On random walks

Random walk in dimension 1. Let \( S_n = x + \sum_{i=1}^{n} U_i \), where \( x \in \mathbb{Z} \) is the starting point of the random walk, and the \( U_i \)'s are IID with \( \mathbb{P}(U_i = +1) = \mathbb{P}(U_i = -1) = 1/2 \).

1. Let \( N \) be fixed (goal you want to attain). Compute the probability \( p_x \) that the random walk reaches 0 before \( N \), starting from \( x \). (Hint: show that \( p_{x+1} - p_x = \frac{1}{2}(p_x - p_{x-1}) \) for \( x \in \{1, \ldots, N\} \)).

2. Use 1. to compute the probability that, starting from 0, you reach \( a > 0 \) before returning to the origin.

3. Use 1. to compute the probability that, starting from the origin, you reach \( a > 0 \) before returning to the origin.

4.* Use 3. to show that the average number of visits to \( a > 0 \) before returning to the origin is 1 (hint: show that it is closely related to the expectation of some geometric random variable).

5. Do the same 1.2.3.4. problems when the random walk is "lazy": \( \mathbb{P}(U_i = 0) = \mathbb{P}(U_i = +1) = \mathbb{P}(U_i = -1) = 1/3 \).

5bis.

6. Suppose that \( x = 0 \). Recall how to prove that \( \mathbb{P}(S_1 \geq 0, \ldots, S_{2n} \geq 0) = \mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0) \).

Random walk in dimension 2. Let \( Z_n = \sum_{i=1}^{n} W_i \), where the \( W_i \)'s are IID nearest neighbor steps \( \mathbb{P}(W_i = (1, 0)) = \mathbb{P}(W_i = (-1, 0)) = \mathbb{P}(W_i = (0, 1)) = \mathbb{P}(W_i = (0, -1)) = 1/4 \). We denote \( W_i = (W_i^{(1)}, W_i^{(2)}) \), and \( Z_n = (X_n, Y_n) \).

7. Show that if you define \( U_i := W_i^{(1)} + W_i^{(2)} \) and \( V_i := W_i^{(1)} - W_i^{(2)} \), then \( A_n := \sum_{i=1}^{n} U_i \) and \( B_n := \sum_{i=1}^{n} V_n \) are independent one-dimensional simple random walks.

8. Use \( A_n \) and \( B_n \) defined in the previous question to compute \( \mathbb{P}(Z_{2n} = (0, 0)) \).

9. Use questions 6.7.8. to compute the probability that the random walk stays in the cone \( \{x + y \geq 0, x - y \geq 0\} \) up to time \( 2n \).

On Markov Chains

1. A taxicab driver moves between the airport A and two hotels B and C according to the following rules: if at the airport: go to one of the hotels with equal probability, and if at one hotel, go to the airport with probability 3/4, and to the other hotel with probability 1/4.

   (a) Find the transition matrix.

   (b) Suppose the driver starts at the airport. Find the probability for each of the three possible location at time 2. Find the probability that he is at the airport at time 3.

   (c) Find the stationary distribution, and apply the convergence theorem to find \( \lim_{n \to \infty} \mathbb{P}(\text{at the airport at time n}) \).

2. Consider the following transition matrices. Classify the states, and describe the long-term behavior of the chain (and justify your reasoning).

\[
P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0.2 & 0.8 & 0 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0 & 0 & 0.7 \end{pmatrix}, \quad P = \begin{pmatrix} .4 & .4 & .3 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & .3 & 0 & .3 & .4 \end{pmatrix}, \quad P = \begin{pmatrix} 2/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}
\]

3. Six children (A, B, C, D, E, F) play catch. If A has the ball, then he/she is equally likely to throw the ball to B, D, E or F. If B has the ball, then he/she is equally likely to throw the ball to A, C, E or F. If E has the ball, then he/she is equally likely to throw the ball to A, B, D, F. If either C or F gets the ball, they keep throwing it at each other. If D gets the balls, he/she runs
away with it.
(a) Find the transition matrix, and classify the states.

(b) Suppose that A has the ball at the beginning of the game. What is the probability that D ends up with the ball?

4. Find the stationary distribution(s) for the Markov Chains with transition matrices

\[
P = \begin{pmatrix} .4 & .6 & 0 \\ .2 & .4 & .4 \\ 0 & .3 & .7 \end{pmatrix} \quad P = \begin{pmatrix} .4 & .6 & 0 & 0 \\ 0 & .7 & .3 & 0 \\ 1 & 0 & .4 & .5 \\ .5 & 0 & 0 & .5 \end{pmatrix}
\]

(a) Compute \(P^2\). (b) Find the stationary distributions of \(P\) and \(P^2\). (c) Find the limit of \(p^{2n}(x, x)\) as \(n \to \infty\).

6. Folk wisdom holds that in Ithaca in the summer, it rains \(1/3\) of the time, but a rainy day is followed by a second one with probability \(1/2\). Suppose that Ithaca weather is a Markov chain. What is its transition matrix?

7. Consider a Markov chain with transition matrix \(P = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}\). Use the Markov property to show that

\[
\mathbb{P}(X_{n+1} = 1) - \frac{b}{a + b} = (1 - a - b) \left( \mathbb{P}(X_n = 1) - \frac{b}{a + b} \right).
\]

Deduce that if \(0 < a + b < 2\), then \(\mathbb{P}(X_n = 1)\) converges exponentially fast to its limiting value \(b/(a + b)\).

8. Let \(N_n\) be the number of heads observed in the first \(n\) flips of a fair coin, and let \(X_n = N_n \mod 5\). Use the Markov Chain \(X_n\) to find \(\lim_{n \to \infty} \mathbb{P}(N_n\text{ is a multiple of }5)\).

9. A basketball player makes a shot with the following probabilities: \(1/2\) if he misses the last two times, \(2/3\) if he has hit one of his last two shots, \(3/4\) if he has hit both his last two shots. Formulate a Markov Chain to model his shooting, and compute the limiting fraction of the time he hits a shot.

10. Ehrenfest chain. Consider \(N\) particles, divided into two urns (which communicate through a small hole). Let \(X_n\) be the number of particles in the left urn. The transition probabilities are given by the following rule: at each step, take one particle uniformly at random among the \(N\) particles, and move it to the other urn (from left to right or right to left, depending on where the particle you chose is).

   (a) Give the transition probabilities of this Markov Chain.
   (b) Let \(\mu_n := \mathbb{E}[X_n|X_0 = x]\). Show that \(\mu_{n+1} = 1 + (1 - 2/N)\mu_n\).
   (c) Show by induction that \(\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n(x - N/2)\).

11. Random walk on a circle. Consider the numbers 1, 2, \ldots, \(N\) written around a ring. Consider a Markov Chain that at any point jumps with equal probability to one of the two adjacent numbers.

   (a) What is the expected number of steps that \(X_n\) will take to return to its starting position?
   (b) What is the probability that \(X_n\) will visit all the other states before returning to its starting position?

12. Knight’s random walk. We represent a chessboard as \(S = \{(i, j), 1 \leq i, j \leq 8\}\). Then a knight can move from \((i, j)\) to any of eight squares \((i+2, j+1), (i+2, j-1), (i+1, j+2), (i+1, j-2), (i-1, j+2), (i-1, j-2), (i-2, j+1), (i-2, j-1)\), provided of course that they are on the
chessboard. Let \( X_n \) be the sequence of squares that results if we pick one of knight’s legal move at random. Find the stationary distribution, and deduce the expected number of moves to return to the corner \((1,1)\), when starting at that corner.

13. Queen’s random walk. Same questions as 12., but for the Queen, which can move any number of squares horizontally, vertically or diagonally.

14. If we have a deck of 52 cards, then its state can be described by a sequence of numbers that gives the cards we find as we examine the deck from the top down. Since all the cards are distinct, this list is a permutation of the set \( \{1, 2, \ldots, 52\} \), i.e., a sequence in which each number is listed exactly once. There are 52! permutations possible. Consider the following shuffling procedures:

(a) pick the card from the top, and put it uniformly at random in the deck (on the top or bottom is acceptable!);

(b) cut the deck into two parts.

Show that, in the two cases, if one repeats the algorithm, then in the limit, the deck is perfectly shuffled, in the sense that all 52! possibilities are equally likely.