**Problem 1.** (50 points) Consider the Markov Chain on the state space \{1, 2, 3, 4, 5, 6\} with the following transition matrix

\[
P = \begin{pmatrix}
0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1/3 & 2/3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 0 & 2/3 & 1/3 \\
0 & 1/2 & 0 & 0 & 0 & 1/2
\end{pmatrix}
\]

(1) (10 points) Draw the graph associated with that transition matrix. Give the communication classes, and classify the states of the Markov Chain.

(2) (15 points) Compute the probabilities \(\lim_{n \to \infty} P(X_n = 3 | X_0 = 1)\) and \(\lim_{n \to \infty} P(X_n = 3 | X_0 = 4)\).

(3) (15 points) Consider the Markov Chain reduced to \{2, 5, 6\}, with transition matrix

\[
P = \begin{pmatrix}
0 & 1/3 & 2/3 \\
0 & 2/3 & 1/3 \\
1/2 & 0 & 1/2
\end{pmatrix}
\]

What is its stationary distribution?

(4) (10 points) Compute \(\lim_{n \to \infty} P(X_n = i | X_0 = 4)\) for all \(i \in \{1, 2, 3, 4, 5, 6\}\).

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**Classes of communication**

\(\{3\} : \) positive recurrent

\(\{1, 4\} : \) transient

\(\{2, 5, 6\} : \) positive recurrent.

2) Note \(a_k = \lim_{n \to \infty} P(X_n = 3 | X_0 = k)\)

\[a_2 = a_5 = a_6 = 0, \quad a_3 = 1\]

One has the relations

\[a_1 = \frac{1}{4} a_2 + \frac{1}{2} a_4 + \frac{1}{4} a_6, \quad a_4 = \frac{1}{2} a_4\]
and \[ a_4 = \frac{1}{4} a_1 + \frac{1}{4} a_2 + \frac{1}{4} a_3 + \frac{1}{4} a_5 \]
\[ = \frac{1}{4} a_1 + \frac{1}{4} = \frac{1}{8} a_4 + \frac{1}{4} \]
\[ \rightarrow a_4 = \frac{2}{7} \quad ; \quad a_1 = \frac{1}{7} \]

3) \text{Irreducible, positive rec., aperiodic (SQR)}

Stationary distribution: \[ \left( \frac{1}{4} , \frac{1}{4} , \frac{1}{2} \right) \]

\[
\begin{bmatrix}
\text{solve} \\
\pi_2 , \pi_5 , \pi_6 \\
\end{bmatrix}
= 
\begin{bmatrix}
\pi_2 , \pi_5 , \pi_6 \\
\end{bmatrix}
\begin{bmatrix}
0 & \frac{3}{5} & \frac{2}{5} \\
0 & \frac{2}{5} & \frac{3}{5} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\end{bmatrix}
\]

\[
\begin{aligned}
\pi_2 &= \frac{1}{2} \pi_6 \\
\pi_5 &= \frac{1}{3} \pi_2 + \frac{2}{3} \pi_5 \\
\pi_6 &= \frac{1}{3} \pi_2 + \frac{2}{3} \pi_5 \\
\end{aligned}
\]

4) \[ \lim \ P \left( X_n = i \mid X_0 = 4 \right) = \begin{cases} 
0 & \text{if } i = 1, 4 \quad (\text{Transience}) \\
\frac{2}{7} & \text{if } i = 3 \quad (\text{Recurrence}) \\
\end{cases} \]

By the limit theorem.

\[
\begin{aligned}
\pi_2 &= \frac{1}{4} \\
\pi_5 &= \frac{1}{4} \\
\pi_6 &= \frac{1}{2} \\
\end{aligned}
\]
Problem 2. (80 points) Let $(U_k)_{k \geq 1}$ be a sequence of IID random variables, with values in \{1, 2, \ldots\}. The random variable $U_k$ is interpreted as the lifetime of the $k$th component. A component is replaced as soon as it breaks: at time 0, the 1st component is put in service, and lasts until time $U_1$, when it is immediately replaced by the 2nd component which lasts an interval of time $U_2$ (thus until time $U_1 + U_2$), immediately replaced by the 3rd component, etc...

We denote by $X_n$ the time elapsed since the last component was replaced ($X_n = 0$ means that the component has just been replaced, $X_n$ is also the time the current component has been in use). We assume for simplicity that $\mathbb{P}(U_1 = k) > 0$ for all $k \geq 1$, and $\mathbb{P}(U_1 = 0) = 0$.

(1) (15 points) Show that $X_n$ is a homogeneous Markov chain, with transition probabilities

$$p(i, j) = \begin{cases} \frac{\mathbb{P}(U_{i+2} \geq i+1)}{\mathbb{P}(U_{i+1} \geq i+1)} & \text{if } j = i+1; \\ 1 - p(i, i+1) & \text{if } j = 0; \\ 0 & \text{otherwise}. \end{cases} \quad \forall i, j \in \{0, 1, \ldots\}$$

(2) (10 points) Is the chain irreducible? Is it aperiodic?

(3) (10 points) Show that the chain is recurrent. Show that it is positive recurrent if and only if $\mathbb{E}[U_1] < +\infty$.

(4) (10 points) Let us focus on the case $\mathbb{E}[U_1] < +\infty$, so that the stationary distribution $\pi$ exists. What is $\pi(0)$? Approximate the probability that you have to replace a component at time $n$, when $n$ is really large. [Hint: no calculation is needed]

(5) (10 points) Show that the stationary distribution is $\pi(i) = \frac{1}{\mathbb{E}[U_1]} \mathbb{P}(U_1 = i + 1)$ (recall that one has $\mathbb{E}[U_1] = \sum_{k=1}^{+\infty} \mathbb{P}(U_1 \geq k)$, because $U_1$ is $\mathbb{N}$-valued). Is $\pi$ reversible?

(6) (10 points) Approximate (for $n$ large) the probability that the component in use at time $n$ will still last more than $k$ units of time.

(7) (15 points) Let us now assume that the distribution of $U_1$ is geometric, with parameter $q$ (probability of failure of a component): $\mathbb{P}(U_1 = j) = (1-q)^{j-1}q$. What is the stationary distribution in that case? What is the probability that the component currently in use will still last more than $k$ units of time? (Note that in that case, the computation is exact, and one does not need the limit theorem).

For $j = i+1$

$$\mathbb{P}(X_{n+1} = j \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = i) = \mathbb{P}(\text{last component survived } i \text{ more unit of time})$$

$$= \mathbb{P}(\text{last component survived } i \text{ units} \mid \text{given that it survived for } i \text{ units})$$

$$= \frac{\mathbb{P}(U_i > i+1 \mid U_i > i)}{\mathbb{P}(U_i > i+1)} = \frac{\mathbb{P}(U_i > i+2)}{\mathbb{P}(U_i > i+1)} \quad \text{time a component lasts has same distrib. as } U_i \quad \text{as } U_1$$

The only other possibility is $j = 0$,

because $X_n$ can only increase by 1 or be reset to 0.

$$p_{ij} = 0 \quad \text{if } j \neq i+1 \quad \text{if } j = 0, \quad p_{io} = 1 - p_{i,i+1}$$
2) \( p_{ij} > 0 \) because \( p_i(j) = P(U_i \geq j+1) > 0 \) and \( p_{ij} > 0 \) so that \( 0 \leftrightarrow j \) for all \( j \in \mathbb{N} \)

\( \rightarrow \) the chain is irreducible.

\( \otimes \) \( p_{00} = P(U_1 = 1) > 0 \), so that the period of 0 is 1.

\( \rightarrow \) the chain is aperiodic (all states have the same period because of irreducibility)

3) Let \( T_0 = \inf \{ n > 0 \mid X_n = 0 \} \) be the first return time to 0 starting from \( X_0 = 0 \), \( T_0 = U_1 \)

\( \rightarrow \) \( P(T_0 < +\infty \mid X_0 = 0) = P(U_1 < +\infty) = 1 \)

\( \rightarrow \) 0 is recurrent, so that the chain is recurrent (because of irreducibility)

The mean return time to 0 is \( \frac{1}{P_0} E[T_0 \mid X_0 = 0] = E[U_1] \)

\( \rightarrow \) 0 is null recurrent (and so is the chain) iff \( E[U_1] = +\infty \)

4) Chain irreducible + positive recurrent \( \rightarrow \pi \exists \)

\( \pi(i) = \frac{1}{\mu_i} \), we already have \( \pi(0) = \frac{1}{P_0} = \frac{1}{E[U_1]} \)

Using the limit theorem (chain is also aperiodic)

\( \lim_{n \to \infty} P(\text{need to change component at time } n) = \lim_{n \to \infty} P(X_n = 0) = \pi(0) = \frac{1}{E[U_1]} \)
5) One has by the stationarity condition
\[ \pi(i) = \frac{P(i-1, i) \pi(i-1)}{P(i, i-1)} = \frac{P(U_1 > i+1) \pi(i-1)}{P(U_1 > i)} \]
Iterating, one gets
\[ \pi(i) = \frac{\sum_{j=0}^{\infty} \frac{P(U_1 = j) \pi(j)}{\mathbb{E}[U_1]}} \]
and
\[ P(U_1 > i) = \sum_{i=0}^{\infty} \pi(i), \quad \pi(0) = \frac{1}{\mathbb{E}[U_1]} \]

6) Using the limit theorem. One decomposes the probability according to how long the component has been in use:
\[ P(\text{component will last } k \text{ more units of time}) = \sum_{i=0}^{\infty} P(X_n = i) P(\text{component lasts } k \text{ more units of time} | X_n = i) \]

Use limit theorem + dominated convergence theorem:
\[ \sum_{i=0}^{\infty} \pi(i), \quad P(U_1 = i+1) / P(U_1 > i) \]
\[ = \frac{1}{\mathbb{E}[U_1]} \sum_{i=0}^{\infty} P(U_1 > i+1) \]

7) If \( U_i \) is geometric parameter \( q \) (prob of failure of a component):
\[ \mathbb{E}[U_i] = \frac{1}{q}; \quad P(U_i > i+1) = P(\text{i failures in a row}) = (1-q) \]
So
\[ \pi(i) = q(1-q)^i \]
[shifted geometric]

From 6), the probability is
\[ q \sum_{i=0}^{\infty} (1-q)^i = q (1-q) \frac{1}{q} = (1-q) \]