Problem 1. Let \((X_n)_{n \geq 0}\) the Markov Chain on the state space \(S = \{1, 2, 3, 4, 5\}\), with transition matrix
\[
P = \begin{pmatrix}
  1/2 & 1/3 & 0 & 0 & 1/6 \\
  0 & 0 & 1 & 0 & 0 \\
  1/2 & 0 & 0 & 1/2 & 0 \\
  1 & 0 & 0 & 0 & 0 \\
  0 & 1/3 & 0 & 2/3 & 0
\end{pmatrix}
\]
(1) Draw the graph associated to this transition matrix.
(2) Determine the communication classes and classify the states (transient or recurrent)
(3) Is the Chain irreducible?
(4) Let \(T_x = \inf\{n \geq 0, X_n = x\}\). Compute \(\mathbb{P}(X_3 = 1|X_0 = 2)\) and \(\mathbb{P}(T_2 < T_3|X_0 = 1)\).
(5) Let \(u(x) = \mathbb{P}(T_2 < T_3|X_0 = x)\), for all \(x \in S\). Determine the linear equation satisfied by \(u\) (and solve it).
(6) Let \(\lambda \geq 0\), and consider \(v(x) = \mathbb{E}[e^{-\lambda T}|X_0 = x]\). Determine the linear equation satisfied by \(v\) (and solve it).

Problem 2. Let \((X_n)_{n \geq 0}\) the Markov Chain on the state space \(S = \{1, 2, 3, 4, 5, 6\}\), with transition matrix
\[
P = \begin{pmatrix}
  1/4 & 1/2 & 0 & 0 & 0 & 1/4 \\
  1/2 & 0 & 1/4 & 1/4 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\
  0 & 0 & 0 & 1/2 & 1/2 & 0 \\
  0 & 0 & 0 & 1/4 & 1/4 & 1/2
\end{pmatrix}
\]
(1) Draw the graph associated to this transition matrix.
(2) Determine the communication classes and classify the states (transient or recurrent)
(3) Compute \(\mathbb{P}(X_n\) is eventually equal to 3\(|X_0 = i\)) for all \(i\) in the state space.
(4) Determine \(\lim_{n \to +\infty} \mathbb{P}(X_n = j|X_0 = i)\) for all \(i, j\) in the state space.

Problem 3. Two players A and B are betting $1 repeatedly, and at every bet, the probability that A wins is \(p \in (0, 1)\). The successive bets are independent. Let \(X_n\) be the amount of money player A has after \(n\) bets, and let \(a\) be the initial fortune of A, and \(b\) the initial fortune of B. The game ends when one of the two players loses all his/her fortune. We specify that if the game stops at time \(n\), then \(X_k = X_0\) for all \(k \geq n\). Hence, \(X_0 = a\), and the game ends as soon as \(X_n \in \{0, a+b\}\).
Let \(T = \inf\{n \geq 0, X_n = 0 \text{ or } X_n = a+b\}\) be the random time the game lasts. The probability that A wins given that his/her initial fortune is \(x\) is denoted \(u(x) = \mathbb{P}(X_T = a+b, T < +\infty|X_0 = x)\).

1. Show that \((X_n)_{n \geq 0}\) is a Markov Chain, give its state space \(S\), and its transition matrix \(P\).
2. Show that \(u(0) = 0, u(a+b) = 1, \) and
\[
u(x) = p u(x+1) + (1-p) u(x-1), \quad 0 < x < a+b
\]
3. Determine \(u(x)\) and \(v(x) = \mathbb{P}(X_T = 0, T < +\infty|X_0 = x)\), and conclude that \(\mathbb{P}(T = +\infty|X_0 = x) = 0\) for all \(x \in S\).
4. Show that, when \(b = +\infty\) (player against casino) and \(p = 1/2\) (fair game), one has \(v(x) = 1\) for all \(x\). Interpret.
5. Let \(m(x) = \mathbb{E}[T|X_0 = x]\) the average time the game lasts, if initial fortune of A is \(x\). Show that \(m(x)\) satisfy the relation
\[
m(x) = 1 + pm(x+1) + (1-p) m(x-1)
\]
for all \(0 < x < a+b\), with boundary conditions \(m(0) = 0, m(a+b) = 0\).
(6) Show that, if \( p = 1/2 \), the unique solution of the equation is \( m(x) = x(a + b - x) \).
(7) What happens in the case \( b = +\infty \). Interpret.

**Problem 4.** Here is the following game. A every time \( n \geq 1 \), and for every point \( i \in \mathbb{Z}^2 \), independent random variables \( U_{n,i} \) are drawn, with \( \mathbb{P}(U_{n,i} = (1,0)) = \mathbb{P}(U_{n,i} = (-1,0)) = \mathbb{P}(U_{n,i} = (0,1)) = \mathbb{P}(U_{n,i} = (0,-1)) = 1/4 \). Let us now consider two random walks \( (X_n, Y_n) \), defined by the recurrence relations

\[
\begin{align*}
X_{n+1} &= X_n + U_{n+1},X_n; \\
Y_{n+1} &= Y_n + U_{n+1},Y_n.
\end{align*}
\]

The starting point of \( X_n \) and \( Y_n \) is not specified. Note that the two random walks \( X_n \) and \( Y_n \) use the same \( U \)’s.
(1) Show that taken individually, \( X_n \) and \( Y_n \) are simple random walk. Are \( X_n \) and \( Y_n \) independent for all \( n \)? Why?
(2) Determine \( \mathbb{P}(X_n = Y_n|X_0 = Y_0 = x) \), for all \( n \in \mathbb{N} \).
(3) Show that the process \( D_n = X_n - Y_n \) is a Markov Chain on \( \mathbb{Z}^2 \). Give its transition matrix, and classify its states.
(4) Let \( T = \inf\{n \geq 0, X_n = Y_n\} \) (with \( T = +\infty \) if \( X_n \neq Y_n \) for all \( n \geq 1 \)). Show that \( \mathbb{P}(T = +\infty|X_0 = x, Y_0 = y) = 0 \), for all \( x, y \in \mathbb{Z} \) with same parity.
(5) Show that \( \mathbb{P}(X_{T+n} = Y_{T+n}|X_0 = x, Y_0 = y) = 1 \) for all \( x, y \in \mathbb{Z} \) with same parity, and all \( n \in \mathbb{N} \). Interpret.
This is called a coupling of the two random walks \( X \) and \( Y \): taken individually, they look like random walks, but they are coupled, in the sense that when they merge, they stick together. It enables us to prove some nice results.
(6) Show that, for any bounded function \( f : \mathbb{Z}^2 \to \mathbb{R} \),

\[ \mathbb{E}[|f(X_n) - f(Y_n)|] \leq 2C\mathbb{P}(T > n), \]

where \( C = \sup_{x \in \mathbb{Z}^2} |f(x)| \).

**Problem 5.** Two urns communicate through a very small hole. There are \( N \) particles distributed in the two urns. At each unit of time, a particle is chosen at random (among the \( N \) particles), and moved to the other urn with probability \( q \in (0,1) \). Let \( X_n \) denote the number of particles in the left urn after \( n \) units of time.
(1) Give the state space of the Markov Chain \( X \), and its transition matrix.
(2) Classify the states of the Markov Chain. Deduce that a stationary distribution exists.
(3) Describe the equations a reversible distribution satisfy. Deduce what is the (reversible) stationary distribution \( \pi \).
(4) Is the chain aperiodic? Tell what is \( \lim_{n \to \infty} \mathbb{P}(X_n = 0) \), with any initial condition.
(5) Set \( \mu_n := \mathbb{E}[X_n|X_0 = 0] \). Show that \( \mu_{n+1} = q + (1 - 2q/N)\mu_n \), and deduce the expression of \( \mu_n \). What is the limit of \( \mu_n \)? How fast does the convergence occur (find the time at which \( |\mu_n - N/2| \leq 1 \))?

**Problem 6.**


Let \( W_n \) be a (possibly lazy) random walk on the \( \{0, \ldots, N\} \), defined as follow: if \( W_n = 0 \) or \( W_n = N \), then \( W_{n+1} = W_n \), otherwise

\[
W_{n+1} = \begin{cases} 
W_n & \text{with probability } (1 - q) \\
W_n + 1 & \text{with probability } q/2 \\
W_n - 1 & \text{with probability } q/2
\end{cases}
\]
for \( q \in (0, 1] \) (\( q = 1 \) is the standard case).

- Specify the state space and the transition matrix. Then, classify the states.
- Deduce that \( \mathbb{P}(T < +\infty) = 1 \), where \( T := \inf\{ n \mid W_n = 0 \text{ or } W_n = N \} \).
- We call \( m_i := \mathbb{E}[T|X_0 = i] \). Find a set of equations (and boundary conditions) satisfied by \((m_i)_{i \in \{0, \ldots, N\}}\). Deduce that \( m_i = q^{-1}(N - i) \).

**Part 2. Random walk on a circle**

We consider the lazy random walk on a circle \( \mathbb{Z}/N\mathbb{Z} \), that we call \( X_n \), defined by

\[
X_{n+1} = \begin{cases} 
X_n & \text{with probability } (1 - q) \\
X_n + 1 \mod N & \text{with probability } q/2 \\
X_n - 1 \mod N & \text{with probability } q/2 
\end{cases}
\]

- Specify the state space and the transition matrix.
- Show that a stationary distribution exist, find it, and show that it is reversible.

**Part 3. Mixing time**

We try to estimate how fast the distribution converges to the equilibrium. We start two (possibly lazy) random walks \( X_n \) and \( Y_n \) on the circle which cannot move at the same time: at each turn, we flip a fair coin to decide whether \( X_n \) or \( Y_n \) moves. We call \( U := \inf\{ n \mid X_n = Y_n \} \).

- Explain why, if \( U \leq n \), then \( X_n \) and \( Y_n \) have the same distribution.
- Deduce that \( |p^n(i, k) - p^n(j, k)| \leq \mathbb{P}(U > n|X_0 = i, Y_0 = j) \leq \frac{1}{n}\mathbb{E}[U|X_0 = i, Y_0 = j] \).
- Using Part 1 (3rd point), and considering \( W_n := X_n - Y_n \mod N \), estimate \( \mathbb{E}[U|X_0 = i, Y_0 = j] \). Deduce that \( \max_{i,j \in \{1, \ldots, N\}} |p^n(i, k) - p^n(j, k)| \leq \frac{N^2}{4qn} \) whatever \( k \) is.
- Making \( Y \) starts from the stationary measure, show that \( \max_{i \in \{1, \ldots, N\}} |p^n(i, k) - \frac{1}{N}| \leq \frac{N^2}{4qn} \) whatever \( k \) is. Deduce the time it takes to approach equilibrium.