Math 505b : Projects

General Remarks.

Here is a list of different projects. You will have to work on them as a group (minimum 2, maximum 4), and write down a report (less than 10 pages), and make an oral presentation of it, of 15/20 minutes (depending on the number of projects and the number of people in the group). Talk together about the different projects you would be interested in before choosing one.

For each project, the goal is to:

1. Give some background on the problem: explain (briefly) why it is studied, possible applications;
2. Define the problem properly, state the result;
3. Write down the solution of the problem;
4. Run some numerical simulation (if possible);
5. Mention what would be interesting to further study if you had more time.

Note that I am not giving you some clear frame of what you are supposed to do. You need to find out, by looking at the references, what are the interesting questions, and the ones that you can solve. Here are some suggested steps for the project (and some schedule)

- Getting familiar with the subject, understanding what it is about, to be able to tell what are the interesting questions. Then, make a list of possible questions to answer: do not hesitate to do simple ones at first, so that you don’t get stuck on some too difficult ones. Look in the references where these questions are treated (2/3 weeks)
- Look at the proofs, understand how they work. Discuss with each other your thoughts. Come to my office for some clarifications if needed (1 1/2 month)
- Write down your report (2/3 weeks).

Register for the projects on the following sheet, before Monday, February 3rd.
https://docs.google.com/spreadsheet/ccc?key=0AqTgB3sL6CQvdFpkxalJXe1RxYkpZZEtXmVQXc#gid=0 (link on Blackboard, website section)

The deadline for the report is Wednesday, April 23rd. The oral presentation will be during the last week of class (probably on Monday/Tuesday, April 28/29th).

All the references mentioned can be found in the content section of blackboard.
Project: Pólya Urn

In the basic Pólya urn model, the urn contains $x$ white and $y$ black balls; one ball is drawn randomly from the urn and its color observed; it is then replaced in the urn, and an additional ball of the same color is added to the urn, and the selection process is repeated.

Questions of interests are:
- The limiting ratio of white balls vs. black balls;
- The drawn sequence;
- Variants of the model, application to reinforced random walk...

References: Polya Urns, Random processes with reinforcement

Project: Mixing cards

When shuffling a deck of $n$ cards, one can employ different techniques. The most basic one is to take the card on the top, and place it randomly in the deck. Another one is the so-called riffle-shuffle, used in the casinos. The question is then to know how long you have to repeat the shuffling process to obtain a sufficiently well shuffled deck of cards.

1. How do you make mathematical sense of "sufficiently well shuffled"?
2. Find the mixing time: for the top-to-bottom, it is $n \ln(n)$, for the riffle-shuffle, it is $\frac{3}{2} \ln(n)$

Reference: Markov Chain mixing, Lectures on mixing time

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Project: the 2D Ising model (two different projects)

A spin system is a probability distribution on $\Omega = \{-1, 1\}^V$, where $V$ is the vertex set of a graph $G = (V, E)$. The value $\sigma(v) \in \{-1, 1\}$ is called the spin at vertex $v$. The physical interpretation is that magnets, each having one of the two possible orientations represented by $+1$ and $1$, are placed on the vertices of the graph; a configuration specifies the orientations of these magnets. The nearest-neighbor Ising model is the most widely studied spin system. In this system, the energy of a configuration $\sigma$ is defined to be $H(\sigma) = -\sum_{x \sim y} \sigma(x)\sigma(y)$, and one define the Gibbs measure $\mu(\sigma) = \frac{1}{Z} e^{-\beta H}$ on the set of all configurations, $\beta$ being the inverse temperature.

1. **Markov chain Monte-Carlo simulation of the Ising model.** It is really hard to sample a random spin-configuration for the Ising model (why?). The goal of this project is to use the Monte Carlo method to do so. Questions of interest are:
   - How does the Monte-Carlo Markov Chain works?
   - Numerical simulations.
   - How long do you have to run the Monte-Carlo algorithm?

   References: Markov Chain mixing, Ising Model

2. **Ising model’s Phase transition** The 2D Ising model has be shown to exhibit a phase transition when temperature evolves. The model has been solved exactly by Onsager. Questions of interest are
   - What is the critical temperature where the phase transition happens?
   - Why can’t we solve the 3D Ising model?

References: Ising model
Project: Random walks in random environment

Here is how you can think about a simple random walk in one dimension: at each step, you flip a coin with parameter \( p \), and take a step to the right if Head, and to the left if Tail. You know that the random walk is recurrent if and only if \( p = 1/2 \). Now imagine that the parameter \( p \) is random and depends on where you stand (a set of coins with different parameters have been put all along the path). Another way of thinking about it is that each edge of \( \mathbb{Z} \) to be crossed has a random conductance. Questions of interest are

- Is the walk transient or recurrent?
- What about higher dimensions?

References: Subjects in Random Walks in Random Environment, Genealogy and RWRE

Project: Brownian motion solves Dirichlet problem

The Dirichlet problem is the problem of finding an harmonic function in the interior of a given region that takes prescribed values on the boundary of the region. Some properties of the Brownian motion allows to solve this problem. Questions of interest are

- Studying various properties of Brownian Motion, solve Dirichlet problem
- Which other classical analysis problems can be solved using Brownian Motion?

References: Brownian Motion, Brownian motion in classical analysis

Project: Mixing time of random walks

When running a random walk, the question is to know how long you have to wait until you lose track of the position of the random walk? Some interesting mathematical tools are used to answer this question, in different contexts:

- Random walk on the circle;
- Random walk on the hypercube;

Reference: Lectures on Mixing Time, Markov Chain Mixing time

Project: Branching random walk (several projects possible)

A Branching Random Walk is a branching process, where each of the descendants are moving according to random walks. Questions of interest are

- What is the speed of the front (left-most descendant)?
- Properties of the repartitions of descendants;
- Spread of the population

References: Notes on Branching Random Walks; Random Walks and Trees; Shape of BRW; Thesis on BRW.

Project: Random matrices

Take a (large) symmetric matrix whose entries are random, and study the properties of its eigenvalues. Questions of interest are:

- What is the limiting distribution of the eigenvalues?
- Any applications of Random Matrices?

References: Random Matrices
Project: Percolation’s phase transition

The percolation model is used to describe porous media. Many interesting questions arise: in particular, there exists some threshold porosity under which a liquid can percolate through the medium and above which it cannot. Questions of interest are

- Mathematical description of the model and proof of the existence of the phase transition;
- Properties of the medium at different porosity (e.g. of the clusters)

References: Notes on Percolation Theory, Probability on Graphs

Project: Wright Fisher model

The simplest model of random genetic drift is known as the Wright Fisher model. We consider a population in which every individual is equally likely to mate with every other and in which all individuals experience the same conditions, and we study the evolution of the alleles frequencies. Questions of interest are:

- Is there some winning allele?
- Can you describe the underlying genealogical trees?
- What about if some mutations appear?

References: Population Genetics, Genealogy and RWRE.

Project: Coalescent process

The coalescent process is a simple model to describe genealogic trees (used for genetics), seen as a system of coalescing lineages. Questions of interest are:

- How long does it take to go back to the most recent common ancestor?
- How do mutations spread along the coalescent tree?

References: Population Genetics, Genealogy and RWRE.

Project: Queues’ waiting time

In the model of Queues, customers arrive at some given rate, and are served at another rate. Questions of interest are

- The stability of the Queue,
- If the Queue is stable, the time a customer will wait,
- Variants of this simple model (adding some servers, first in first out or first in last out)...

References: Applied Probabilities and Queues.

Project: Exclusion process

The Totally Asymmetric Exclusion Process is used to model a flow of particle moving in one direction. At each step, a particle is chosen at random, and moved to the right if the site is empty. Questions of interest are:

- Occupation probability of a site;
- Speed of the flow.
- Variants of the model?

References: Exclusion process solution, Interacting Particle Systems.
Project: Kolmogorov’s test to determine distributions

If you observe some data, you might want to know if it follows some particular distribution (for example: is it Standard Normal?). Using Glivenko-Cantelli Lemma (prove it), design a test to determine if a random variable is Standard Normal or not, giving also the confidence of the test.

Hat about other distributions?

References: Kolmogorov’s test, Glivenko-Cantelli Theorem (look on the internet, I am not aware of good references...), look also at Grimmett Stirzaker p504.