdot (\tau_0 + n - 1)} \binom{n}{y_1, \ldots, y_m}$$
Proof makes use of the markov property by writing the probability as the product of probability of n consecutive draws and also notice there are \( \binom{n}{y_1, \ldots, y_m} \) ways of picking \( n \) balls. Details are in the appendix.

While we see the transition probability is additive in time, Theorem 2.4 gives its closed form based on starting and finishing numbers of balls. Starting from this, we could derive the expectation and variance of \( \bar{X}_{jn} \) and also the Polya urn’s limiting distribution stated as follows.

**Theorem 2.5.** Suppose a Polya urn starts with \( X_{j0} \) balls of \( j \)th color \( (j = 1, \ldots, K) \). Let \( \bar{X}_{jn} \) be the number of \( j \)th \( (j = 1, 2, \ldots, K) \) color balls drawn in the Polya urn after \( n \) draws, when \( n \to \infty \),

\[
\left( \frac{\bar{X}_{1n}}{n}, \ldots, \frac{\bar{X}_{Kn}}{n} \right) \xrightarrow{D} \text{Dir}(X_{10}, \ldots, X_{K0}). \tag{2}
\]

Proof follows Lemma 2.4 and Stirling approximation. Details are given in the appendix.

**Remark** Theorem 2.5 tells what proportion of colors of balls we are supposed to get if we start with some known number of balls and draw infinitely many times. This depends on how many balls of each color we start from. Asymptotically, we will have a fixed proportion to sample different colors of balls and this proportion should follow Dirichlet distribution with parameters the initial numbers of balls.

### 2.5 An alternative perspective – Bayesian generative model

So far, we have reviewed Polya urn in its classical scheme where one repeatedly picks balls with different colors from one urn. Each time picking a specific color relies on what we already have in the urn at that time. In this section we provide another perspective reviewing Polya urn. It is more of a Bayesian treatment and it’s easy to be extended to more general case, as we will show in section 2.6 where \( K = +\infty \).

We assume balls are drawn from one distribution \( G \) independently instead of drawn from one time-adaptive distribution. \( G = \text{multi}(p_1, \ldots, p_K) \) is a probability measure over all \( K \) colors and specifies a multinomial distribution. As standard Bayesian treatment, we treat \( G \) as a random variable and assign a prior distribution \( \text{Dir}(\alpha_1, \ldots, \alpha_K) \) to it where \( \alpha_i = X_{i0} \quad \forall i \). Then the evolving process of this new urn is described as follows,

\[
G \sim \text{Dir}(\alpha_1, \ldots, \alpha_K) \\
\text{color}(i) \sim G \quad \forall i. \tag{3}
\]

Now it is no longer sequential picking, but all balls are generated independently. Since the true distribution of \( G \) is unknown, We need to infer the conditional distribution of \( G \) given the observed balls. Whenever a new ball is observed, we update our belief on \( G \) and then make prediction on the upcoming ball. Note that the underlying distribution of \( G \) is assumed to be unchanged.

So how does this relate to standard Polya urn model? The next Lemma shows that after observing a given number of balls, one has the same prediction on the color of the next ball as the standard Polya urn model.

**Theorem 2.6.** By choosing prior \( \text{Dir}(X_{10}, \ldots, X_{K0}) \) where \( X_{j0} \) denotes the initial number of balls with \( j \)th color \( (j = 1, 2, \ldots, K) \), the posterior distribution of \( G \) given \( X_j \) \( j \)th color balls, is \( \text{Dir}(X_1, \ldots, X_K) \). The expected probability of picking an \( i \)th color ball is \( \frac{X_i}{\sum_j X_j} \), which is the same as given by Polya urn model rule.

The proof follows the fact that Dirichlet and multinomial are conjugate and that it’s easy to compute expectation of random variable from dirichlet distribution.

**Remark** While the probability of picking one color ball is changing during the process in both views of Polya urn, the mechanisms are different. In the original view of polya urn, it relies on the assumption that we always pick balls uniformly in the urn. Since number of balls in the urn is changing, we have time-adaptive probability. In the alternative view, balls are assumed to be drawn from one underlying probability distribution. What sequentially changes is the belief we have on its distribution.
Not surprisingly, as in non-Bayesian analysis, we could derive the probability of observing a given number of balls starting from some balls as stated in the following lemma.

**Lemma 2.7.** Let \( X_j \) be the number of \( j \)th \((j = 1, 2, \ldots, K)\) color balls observed, \( \tilde{X}_{jn} \) be the number of \( j \)th color balls appeared in the next \( n \) draws according to the conditional probability of \( G \), then

\[
p(\tilde{X}_{jn} = y_j, \forall j | \sum_j y_j = n) = \frac{\Gamma(\sum_j X_j) \prod_{j=1}^K \Gamma(X_j + y_j)}{\Gamma(\sum_j X_j + y_j) \prod_{j=1}^K \Gamma(X_j)} \left( \frac{n}{y_1, \ldots, y_m} \right)
\]

**Proof.** For a fixed \( G = (p_1, \ldots, p_K) \), we know the distribution of \((\tilde{X}_{1n}, \ldots, \tilde{X}_{Kn})\) is multinomial. Since \( G \) is a random variable with conditional probability \( \text{Dir}(X_1, \ldots, X_K) \), we integrate over \( G \) and get the result in equation (4).

**Remark** Lemma 2.8 recovers the result in Lemma 2.4 but in a different way. The proof is more straightforward since all subsequential samples are considered drawn independently from conditional distribution of \( G \), which could be computed once given \( X_j \).

**Corollary 2.8.** Let \( X_j \) be the number of \( j \)th \((j = 1, 2, \ldots, K)\) color balls observed, \( \tilde{X}_{jn} \) be the number of \( j \)th color balls appeared in the next \( n \) draws according to the conditional probability of \( G \), when \( n \to \infty \)

\[
p(\tilde{X}_{jn}/n = y_j, \forall j | \sum_j y_j = 1) \overset{n \to \infty}{\to} \text{Dir}(X_1, \ldots, X_K).
\]

Apparently this corollary recovers Theorem 2.5.

### 2.6 Extension: Polya urn with unbounded color numbers

We extend Polya urn with fixed number of color balls to Polya urn with possibly infinitely many colors via the generative model discussed in 2.5.

Our motivation is that, in real life data, we don’t know the total number of “colors” beforehand. Therefore it is unwise to set a fixed \( K \) before we observe the whole data population. Moreover, even if we observe the entire population, we have no clue whether the upcoming “ball” has a different “color” than the balls observed so far. For example, in Brain Image research, one models an unknown number of spatial activation patterns in fMRI images. Another example, in text modeling, one models an unknown number of topics across several corpora of documents. Current Polya urn type model is limited in these flexible situations. **Is there a principled way to deal with this more flexible model?**

It turns out to be highly non-trivial to do that due to the fact in equation (3) \( G \) is defined in a finite space, thus too restricted to deal with unbounded number of colors. One needs to extend its space to infinite space. Particularly, one defines a proper prior of \( G \) that has a support over infinite space.

Our solution resorts to Dirichlet Process (DP) [4, 3]. DP defines a distribution over distribution where the space could be infinite. We refer the formal definition of DP with parameter to [3] and give an equivalent one here.

**Definition** Let \( \mathcal{X} \) be a set and \( \mathcal{A} \) be a \( \sigma \)-field of subsets of \( \mathcal{X} \). \( \alpha \) is a positive number and \( G_0 \) is a probability measure on \((\mathcal{X}, \mathcal{A})\). We say a Dirichlet Process has sample path \( G \) to be a probability on \((\mathcal{X}, \mathcal{A})\). Particularly, for any measurable partition of \( \mathcal{X} \), say \( \{A_i\}_{i=1}^n \), we have \((G(A_1), \ldots, G(A_n)) \sim \text{Dirichlet}(\alpha G_0(A_1), \ldots, \alpha G_0(A_n))\).

To make sure this indeed defines a proper random process, we need to examine that the Kolmogorov consistency conditions are satisfied. We refer to [3] for details of examination. In what follows we assume it defines a proper random process.

Now we are ready to define the **generalized Polya urn model**

\[
G \sim DP(\alpha, G_0) \quad \text{color}(i) \sim G \quad \forall i
\]
where the \( G_0 \) could be any distribution over colors, and scalar \( \alpha \) controls how likely one is to observe a new color. Dirichlet process has the following two good properties.

**Lemma 2.9.** A sample from Dirichlet Process is a discrete distribution, made of up countably infinite number of point masses.

**Lemma 2.10.** Let \( G \) be a sample from Dirichlet process \( DP(\alpha, G_0) \) on the color space and let \( c_n \) be the color of \( n \)th ball observed, then the conditional probability of \( G \) is still Dirichlet Process with parameters \( \alpha \) and \( G_0 + \sum_{n=1}^{N} \delta_{c_n} \).

**Remark** Lemma 2.9 makes it possible for us to use DP to model clustering problem. If not discrete, then the possibility of picking two identical points from \( G \) is zero. Lemma 2.10 tells that Dirichlet Process is conjugate with Multinomial distribution, also, making computation tractable.

**Theorem 2.11.** Let \( X_i \) denote the number of balls with \( i \)th color drawn independently from generalized Polya urn model defined in (6), then the probability of observing the next ball with \( i \)th color is \( \frac{x}{\sum x_i + \alpha} \); the probability of observing the next ball with a new color is \( \frac{\alpha}{\sum x_i + \alpha} \).

The proof follows Lemma 2.10 and expectation of random variable from Dirichlet Process.

**Remark** It is easy to see that (6) is an extension of standard Polya urn model since it allows us to draw a new color with non-zero probability. When \( \alpha \) goes to 0, we recover standard Polya urn model (3); when \( \alpha \) goes to infinity, (6) becomes drawing balls from stationary distribution \( G_0 \). Meanwhile, it also explicitly allows flexible choice of \( G_0 \), which is the base probability distribution over color space. For instance, it is easy to incorporate other attributes of colors in order to pick a new color.

## 3 Generalized Polya Urn with a Replacement Vector

We generalize the simple Polya urn containing two colors, black and white. Instead of replacing 2 balls of the observed color, a deterministic number of balls of the observed color are returned to the urn. Define replacement vectors \( \vec{r} = (r_0, r_1, \ldots) \) and \( \vec{w} = (w_0, w_1, \ldots) \), for the \( n \)-th time of observing a black ball, \( r_n \) number of black balls are returned to the urn; similarly, for the \( n \)-th time of observing a white ball, \( w_n \) number of white balls are returned.

In the original Polya urn problem with replacement of 2(or increment of 1), it is easy to show that there are infinitely many times we draw either ball.

**Lemma 3.1.** In the basic Polya’s urn problem with increment 1, there are infinitely many times we choose a ball of either color.

**Proof.** Assume initially there are \( t \) balls in total. W.L.O.G. assume we identify one red ball initially in the urn. Then in step \( n \), \( p_n \), the probability of drawing the one we identified is \( 1/(t+n) \). \( \sum_n p_n = \infty \) and from Borel-Cantelli we conclude we draw the identified red balls infinitely many times. The rest follows.

Lukcily we have a more general result for generalized Polya urn from Herman Rubin.

**Theorem 3.2.** Let \( \vec{r} = (r_0, r_1, \ldots), \vec{w} = (w_0, w_1, \ldots) \) be two sequences of nonnegative numbers such that \( r_0 > 0 \) and \( w_0 > 0 \). Put \( R_k = \sum_{i=0}^{k} r_i \) and \( W_k = \sum_{i=0}^{k} w_i \). The first entry of the infinite sequence is type \( r \) with probability \( R_0/(R_0 + W_0) \), type \( w \) with probability \( W_0/(R_0 + W_0) \). Given the first \( n \) entries consist of \( x \) \( r \)'s and \( y = n - x \) \( w \)'s, in a given order, the probability that the \( n+1 \)st entry is \( r \) equals \( R_x/(R_x + W_y) \), and the probability it is \( w \) equals \( W_y/(R_x + W_y) \). Let \( p_r = P(\text{all but finitely many elements of the sequence are red}) \) and \( p_w = P(\text{all but finitely many elements of the sequence are white}) \).

Put \( \phi(r) = \sum_{i=1}^{\infty} R_i^{-1} \), \( \phi(w) = \sum_{i=1}^{\infty} W_i^{-1} \), then

1. If \( \phi(r) < \infty \) and \( \phi(w) < \infty \) then \( p_r > 0, p_w > 0 \) and \( p_r + p_w = 1 \).
2. If \( \phi(r) < \infty \) and \( \phi(w) = \infty \), \( p_r = 1 \).
3. If \( \phi(r) = \infty \) and \( \phi(w) = \infty \), both \( p_r \) and \( p_w \) equal 0.
Example Assume the initial urn is \((1,1)\). \(r_i = w_i = 1, \forall i > 0\). Then \(R_k = 2k + 1, W_k = 2k + 1\). We have the same conclusion as above: there are infinitely many times we draw the balls of each color.

Example \(a_i = i, \forall i > 0\). Then \(R_k = O(k^2), W_k = O(k^2), \phi(r) < \infty, \phi(w) < \infty\). So \(p_r + p_w = 1\), which means either we draw only finitely many red balls, or only finitely many white balls.

4 Application on Edge-Reinforced Random Walk

Recall that a simple random walk \(\bar{X}\) on \(\mathbb{Z}\) is a process that takes values on the \(\mathbb{Z}\) at any time \(i \geq 0\), i.e., \(X_i \in \mathbb{Z}\), and

\[
P(X_{i+1} = X_i + 1) = P(X_{i+1} = X_i - 1) = \frac{1}{2}.
\]

Further, the edge-weighted random walk \(\bar{X}\) on \(\mathbb{Z}\) is a process that takes values on the \(\mathbb{Z}\) at any time \(i \geq 0\), and

\[
P(X_{i+1} = X_i + 1) = \frac{w((i, i + 1))}{w((i, i + 1)) + w((i, i - 1))},
\]

\[
P(X_{i+1} = X_i - 1) = \frac{w((i, i - 1))}{w((i, i + 1)) + w((i, i - 1))},
\]

where \(w((i, i - 1))\) is the weight of the edge \((i, i - 1)\).

An Edge-Reinforced Random Walk (ERRW) on \(\mathbb{Z}\) is an edge-weighted random walk on \(\mathbb{Z}\) but every time the edge is crossed, the weight of the crossed edge increases so that the process tends to revisit the edge. Thus, an ERRW process \(\bar{X}\) is not a Markovian process because the probability of traversing an edge depends on the number of times it has traversed in its entire history.

Analogous with a weighted random walk, let \(w(n, i)\) be the weight of edge \(e_i = (i, i + 1)\) at time \(n\). We say that \(\bar{X}\) traverses \(e_i\) at some time \(n > 0\), if \((X_{n-1}, X_n) \in \{(i, i + 1), (i + 1, i)\}\). After the process traverses edge \(e_i\) at time \(n\), we add a nonnegative increment \(a\) to \(w(n, i)\) so that the weight of the edge \(e_i\) at time \((n + 1)\) is \(w(n + 1, i) = w(n, i) + a \geq w(n, i)\). Here are some notations.

Definition A walk is of sequence type if there is a sequence \(\bar{a} = \alpha_1, \alpha_2, \ldots\), of nonnegative numbers such that at the \(k\)-th crossing of an edge it is reinforced with \(a_k\). Thus, if \(\gamma(n, j)\) is the number of times \((X_0, X_1, \ldots, X_n)\) crosses \(e_j\) then

\[
w(n, j) = w(0, j) + \sum_{i=1}^{\gamma(n,j)} \alpha_i, \ a.s
\]

Definition A walk has i.i.d reinforcement if there exists a sequence \(Z_1, Z_2, \ldots\), of i.i.d. nonnegative random variables such that

\[
w(n, j) = w(0, j) + \sum_{0 \leq i \leq n: (X_i, X_{i+1}) \in \{(j, j+1), (j+1, j)\}} Z_i.
\]

Definition Let \(v(l, j)\) denote the weight of edge \(j\) when it has been traversed \(l\) times. Thus

\[
v(l, j) = w(0, j) + \sum_{i=1}^{l} \alpha_i, \ \text{for sequence type reinforcement},
\]

\[
v(l, j) = w(0, j) + \sum_{i=1}^{l} Z_i, \ \text{for i.i.d type reinforcement}.
\]

One key question for the ERRW is the recurrence of the process. Compared with the simple random walk on \(\mathbb{Z}\), it is not hard to imagine the ERRW is more likely to be recurrent or even stuck on a finite subgraph of \(\mathbb{Z}\). Before showing the results on recurrence of the process, we first relate the ERRW process to the Polya’s urns.
4.1 Edge-Reinforced Random Walk via Polya’s urn

Imagine for each vertex $j \in \mathbb{Z}$ we associate it with a Polya’s urn $U_j$ with two colors black and white. The black color indicates ‘to go up’ and the white color indicates ‘to go down’. The number of white and black balls in urn $U_j$ at time $n$ keeps track of the weight of edge $e_{j-1} = (j-1, j)$ and $e_j = (j, j+1)$, respectively.

For example, if we are to model a ERRW process with all initial weights and all increments equal to 1(which is called Diaconis walk), we start at vertex 0 with 1 ball of each color in urn $U_0$. We draw a ball from the urn $U_0$ and walk in the direction indicated by the color of the ball. We then come to a new vertex with an associated urn, which will decide the next step. Once we reach a new vertex $j$ for the first time, we visit $e_{j-1}$ but never $e_j$. So if $j > 0$, we add one more black ball to the urn $U_j$ at vertex $j$, and it should have 2 black balls and 1 white ball at the first time we reach the vertex $j$. Similarly, if $j < 0$, then the urn $U_j$ at vertex $j$ has 1 black balls and 2 white balls at the first time we reach vertex $j$.

However, in the above setting, the urns associating with vertices are not independent of each other. Consider the following scheme in the Diaconis walk. The first move is chosen by randomly drawing a ball from the urn $U_0$, and then we walk along the edge $e_1$ or $e_0$ in the direction corresponding to the resulting color. Once we leave the vertex 0, we put back the ball the drawn ball into the urn, along with TWO more balls of the same color. The idea behind this scheme is that at the time the walk ever returns to the vertex 0 and want to use urn $U_0$ to make the next decision, it must have already traversed the edge $e_1$ or $e_0$ twice, once from the vertex 0 to vertex 1 and once in the reversed direction back. Thus the weight of the chosen edge will have increased by exactly two. For the other vertex $j \neq 0$, they behave in the same way as that for the vertex 0 once after the walk reaches it. Initially, we put 2 black balls and 1 white ball into the urn $U_j$ for $j > 0$, and we put 1 black ball and 2 white balls into the urn $U_j$ for $j < 0$. It is so because upon the first time the walk arrives at the vertex $j$, the edge $e_{j-1}$ must have already traversed once from vertex $j-1$ to $j$. See Figure 4.1 for the demonstration of the initial setting of the Diaconis walk via Polya’s urn. Under this scheme, the urns at each vertex are independent because the outcome of drawing from one urn cannot affect any other, which is a very nice property.

![Figure 1: The initial setting of the Diaconis walk via Polya’s urn](ref [2])

For a general ERRW with reinforcement of a sequence $\alpha = \alpha_1, \alpha_2, \ldots$. Define $T_{k,i} = \min\{T_{k,i} > T_{k-1,i}; X_{T_k,i} = i\}$, i.e. the $k$-th time the walk stops at $i$. The setting of the Polya’s Urns model corresponding to the ERRW is as follows.

1. Now first let’s look at the origin. $T_{0,0} = 0$, and the weight is $(1, 1)$. If the first move is to the right, then the weight becomes $(1, 1 + a_1)$, and at time $T_{1,0}$, the first time the walk comes back, the weight becomes $(1, 1 + a_1 + a_2)$. Similarly, if the first move was to the left, the weight at $T_{1,0}$ will be $(1 + a_1 + a_2, 1)$. Actually this means the origin can be thought of as a Polya’s urn, with 1 ball of each color. And the replacement vector will be $(a_1 + a_2, a_3 + a_4, \cdots)$.

2. Now we extend the concept of associating an urn with each point to the rest. For the points to the right of the origin, they can also be thought of as Polya’s urn each. The replacement vector will be exactly the same $(a_1 + a_2, a_3 + a_4, \cdots)$ for the ball with color representing going to the right, the initial weight is $(1 + a_1, 1)$, and the replacement vector representing going to the left is now $(a_2 + a_3, a_4 + a_5, \cdots)$.

3. For the points to the left of the origin, the situation just a mirror of the above.
4.2 Asymptotics of ERRW on $\mathbb{Z}$

First we give the mathematical formulation of ERRW process.

**Definition** An ERRW on $\mathbb{Z}$ is a sequence $\vec{X} = \{X_n, n \geq 0\}$ of integer valued r.v’s and a matrix $[w] = w(n,j), n \in \mathbb{N}, j \in \mathbb{Z}$ of positive r.v’s all defined on the same probability space. Let $\mathcal{G}_n$ be the $\sigma$-field $\sigma(\{X_i, w(n,j), 0 \leq i \leq n, j \in \mathbb{Z}\})$. Then the following relations hold

1. $w(n+1,j) \geq w(n,j)$ with equality if $\vec{X}$ does not traverse $e_j$ at time $n+1$,
2. $P(X_{n+1} = j+1 | X_n = j, \mathcal{G}_n) = 1 - P(X_{n+1} = j-1 | X_n = j, \mathcal{G}_n) = \frac{w(n,j)}{w(n,j+1)+w(n,j-1)}$

A walk is initially fair if all the initial weights $w(0,j)$ are 1. Then we have the following

**Theorem 4.1.** Let $\vec{X}$ be an initially fair ERRW. Then

$$P(\vec{X} \text{ is recurrent}) + P(\vec{X} \text{ has finite range}) = 1$$

**Proof.** See [5] \qed

Basically, the Theorem 4.1 states that for an initially fair ERRW process, the process will either be recurrent or stuck in a finite subgraph on $\mathbb{Z}$. So the next interesting question is under which conditions are we sure that the process will be recurrent?

**Definition** For the initially fair ERRW, if the reinforcement is a sequence type with a nonnegative sequence $\vec{\alpha}$, then let

$$\phi(\vec{v}_j) = \sum_{n=0}^{\infty} v(n,j)^{-1} = \sum_{n=1}^{\infty} \left(1 + \sum_{i=1}^{n} \alpha_i\right)^{-1};$$

and if the reinforcement if i.i.d, then

$$\phi(\vec{v}_j) = \sum_{n=0}^{\infty} v(n,j)^{-1} = \sum_{n=1}^{\infty} \left(1 + \sum_{i=1}^{n} Z_i\right)^{-1}.$$

The quantity $\phi(\vec{a})$ will have great effect on the limit behavior of the ERRW process, determining whether the process will be recurrent or stuck in a finite subgraph.

**Theorem 4.2.** Let $\vec{X}$ be an initially fair ERRW, with reinforcing sequence $\vec{\alpha}$, or i.i.d reinforcement with associated r.v’s $Z_1, Z_2, \ldots$. Then

1. $\phi(\vec{v}_j) < \infty \ a.s. \Rightarrow \vec{X} \text{ has finite range } a.s.$
2. $\phi(\vec{v}_j) = \infty \ a.s. \Rightarrow \vec{X} \text{ is recurrent } a.s.$
3. If $\vec{X}$ has finite range, there exist (random) integers $N$ and $j$ such that $n > N \Rightarrow X_n \in j, j+1$.

**Proof.** See [5] \qed

Given theorem 3.2 for the generalized Polya urn with a replacement vector $\vec{a}$, we find the its connection to theorem 4.2 for the ERRW with the reinforcement of the same vector $\vec{a}$. Since the reinforced random walk always has a tendency to come back to origin initially $(1 + \alpha_1 > 1)$ and the vertices on the $\mathbb{Z}$ are symmetric, we thus intuitively suggest that if we can draw infinitely many times balls of each color, we should not be bounded in any range in the walk, since each position/urn is essentially similar to any other. Based on this intuition, the relationship between $\phi(\vec{a})$ and $\infty$, where $\vec{a} = (a_0, a_1, a_2, \ldots)$, should determine the recurrence of the reinforced random walk.

5 Simulation

We will use numerical simulation trying to validate our result.
5.1 Polya urn with two colors

We start one run with an urn with 1 red ball and 1 blue ball, and perform 100,000 draws and obtain the proportion of red balls in the urn. We performed 1000 runs to see the distribution of the final proportion. We then compare the empirical distribution of the final portion to the theoretical counterpart, which should be uniform. The data are summarized as a histogram after adjusting as probability (area enforced to be 1). Empirical density function is a smooth estimate of the density of the data using `density` function in R language.

Now we change the scheme to urn initially with 6 red balls and 3 blue balls, with other setting unchanged. The following results are obtained.

5.2 Recurrence of edge-reinforced random walk

Because of the difficulty of simulating $Z$ numerically, we will restrict ourselves to the range $|i| \leq 200, i \in \mathbb{Z}$. The simple random walk is clearly recurrent, since in this case $a_i = 0, i \geq 0$, which is consistent with the result in Theorem 3.2. We performed 500,000 steps in the walk and then show the number of times each point has been visited.
Now we change $a_1 = 0, a_i = \frac{1}{\sqrt{i}}, \forall i \geq 1$. Clearly $\phi(a)$ diverges, and the walk should be recurrent. We still do 500,000 steps in the random walk, and get the following.

It looks like the range is confined. But if we continue with more steps (5,000,000 steps), we see that the walk goes all the way to the two extremes.

As $a_i$ becomes bigger, more steps are needed to cover the interval $[-200, 200]$, assuming the walk is recurrent. The transition point from recurrence to a local walk is determined by $\phi(a)$. The walk is
restricted in finite range when \( \phi(a) < \infty \). Take \( a_1 = 0, a_i = i, \forall i \geq 2 \), then \( \phi(a) < \infty \), and use 5,000,000 steps, we get the following walk (here we chose \( a_1 = 0 \) to make the walk less origin centered)

Further inspection shows that 4,999,995 out of 5,000,000 steps happened on the edge < 2, 3 >.

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References

7 Appendix
Lemma 2.4
Suppose a Polya urn starts with \( X_{j0} \) balls with \( j \)th color \( (j = 1, ..., K) \) and \( \tau_0 = \sum_j X_{j0} \). Let \( \tilde{X}_{jn} \) be the number of \( j \)th \( (j = 1, 2, ..., K) \) color balls drawn in the Polya urn after \( n \) draws, then

\[
P(\tilde{X}_{jn} = y_j, j = 1, ..., m) = \frac{\prod_{j=1}^{K} [X_{j0}(X_{j0} + 1) \ast \cdots \ast (X_{j0} + y_j - 1)]}{\tau_0(\tau_0 + 1) \ast \cdots \ast (\tau_0 + n - 1)} \binom{n}{y_1, \ldots, y_m}
\]

Proof. In a string of \( n \) draws achieving \( k_j \) \( j \)-th-color ball drawings, \( k_j \) needs to sum to \( n \). Suppose \( 1 \leq t_{j1} \leq t_{j2} \leq \cdots \leq t_{jk_j} \leq n \) are the time indexes of the \( j \)-th-color ball draws. The probability of this particular string is

\[
\prod_{j=1}^{K} \frac{X_{j0}}{\tau_{t_{j1}}} \times \frac{X_{j0} + 1}{\tau_{t_{j2}}} \times \cdots \times \frac{X_{j0} + k_j - 1}{\tau_{t_{jk_j}}}
\]  

(7)
Note that the expression does not depend on the indexes. The time indexes can be chosen in \( \binom{n}{y_1, \ldots, y_m} \) ways.

**Theorem 2.5**

Suppose a Polya urn starts with \( X_{j0} \) balls with \( j \)th color \((j = 1, \ldots, K)\). Let \( \tilde{X}_{jn} \) be the number of \( j \)th \( (j = 1, 2, \ldots, K) \) color balls drawn in the Polya urn after \( n \) draws, when \( n \to \infty \),

\[
\left( \frac{\tilde{X}_{1n}}{n}, \ldots, \frac{\tilde{X}_{Kn}}{n} \right) \overset{D}{\to} Dir(X_{10}, \ldots, X_{K0}).
\]

**Proof.** By Lemma 2.4,

\[
P(\tilde{X}_{jn} = y_j, j = 1, \ldots, m) = \frac{\prod_{j=1}^{m}[X_{j0}(X_{j0}+1) \ast \ldots \ast (X_{j0}+y_j-1)]}{\tau_0(\tau_0+1) \ast \ldots \ast (\tau_0+n-1)} \binom{n}{y_1, \ldots, y_m}
\]

\[
= \frac{\prod_{j=1}^{m} \Gamma(X_{j0}+y_j) \Gamma(\tau_0)}{\prod_{j=1}^{m} \Gamma(X_{j0}) \Gamma(\tau_0+n)} \frac{\Gamma(n+1)}{\prod_{j=1}^{m} \Gamma(y_j+1)} = \frac{\Gamma(n+1)}{\prod_{j=1}^{m} \Gamma(y_j+1)} \prod_{j=1}^{m} \frac{\Gamma(\tau_0)}{\Gamma(\tau_0+n)}
\]

Applying Stirling approximation \( \frac{\Gamma(x+r)}{\Gamma(x+s)} \approx x^{r-s} \) as \( x \to \infty \) gives

\[
P(\frac{\tilde{X}_{jn}}{n} \leq z_j, j = 1, 2, \ldots, m) \to \frac{\Gamma(\tau_0)}{\prod_{j=1}^{m} \Gamma(X_{j0})} \int_z \prod_{j=1}^{m} z_j^{x_{j,0}-1}(\sum z_i = 1)
\]

which is the c.d.f. of Dirichlet distribution with parameter \((X_{10}, \ldots, X_{m0})\).

**Lemma 2.9**

A sample from Dirichlet Process is a discrete distribution, made of up countably infinite number of point masses.

**Lemma 2.10**

Let \( G \) be a sample from Dirichlet process \( DP(\alpha, G_0) \) on the color space and let \( c_n \) be the color of \( n \)th ball observed, then the conditional probability of \( G \) is still Dirichlet Process with parameters \( \alpha \) and \( G_0 + \sum_{n=1}^{N} \delta_{c_n} \).

We refer the proofs of Lemma 2.9 and 2.10 to paper [1, 3]