On random walks

Random walk in dimension 1. Let $S_n = x + \sum_{i=1}^{n} U_i$, where $x \in \mathbb{Z}$ is the starting point of the random walk, and the $U_i$’s are IID with $\mathbb{P}(U_i = +1) = \mathbb{P}(U_i = -1) = 1/2$.

1. Let $N$ be fixed (goal you want to attain). Compute the probability $p_x$ that the random walk reaches 0 before $N$, starting from $x$. (Hint: show that $p_{x+1} - p_x = \frac{1}{2}(p_x - p_{x-1})$ for $x \in \{1, \ldots, N-1\}$).

Solution: $p_x = \frac{N-x}{N}$

2. Use 1. to compute the probability that, starting from 0, you reach $a > 0$ before $-b < 0$.

Solution: from previous question: $\frac{a}{a+b}$

3. Use 1. to compute the probability that, starting from the origin, you reach $a > 0$ before returning to the origin.

Solution: from previous questions: $\frac{1}{2}a$ (first step is up, then reach $a$ before returning to 0).

4.* Use 3. to show that the average number of visits to $a > 0$ before returning to the origin is 1 (hint: show that it is closely related to the expectation of some geometric random variable).

Solution: let $N_a$ be the number of visits to $a$ before returning to 0.

Using question 3. one has $\mathbb{P}(N_a = 0) = \frac{1}{2} + \frac{1}{2}(1 - 1/a)$ (call that probability $q$).

Note that once you’ve made a visit to $a$ (that is given $N_a \geq 1$), which happens with probability $1 - q$, the number of returns is a geometric random variable, with probability of success $1 - q$: the probability of coming back to $a$ before 0 (failure) is $\frac{1}{2} + \frac{1}{2}(1 - 1/a) = q$. Therefore, $\mathbb{E}[N_a | N_a \geq 1] = \frac{1}{1-q}$.

$$\mathbb{E}[N_a] = \mathbb{P}(N_a = 0)\mathbb{E}[N_a | N_a = 0] + \mathbb{P}(N_a \geq 1)\mathbb{E}[N_a | N_a \geq 1] = (1 - q) \frac{1}{1-q} = 1$$

5. Do the same 1.2.3.4. problems when the random walk is "lazy": $\mathbb{P}(U_i = 0) = \mathbb{P}(U_i = +1) = \mathbb{P}(U_i = -1) = 1/3$

Solution: same answers for 1, 2; for 3. one gets $\frac{1}{3}a$; same reasoning for 4. to get that the average number of visits is 1.

6. Suppose that $x = 0$. Recall how to prove that $\mathbb{P}(S_1 \geq 0, \ldots, S_{2n} \geq 0) = \mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0)$.

Solution:

$$\mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = 2\mathbb{P}(S_1 > 0, \ldots, S_{2n} > 0) \quad \text{by symmetry}$$

$$= 2\mathbb{P}(S_1 = 1, S_2 \geq 1, \ldots, S_{2n} \geq 1) = 2\frac{1}{2}\mathbb{P}(S_2 \geq 1, \ldots, S_{2n} \geq 1|S_1 = 1)$$

$$= \mathbb{P}(S_1 \geq 0, \ldots, S_{2n-1} \geq 0) \quad \text{(vertical translation)}$$

$$= \mathbb{P}(S_1 \geq 0, \ldots, S_{2n} \geq 0) \quad \text{bc. then } S_{2n-1} \geq 1 \text{ (parity matter), and } S_{2n} \geq 0$$
\[ P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = 2P(S_1 > 0, \ldots, S_{2n} > 0) = \sum_{k \geq 1} P(S_2 > 0, \ldots, S_{2n} > 0, S_{2n} = 2k|S_1 = 1) \]

\[ = \sum_{k \geq 1} (P(S_{2n} = 2k|S_1 = 1) - P(\exists m, \text{ s.t. } S_m = 0, S_{2n} = 2k|S_1 = 1)) \]

\[ = \sum_{k \geq 1} (P(S_{2n} = 2k|S_1 = 1) - P(S_{2n} = -2k|S_1 = 1)) \text{ by a reflexion argument} \]

\[ = \sum_{k \geq 1} (P(S_{2n} = 2k|S_1 = 1) - P(S_{2n} = 2k|S_1 = -1)) \text{ by symmetry} \]

\[ = \sum_{k \geq 1} (P(S_{2n-1} = 2k - 1|S_0 = 0) - P(S_{2n-1} = 2k + 1|S_0 = 0)) \text{ by a vertical translation} \]

because \( P(S_{2n} = 0) = \frac{1}{2} P(S_{2n-1} = -1) + \frac{1}{2} P(S_{2n-1} = 1). \)

**Random walk in dimension 2.** Let \( Z_n = \sum_{i=1}^{n} W_i, \) where the \( W_i \)'s are IID nearest neighbor steps \( P(W_i = (1, 0)) = P(W_i = (-1, 0)) = P(W_i = (0, 1)) = P(W_i = (0, -1)) = 1/4. \) We denote \( W_i = (W_{i}^{(1)}, W_{i}^{(2)}), \) and \( Z_n = (X_n, Y_n). \)

7. Show that if you define \( U_i := W_i^{(1)} + W_i^{(2)} \) and \( V_i := W_i^{(1)} - W_i^{(2)} \), then \( A_n := \sum_{i=1}^{n} U_i \) and \( B_n := \sum_{i=1}^{n} V_i \) are independent one-dimensional simple random walks.

**Solution:** Just verify that \( P(U_i = +1, V_i = -1) = P(W_i^{(1)} = 0, W_i^{(2)} = 1) = 1/4 \) (for all possibilities: \( U_i = +1, V_i = +1, U_i = -1, V_i = +1, U_i = -1, V_i = -1 \)), and that \( P(U_i = +1) = P(U_i = -1) = P(V_i = +1) = P(V_i = -1) = 1/2. \)

8. Use \( A_n \) and \( B_n \) defined in the previous question to compute \( P(Z_{2n} = (0, 0)). \)

**Solution:** \( P(Z_{2n} = (0, 0)) = P(A_{2n} = 0)P(B_{2n} = 0) = (2^n)^2, \) because \( A_n \) and \( B_n \) are independent.

9. Use questions 6.7.8. to compute the probability that the random walk stays in the cone \( \{x + y \geq 0, x - y \geq 0\} \) up to time 2n.

**Solution:**

(1)

\[ P(Z_k \in \{x + y \geq 0, x - y \geq 0\} \forall 0 \leq k \leq 2n) = P(A_k \geq 0, \forall 0 \leq k \leq 2n)P(K \geq 0, \forall 0 \leq k \leq 2n) \]

\[ = P(A_{2n=0})P(B_{2n=0}) = \left(\frac{2n}{2}\right)^2, \]

using question 6. on random walks in dimension 1.

**On Markov Chains**

1. A taxicab driver moves between the airport A and two hotels B and C according to the following rules: if at the airport: go to one of the hotel with equal probability, and if at one hotel, go to the airport with probability 3/4, and to the other hotel with probability 1/4.

(a) Find the transition matrix.

(b) Suppose the driver starts at the airport. Find the probability for each of the three possible location at time 2. Find the probability that he is at the airport at time 3.
(c) Find the stationary distribution, and apply the convergence theorem to find 
\( \lim_{n \to \infty} \mathbb{P}(\text{at the airport at time } n) \).

**Solution:** 1= airport, 2= 1st hotel, 3=2nd hotel 
\[ P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}. \]

\( \mathbb{P}(X_2 = \text{airport}|X_0 = \text{airport}) = 3/4, \mathbb{P}(X_2 = \text{hotel1}|X_0 = \text{airport}) = \mathbb{P}(X_2 = \text{hotel2}|X_0 = \text{airport}) = 1/8. \)

\( \mathbb{P}(X_3 = \text{airport}|X_0 = \text{airport}) = 3/16. \)

Stationary distribution \( \pi(\text{airport}) = 3/7, \pi(\text{hotel1}) = \pi(\text{hotel2}) = 2/7. \) Since irreducible, aperiodic, 
\[ \lim_{n \to \infty} \mathbb{P}(\text{at the airport at time } n) = 3/7 \]

2. Consider the following transition matrices. Classify the states, and describe the long-term behavior of the chain (and justify your reasoning).

\[ P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & .8 \\ .1 & .2 & .3 & .4 \\ 0 & .6 & 0 & .4 \end{pmatrix}, \quad P = \begin{pmatrix} .4 & .4 & .3 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & .3 & 0 & .3 & .4 \end{pmatrix}, \quad P = \begin{pmatrix} 2/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \end{pmatrix}. \]

**Solution:**

1. Communication classes \{1, 5\} (positive recurrent), \{2, 3, 4\} (transient, because communicates with exterior). In the long run, we end up in the two states \{1, 5\}, with equilibrium distribution \( \pi(1) = 3/13, \pi(5) = 10/13. \)

2. Communication classes \{2, 4\} (positive recurrent), \{1, 3\} (transient, because communicates with exterior), \{5\} (transient). In the long run, we end up in the two states \{2, 4\}, with equilibrium distribution \( \pi(2) = \pi(4) = 1/2. \)

3. Communication classes \{1, 4\} (positive recurrent) \{2, 5\} (positive recurrent) \{3\} and \{6\} (transient). In the long run, we end up in one of the two classes \( C_1 := \{1, 4\} \) or \( C_2 := \{2, 5\}. \) Starting from 1, 4, 6 one ends up in \( C_1 \), starting from 2, 5 one ends up in \( C_2 \), starting from 3, one ends up in \( C_1 \) with probability 1/2 and in \( C_2 \) with probability 1/2.

Stationary distribution in \( C_1: \pi(1) = 3/5, \pi(4) = 2/5. \)
Stationary distribution in \( C_2: \pi(2) = 1/2, \pi(5) = 1/2. \)

3. Six children (A, B, C, D, E, F) play catch. If A has the ball, then he/she is equally likely to throw the ball to B, D, E or F. If B has the ball, then he/she is equally likely to throw the ball to A, C, E or F. If E has the ball, then he/she is equally likely to throw the ball to A, B, D, F. If either C or F gets the ball, they keep throwing it at each other. If D gets the balls, he/she runs away with it.

(a) Find the transition matrix, and classify the states.
(b) Suppose that A has the ball at the beginning of the game. What is the probability that D ends up with the ball?

**Solution:** Communication classes: \( \{C, F\} \) (positive recurrent) \( \{D\} \) (positive recurrent) \( \{A, B, E\} \) (transient).

Call \( p_X \) the probability that D ends up with the ball staring from \( X (X = A, B, C, D, E \) or \( F). \) Clearly, \( p_D = 1 \) and \( p_C = p_F = 0. \) Also, \( p_A = \frac{1}{4}p_B + \frac{1}{4}p_D + \frac{1}{4}p_E + \frac{1}{4}p_F, \) \( p_B = \ldots. \) One finds \( p_A = p_E = 2/5, p_B = 1/5. \)
4. Find the stationary distribution(s) for the Markov Chains with transition matrices

\[ P = \begin{pmatrix} .4 & .6 & 0 \\ .2 & .4 & .4 \\ 0 & .3 & .7 \end{pmatrix} \quad P = \begin{pmatrix} .4 & .6 & 0 & 0 \\ 0 & .7 & .3 & 0 \\ .1 & 0 & .4 & .5 \\ .5 & 0 & 0 & .5 \end{pmatrix} \]

**Solution:**

(1) Unique stationary distribution (bc. irreducible, positive recurrent) \( \pi = (1/8, 3/8, 1/2) \).
(2) Unique stationary distribution (bc. irreducible, positive recurrent) \( \pi = (1/5, 2/5, 1/5, 1/5) \)

5. Consider the Markov Chain with transition matrix

\[ P = \begin{pmatrix} 0 & 0 & 3/5 & 2/5 \\ 0 & 0 & 1/5 & 4/5 \\ 1/4 & 3/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \]

(a) Compute \( P^2 \). (b) Find the stationary distributions of \( P \) and \( P^2 \). (c) Find the limit of \( p^{2n}(x, x) \) as \( n \to \infty \).

**Solution:**

\[ P^2 = \begin{pmatrix} 7/20 & 13/20 & 0 & 0 \\ 9/20 & 11/20 & 0 & 0 \\ 0 & 0 & 3/10 & 7/10 \\ 0 & 0 & 2/5 & 3/5 \end{pmatrix} \]

(b) \( P^2 \) is not irreducible. Stationary distribution on \( \{1, 2\} \): \( \bar{\pi}(1) = 9/22, \bar{\pi}(2) = 13/22 \) (and \( \bar{\pi}(3) = \bar{\pi}(4) = 0 \)). Stationary distribution on \( \{3, 4\} \): \( \bar{\pi}(3) = 4/11, \bar{\pi}(4) = 7/11 \) (and \( \bar{\pi}(1) = \bar{\pi}(2) = 0 \)). Any convex combination of \( \bar{\pi} \) and \( \bar{\pi} \), of the type \( \theta \bar{\pi} + \bar{\pi} \), is a stationary distribution for \( P^2 \).

\( P \) is irreducible: unique stationary distribution, \( \pi \), which must be a stationary distribution for \( P^2 \), thus of the form \( \theta \bar{\pi} + \bar{\pi} \). From the first column, we must get \( \theta \bar{\pi}(1) = \frac{1}{4}(1-\theta)\bar{\pi}(3) + \frac{1}{2}(1-\theta)\bar{\pi}(4) \), so that \( \theta = 1/2 \), and \( \pi = (9/44, 13/44, 4/22, 7/22) \).

(c) \( \lim_{n \to \infty} p^{2n}(x, x) = 9/22 \) if \( x = 1, 13/22 \) if \( x = 2, 4/11 \) if \( x = 3, 7/11 \) if \( x = 4 \).

6. Folk wisdom holds that in Ithaca in the summer, it rains 1/3 of the time, but a rainy day is followed by a second one with probability 1/2. Suppose that Ithaca weather is a Markov chain. What is it transition matrix?

**Solution:** If rain=1, sunny=2, the stationary distribution is \( \pi = (1/3, 2/3) \). We have that \( p(1, 1) = 1/2, p(1, 2) = 1/2 \). To be a stationary distribution, \( \pi \) must verify \( \pi(1) = p(1, 1)\pi(1) + p(2, 1)\pi(2) \), so that \( p(1, 1) = 1/4 \), and \( p(2, 2) = 3/4 \).

7. Consider a Markov chain with transition matrix \( P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \). Use the Markov property to show that

\[ \mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b) \left( \mathbb{P}(X_n = 1) - \frac{b}{a+b} \right) . \]

Deduce that if \( 0 < a + b < 2 \), then \( \mathbb{P}(X_n = 1) \) converges exponentially fast to its limiting value \( b/(a+b) \).

**Solution:** \( \mathbb{P}(X_{n+1} = 1) = \mathbb{P}(X_n = 1)(1-a) + \mathbb{P}(X_n = 0)b = b + (1-a-b)\mathbb{P}(X_n = 1) \) which gives the equation needed. Then, one gets \( \mathbb{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)^{n+1} (\mathbb{P}(X_0 = n) - b/(a+b)) \), converges exponentially fast to 0.

8. Let \( N_n \) be the number of heads observed in the first \( n \) flips of a fair coin, and let \( X_n = N_n \mod 5 \). Use the Markov Chain \( X_n \) to find \( \lim_{n \to \infty} \mathbb{P}(N_n \text{ is a multiple of } 5) \).
9. A basketball player makes a shot with the following probabilities: 1/2 if he misses the last two times, 2/3 if he has hit one of his last two shots, 3/4 if he has hit both his last two shots. Formulate a Markov Chain to model his shooting, and compute the limiting fraction of the time he hits a shot.

Solution: call $X_n$ the indicator function that he hits at time $n$, and call $Y_n = (X_{n-1}, X_n)$. There are 4 states $(0, 0), (0, 1), (1, 0), (1, 1)$. The transition matrix is the following $(0, 0)$ is the first state, $(0, 1)$ the second one, $(1, 0)$ the third one, $(1, 1)$ the fourth) $P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}$.

This is an irreducible, positive recurrent, aperiodic chain. Stationary distribution: $\pi(0, 0) = 1/8, \pi(0, 1) = \pi(1, 0) = 3/16, \pi(1, 1) = 1/2$. Proportion of the time he hits one = limit of the probability he hits one at a given time = $\lim_{n \to \infty} P(X_n = 1) = \pi(1, 0) + \pi(1, 1) = 11/16$.

10. Ehrenfest chain. Consider $N$ particles, divided into two urns (which communicate through a small hole). Let $X_n$ be the number of particles in the lefturn. The transition probabilities are given by the following rule: at each step, take one particle uniformly at random among the $N$ particles, and move it to the other urn (from left to right or right to left, depending on where the particle you chose is).

(a) give the transition probabilities of this Markov Chain.
(b) Let $\mu_n := \mathbb{E}[X_n | X_0 = x]$. Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$.
(c) Show by induction that $\mu_n = \frac{N}{2} + (1 - \frac{2}{N})^n(x - N/2)$.

Solution: (a) $p(j, j + 1) = \frac{N - j}{N}$ if $0 \leq j \leq N - 1$ and $p(j, j - 1) = \frac{j}{N}$ if $1 \leq j \leq N$. (b) Use that $\mathbb{E}[X_{n+1} | X_n = k] = 1 + k(1 - 2/N)$ for all $k \in \{0, \ldots, N\}$. (c) Induction comes easily from part (b): $\mu_n - N/2 = (1 - 2/N)(\mu_{n-1} - N/2)$.

11. Random walk on a circle. Consider the numbers $1, 2, \ldots, N$ written around a ring. Consider a Markov Chain that at any point jumps with equal probability to one of the two adjacent numbers.

(a) What is the expected number of steps that $X_n$ will take to return to its starting position?
(b) What is the probability that $X_n$ will visit all the other states before returning to its starting position?

Solution: (a) Stationary distribution is $\pi(i) = 1/N = 1/\mu_i$. Thus, $\mu_i = \mathbb{E}[T_i | X_0 = i] = N$.
(b) After one step, it correspond to the probability for a random walk to hit $N - 1$ before 0, starting from 1. Already seen that this probability is $\frac{1}{N - 1}$.

12. Knight’s random walk. We represent a chessboard as $S = \{(i, j), 1 \leq i, j \leq 8\}$. Then a knight can move from $(i, j)$ to any of eight squares $(i + 2, j + 1), (i + 2, j - 1), (i + 1, j + 2), (i + 1, j - 2), (i - 1, j + 2), (i - 1, j - 2), (i - 2, j + 1), (i - 2, j - 1)$, provided of course that they are on the chessboard. Let $X_n$ be the sequence of squares that results if we pick one of knight’s legal move at random. Find the stationary distribution, and deduce the expected number of moves to return to the corner $(1, 1)$, when starting at that corner.

Solution: (see what was done in class) Stationary distribution is $\frac{1}{\mu_x} = \pi(x) = \frac{\text{deg}(x)}{\sum_{x \in S} \text{deg}(x)}$, where $\text{deg}(x)$ is the number of moves possible from $x$.

$\sum_{x \in S} \text{deg}(x) = 336$, expected number of moves to return to the corner $(1, 1) = 168$. 

13. Queen’s random walk. Same questions as 12., but for the Queen, which can move any number of squares horizontally, vertically or diagonally.

**Solution:** Same type of stationary distribution, except \( \sum_{x \in S} \deg(x) = 1452 \), and for the corner \( \deg(x) = 21 \). Expected number of moves to return to the corner \( \approx 69.14 \).

14. If we have a deck of 52 cards, then its state can be described by a sequence of numbers that gives the cards we find as we examine the deck from the top down. Since all the cards are distinct, this list is a permutation of the set \( \{1, 2, \ldots, 52\} \), i.e., a sequence in which each number is listed exactly once. There are 52! permutations possible. Consider the following shuffling procedures:

(a) pick the card from the top, and put it uniformly at random in the deck (on the top or bottom is acceptable!);

(b) cut the deck into two parts and choose one part at random, and put it on top of the other.

In both cases, tell, if one repeats the algorithm, if in the limit the deck is perfectly shuffled, in the sense that all 52! possibilities are equally likely.

**Solution:** Just verify that the chains are irreducible, aperiodic, and that the stationary distribution is the uniform distribution \( \pi(\sigma) = \frac{1}{52!} \). The limit Theorem does the rest.

(a) Irreducible. Start from the cards in order, and try to get any permutation \( \sigma \). We first try to place the right card in the last position (position 52): make it go up by putting the cards one by one on the bottom, and when it’s on the top, put it on the bottom. Then you don’t want to touch that card anymore, and want to put the right card in position 51. Do the same process: make the card you want go up in the pile by putting all the cards one by one in position 51, then stop when you put the right card in position 51. Keep doing that for all cards. You can go back to the cards in order by the same type of procedure.

Aperiodic, bc. you can stay at the same permutation in one step by putting the card on the top.

Stationary distribution. All possible moves are equally likely: \( p(\sigma, \sigma') = \frac{1}{52} \) if there exists a move which transform \( \sigma \) in \( \sigma' \).

\[
\sum_{\sigma} p(\sigma, \sigma') \frac{1}{52!} = \frac{1}{52!} \frac{1}{52!} \#\{\sigma, \sigma \to \sigma' \text{ (in one step)}\} = \frac{1}{52!}
\]

because there are 52 permutations \( \sigma \) which can give \( \sigma' \) by moving the top card into the deck (just choose in \( \sigma' \) which one came from the top).

(b) NOT irreducible. The shuffle is too periodic for that: it keeps the order: if \( i \) is followed by \( j \), then after any number of shuffles, \( i \) is still followed by \( j \) (mod 52). From one permutation, you can reach only 52 other permutations... So there are 51! communication classes.