We consider here a Markov chain \((X_n)_{n \geq 0}\), whose state space is \(\mathbb{N} = \{0, 1, 2, \ldots\}\). At each time, \(X_n\) can either increase by 1, stay the same, or decrease by 1 (it can be for example a queue). For every \(x \geq 1\), the transition probabilities are given by
\[
P(x, x - 1) = p(x), \quad P(x, x) = r(x), \quad P(x, x + 1) = q(x),
\]
where \(p(x) + r(x) + q(x) = 1\) for all \(x\). Naturally, \(p(0) = 0\), and we assume that \(p(x) > 0\) for all \(x \geq 1\), and \(q(x) > 0\) for all \(x \geq 0\).

1. Show that the chain is irreducible. Under which condition is the chain aperiodic?

2. Let us denote \(u_k := \mathbb{P}(X_n \neq 0 \text{ for all } n \geq 0 | X_0 = k)\). Show that, for all \(k \geq 1\), one has
\[
u_{k+1} - u_k = \frac{p(k)}{q(k)} (u_k - u_{k-1}).
\]

3. Use the previous relations to show that if
\[
\sum_{i=1}^{\infty} \frac{p(1) \cdots p(i)}{q(1) \cdots q(i)} = +\infty,
\]
then \(u_1 = 0\). Conclude for a sufficient condition for the recurrence of the chain. (It is actually a necessary and sufficient condition)

4. Let \(\mu(\cdot)\) be a solution of \(\mu = \mu P\) for the Markov Chain \((X_n)_{n \in \mathbb{N}}\). Show that, for all \(x \geq 1\)
\[
\mu(x) = \frac{q(0) \cdots q(x-1)}{p(1) \cdots p(x)} \mu(0).
\]
Deduce a condition under which the chain is positive recurrent. What is then the invariant probability measure? Show that it is also reversible.

5. Application: let \(X_n\) represent the length of a queue. At time \(n\), if the queue is non empty, the request of the first person in the line is treated with probability \(v\) (which does not depend on the size of the queue), and one customer arrives with probability \(w(x)\), independently of the treatment of the request (note that \(w(x)\) possibly depends on the size of the queue \(x\)). What are \(p(x), r(x), q(x)\) in this context?

Let us consider two cases: \(w(x) = w\) (a given constant), or \(w(x) = 1/(1+x)\). In both cases, under which condition (on \(v\) and \(w\)) is the chain positive recurrent? What are the invariant probability measures in both cases? What does the convergence theorem then say?