**Final Exam Solutions**
**Math 118, December 23, 2004**

**Problem 1.** Calculate the following limits. If the limit is infinite, indicate whether it is $+\infty$ or $-\infty$.

(a) $\lim_{x \to 4^+} \sqrt{\frac{4x}{x-4}}$

(b) $\lim_{x \to 2} \frac{x^2-16}{2-\sqrt{x}}$

(c) $\lim_{t \to -3} \frac{t^2 + 6t + 9}{9 - t^2}$

(d) $\lim_{x \to \infty} \frac{\sqrt{x^2-4}}{x}$

(e) $\lim_{t \to 0} \frac{\sqrt{t} + 4 - 2}{t}$

**Solution.**

(a) As $x \to 4^+$, $x - 4 \to 0$ and $x - 4$ remains positive; on the other hand, $4x \to 16$. Thus $4x/(x - 4) \to +\infty$, as does the square root of $(4x)/(x - 4)$:

Answer (a): $+\infty$

(b) As $x \to 4$, the denominator $2 - \sqrt{x} \to 0$; the numerator also $\to 0$. Thus we are first going to have to do some algebraic simplifications. A direct way is to note that

$$x^2 - 16 = (x - 4)(x + 4) = (\sqrt{x} - 2)(\sqrt{x} + 2)(x + 4),$$

and thus

$$\frac{x^2 - 16}{2 - \sqrt{x}} = \frac{-x^2 - 16}{\sqrt{x} - 2} = -\frac{(\sqrt{x} + 2)(x + 4)}{\sqrt{x} - 2}.$$

As $x \to 4$ we have $\sqrt{x} \to 2$ (therefore $\sqrt{x} + 2 \to 4$) and $x + 4 \to 8$; thus by the “limit of a product is the product of the limits” rule,

$$\lim_{x \to 4} \frac{x^2 - 4}{2 - \sqrt{x}} = -4 \times 8 = -32.$$

What is not acceptable is an application of L’Hospital’s Rule,

$$\lim_{x \to 4} \frac{x^2 - 4}{2 - \sqrt{x}} = \lim_{x \to 4} \frac{2x}{-1/2\sqrt{x}} = -8 \frac{1/4}{1} = -32.$$

(If you don’t understand this computation, don’t worry about it: L’Hospital’s Rule isn’t taught in Math 118, which is why no credit was given for this solution.)

Answer (b): $-32$

(c) Once again, the numerator approaches 0 and the denominator approaches 0, so we simplify the fraction by factoring the numerator:

$$\lim_{t \to -3} \frac{t^2 + 6t + 9}{9 - t^2} = \lim_{t \to -3} \frac{(t + 3)^2}{(t - 3)(t + 3)} = \frac{t + 3}{3 - t}.$$

This equality is valid for all $t \neq -3$, which is sufficient for computing the limits:

$$\lim_{t \to -3} \frac{t^2 + 6t + 9}{9 - t^2} = \lim_{t \to -3} \frac{t + 3}{3 - t} = \frac{0}{6} = 0.$$

Answer (c): 0

(d) Looks similar to part (c), but the limit is as $x \to \infty$. The trick now is to divide numerator and denominator by $x^2$, so

$$\frac{x^2 + 6x + 9}{9 - x^2} = \frac{1 + 6/x + 9/x^2}{9/x^2 - 1}.$$

Since $1/x \to 0$ and $1/x^2 \to 0$ as $x \to \infty$, the numerator of this new expression converges to 1 and the denominator to $-1$. Thus by the “quotient of the limits is the limit of the quotients” rule,

Answer (d): $-1$
(e) This one is a little tricky, and goes somewhat against the grain of the way you were taught to simplify things in high-school: we “rationalize the numerator,” not the denominator!

\[
\frac{\sqrt{t+4} - 2}{t} = \frac{(\sqrt{t+4} - 2)(\sqrt{t+4} + 2)}{t(\sqrt{t+4} + 2)}
\]

\[
= \frac{(t+4) - 4}{t(\sqrt{t+4} + 2)}
\]

\[
= \frac{1}{t(\sqrt{t+4} + 2)}
\]

This trick gets rid of the “singularity” which division by zero would cause, by cancelling a \(t\) from numerator and denominator. At any rate, as \(t \to 0\) we have \(t + 4 \to 4\) and therefore \(\sqrt{t+4} \to \sqrt{4} = 2\), hence

\[
\lim_{t \to 4} \frac{\sqrt{t+4} - 2}{t} = \lim_{t \to 4} \frac{1}{\sqrt{t+4} + 2} = \frac{1}{4}.
\]

Answer (e): \(\frac{1}{4}\)

□

Problem 2. Differentiate \(f(x)\), where:

(a) \(f(x) = \frac{2x + 1}{3x^2 + 1}\);
(b) \(f(x) = \ln(x^2 + 1)\);
(c) \(f(x) = x^3e^x\);
(d) \(f(x) = \ln(x^3)\);
(e) \(f(x) = \frac{e^x - 1}{x}\).

Solution.

(a) We have

\[
f'(x) = \frac{(3x^2 + 1)(2x) - 6x(x^2 + 1)}{3x^2 + 1)^2}
\]

\[
= \frac{-6x^2 - 6x + 2}{3x^2 + 1)^2}.
\]

Answer (a); \(-6x^2 - 6x + 2 = \frac{2 \cdot (x - 1)e^x + 1}{x^2}\)

(b) Remember that the derivative of \(\ln u\) with respect to \(u\) is \(1/u\), and the chain rules: \(f'(x) = \frac{1}{x^2 + 1} \cdot 2x\) = \(\frac{2x}{x^2 + 1}\).

Answer (b); \(\frac{2x}{x^2 + 1}\)

(c) Three rules are applied: for the limit of a product, the fact that \(\lim_{x \to 0} x^3 = 1\), and the chain rule:

\[
f'(x) = 4x^3e^x + x^4 \cdot 2xe^x
\]

\[
= (4x^3 + 2x^2)e^x
\]

\[
= 2x^3(2x + 1)e^x.
\]

Answer (c); \(2x^3(2x + 1)e^x\)

Your instructions were to “simplify all expressions.” Points were taken off for not factoring.

(d) There are two ways to do this problem; it was put on in the hope that good students would see that

\[
\ln x^3 = 3 \ln x,
\]

and therefore

\[
f'(x) = \frac{d}{dx}(3 \ln x) = \frac{3}{x}.
\]

Alternatively, it’s not much harder to use the chain rule:

\[
f'(x) = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}.
\]

Points were taken off if you didn’t simplify the last expression to \(3/x\).

Answer (d); \(\frac{3}{x}\)

(e) This is the rule for differentiating a quotient:

\[
f'(x) = \frac{x e^x - (e^x - 1) \cdot 1}{x^2} = \frac{(x - 1)e^x + 1}{x^2}.
\]

Originally, the thought was to ask students to show that \((e^x - 1)/x\) is an increasing function. (This is a general property of functions: if \(g\)
is concave up, then \((g(x) - g(a))/(x - a)\) is increasing.) To do that, you need to show the derivative
\[
\frac{(x - 1)e^x + 1}{x^2}
\]
is positive for \(x > 0\). Since the denominator is positive, that means we have to show the numerator, call it \(h(x)\),
\[
h(x) = (x - 1)e^x + 1,
\]
is positive for \(x > 0\). We’ll show \(h(x)\) is increasing and \(h(0) = 0\); that would mean that \(h(x) > h(0)\) for \(x > 0\), i.e. \(h(x) > 0\) for \(x > 0\). But
\[
h'(x) = e^x + (x - 1)e^x = xe^x,
\]
which is certainly positive for \(x > 0\), and this proves the result.

Since no similar problems were assigned in homework (there aren’t any similar problems in the book), we left the problem at the bare minimum:

Answer (e): \(\frac{(x - 1)e^x + 1}{x^2}\)

\[\square\]

**Problem 3.** Find the equation of the tangent line to the curve
\[x^3 - 3y^3 - x + e^y = 1\]
at \(x = 1, y = 0\).

**Solution.** You really should check that \(x = 1, y = 0\) is a point on the curve:
\[1^3 - 30^3 - 1 + e^0 = 1.
\]Yup. (Never trust a professor.)

OK, the equation of the tangent line is
\[y - 0 = m(x - 1),\]
where the slope \(m\) is the derivative \(dy/dx\) evaluated at \(x = 0\). To evaluate this we implicitly differentiate the equation with respect to \(x\):
\[
\frac{d}{dx} \left( x^3 - 3y^3 - x + e^y \right) = \frac{dy}{dx} 1
\]
i.e.
\[3x^2 - 9y^2 \frac{dy}{dx} - 1 + e^y \frac{dy}{dx} = 0.
\]

We solve this last for \(dy/dx\),
\[
\frac{dy}{dx} = \frac{1 - 3x^2}{e^y - 9y^2}.
\]
Substituting \(x = 1, y = 0\), we obtain
\[m = \frac{-2}{1} = -2\]
Therefore the answer is \(y = -2(x - 1) = -2x + 2\).

Answer: \(y = -2x + 2\)

Figure 1 shows the solution curve for the equation (note that it’s *not* the graph of a function; by wandering about like this, there are several places where the tangent line is vertical and the graph cuts back under or over itself). But near \(x = 1, y = 0\) there is indeed a section of the graph which is the graph of a function. We have plotted the tangent line as a dashed line.

\[\square\]
Problem 4. The graph of the derivative $f'(x)$ of a function $f(x)$ is given below.

Figure 2. Graph of the derivative of $f$, $y = f'(x)$.

Indicate by writing the answer in the provided boxes:

(a) The $x$-coordinate(s) of all the relative maxima of the function $f(x)$.
(b) The $x$-coordinate(s) of all relative minima of the function $f(x)$.
(c) The $x$-coordinate(s) of all inflection points of the function $f(x)$.
(d) Specify the interval(s) on which the function $f(x)$ is concave down.
(e) Given that $f(0) = 0$ and that the shaded region has area 108/5, find the value of $f(-2)$.

Solution. We emphasize that this is the graph of the derivative of $f$, not the graph of $f$!!

The critical numbers of $f$ are where $f'(x) = 0$. From the graph we read off that these are at $x = -2$, $x = 0$, and $x = 3$. We see that $f'$ is negative on the interval $(-2, 0)$, and is positive (or zero) everywhere else. Therefore $f$ is increasing on $(-\infty, -2]$ (because $f'$ is positive there!), decreasing on $[-2, 0]$, and increasing on $[0, +\infty)$. Thus the critical number $x = -2$ is a relative maximum point (the function is increasing just to the left, decreasing just to the right); the critical number $x = 0$ is a relative minimum point (the function is decreasing just to the left, increasing just to the right); and the critical number $x = 3$ is neither a relative maximum nor a relative minimum point.

As for inflection points and intervals of concavity: $f$ is concave up in an interval where $f'' > 0$, i.e. where $f'$ is increasing, and is concave down where $f'' < 0$, i.e. where $f'$ is decreasing. By eyeballing the graph of $f'$, we see that $f'$ is decreasing from $-\infty$ to $-1$; increasing from $-1$ to $1$; decreasing from $1$ to $3$; thence increasing again from $3$ to $+\infty$. Therefore $f$ is concave down on $(-\infty, -1]$ and on $[1, 3]$. Points of inflection are at $x = -1$ ($f$ changes from being concave down to being concave up) and at $x = 1$ (changes from concave up to concave down) and at $x = 3$ (changes from concave down to concave up). Caveat: we have used $"-\infty"$ and $"+\infty"$ as though the graph continued to the left and right. But the instructions are that this is the graph of $f'$. The domain of the function appears to run from about $-2.5$ to about $4.5$. Thus where we have used $"-\infty"$ above, $-2.5$ is more correct; and where we have used $"+\infty"$ the value of $4.5$ is more correct. Points weren’t taken off for this subtlety.

As for the value of $f(-2)$: we are told that the shaded region has area 108/5; since the integral interprets regions below the axis as having negative area, this means we are told that

$$\int_{-2}^{0} f'(x) \, dx = \frac{-108}{5}.$$  

By the Second Fundamental Theorem of Calculus

$$\int_{-2}^{0} f'(x) \, dx = f(0) - f(-2),$$

and we are told that $f(0) = 0$; thus

$$-\frac{108}{5} = 0 - f(-2).$$

Therefore $f(-2) = \frac{108}{5}$.

Answer (a): Relative maximum at $x = -2$

Answer (b): Relative minimum at $x = 0$

Inflection points at $x = -1, x = 1, x = 3$

Concave down on $[-2.5, 1]$ and on $[1, 3]$. 
An aside: how was the graph in Figure 2 created? It is actually the graphs of two functions joined at \( x = 0 \). The left half is the graph of \( y = (9/2)x(x + 2) \) This creates an upright parabola crossing the axis at \( x = -2 \) and \( x = 0 \) (I’ll explain the factor \( 9/2 \) in a moment). The right half is the graph of \( y = x(x - 3)^2 \) (this has roots at \( x = 0 \) and \( x = 3 \), and the square \( (x - 3)^2 \) guarantees that the graph has a root at \( x = 3 \) but doesn’t go negative near \( x = 3 \). Finally, to look right, these two graphs should have the same derivative at \( x = 0 \). Since \( \frac{d}{dx} x(x + 2) = 2x + 2 = 2 \) when \( x = 0 \), while \( \frac{d}{dx} x(x + 3)^2 = 3x^2 - 12x + 9 = 9 \) when \( x = 0 \), we see that a multiplying factor of \( 4.5 \) is appropriate for the first function. (Not multiplying by \( 4.5 \) wouldn’t have changed any of the answers of the problem, and the mismatch of tangents at \( x = 0 \) wouldn’t even be noticeable to the eye, but I’m a perfectionist.)

\[ f(-2) = \frac{108}{5} \]

Problem 5. A local TV station has made a survey of the watching habits of children on Sundays between the hours of 8:00 AM and 11:00 AM. The survey indicates that the percentage of children that are watching the station \( x \) hours after 8:00 AM is given by

\[ f(x) = \frac{1}{2}(2x^3 - 3x^2 - 12x + 72). \]

(a) At what time between 8:00 AM and 11:00 AM are the most children watching the station? What percentage of children is watching at that time?

(b) At what time between 8:00 AM and 11:00 AM are the fewest children watching the station? What percentage of children is watching at that time?

Solution. This is a max/min problem on a closed interval \([0, 3]\) (because \( 3 = 11 - 8 \)). Since a continuous function always has an absolute maximum and absolute minimum on a closed bounded interval, both parts have an answer; furthermore, the max/min must be taken either at an endpoint or an interior point, in which case it will be a critical point. So our first job is to find the critical points:

\[ f'(x) = 3x^2 - 3x - 6 = 3(x + 1)(x - 2), \]

therefore the critical points are \( x = -1 \) and \( x = 2 \). The only one of these in the interval \([0, 3]\) is \( x = 2 \). Thus we consider a minitable

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>31.5</td>
</tr>
</tbody>
</table>

confident that the absolute maximum and absolute minimum must appear in this list. Obviously the maximum is 36, taken when \( x = 0 \), and the minimum is 26, taken when \( x = 2 \). Translating to clock time, the answers are:

Maximum number at: 8:00 AM

Minimum number at: 10:00 AM

You weren’t asked to plot the graph—this illustrates the power of the analytic method, you don’t need the graph—but I have included it in Figure 3.

Problem 6. Jack and Jill have set a wedding date for three years from now. Jill’s parents want to set aside enough money now so they will have £10,000 available for the wedding. If they invest it in an account earning 6% a year, how much do they need to set aside in order to have £10,000 in the account at the end of three years if:
(a) The interest is compounded monthly;  
(b) The interest is compounded continuously.

Solution. This is a “present value” problem. The answers are:

Answer (a): $10,000 \cdot 1.005^{-36}$

Answer (b): $10,000 \cdot e^{-0.18}$

(The monthly interest rate is $0.06/12 = 0.005$ expressed as a decimal; the $-0.18$ is $3 \cdot (-0.06)$.)

The numerical answers are $8,356.45$ and $8,352.70$, but under the conditions of the exam (no calculators) you would have been unable to compute these. If you did, your entire final would have been examined very closely for other signs of cheating. □

Problem 7. Evaluate the integrals, or indicate if they do not exist:

(a) $\int e^{1/x} \frac{x}{x^2} \, dx$.  
(b) $\int_{\ln 2}^{\infty} xe^{-2x} \, dx$.  
(c) $\int \frac{x}{2x-1} \, dx$.

Solution.

(a) This is a substitution problem. Put $u = 1/x$; then $du/dx = -1/x^2$, and thus

$$du = -\frac{1}{x^2} \, dx.$$  

Thus the integral becomes

$$\int e^{1/x} \frac{x}{x^2} \, dx = - \int e^u \, du$$

$$= -e^u + C$$

$$= -e^{1/x} + C.$$  

Answer (a): $-e^{1/x} + C$

(b) This is a definite integral. The best idea is to compute the indefinite integral, that is, find an antiderivative for the integrand; then substitute the values $x = \ln 2$ and $x = 0$ and subtract.

To compute the indefinite integral, do an integration by parts with $dv = e^{-2u} \, du$; a value of $v$ which does this is

$$v = -\frac{1}{2}e^{-2u}$$

(which officially you should get by a substitution; or remember that $de^{kx}/dx = ke^{kx}$). Thus

$$\int u e^{-2u} \, du = \int ud\left(-\frac{1}{2}e^{-2u}\right)$$

$$= u \left(-\frac{1}{2}e^{-2u}\right) - \int -\frac{1}{2} e^{-2u} \, du$$

$$= -\frac{1}{2}ue^{-2u} + \frac{1}{2} \int e^{-2u} \, du.$$  

Finally, recall that $\int e^{-2u} \, du = -\frac{1}{2}e^{-2u}$ (remember how we computed $v$ from $dv$), to get

$$\int u e^{-2u} \, du = -\frac{1}{2}ue^{-2u} - \frac{1}{4}e^{-2u} = -\left(\frac{1}{2}u + \frac{1}{4}\right) e^{-2u}.$$  

However, the problem asks for the definite integral from 0 to $\ln 2$. (This is why we didn’t bother to add the “+C” to the indefinite integral.) The final answer is therefore

$$-\left(\frac{1}{2}u + \frac{1}{4}\right) e^{-2u} \bigg|_0^{\ln 2},$$

which works out to

Answer (b): $\frac{3}{16} - \frac{1}{8} \ln 2$

An alternative is to use the form of the indefinite integral from the tables:

$$\int u e^{au} \, du = \frac{1}{a^2} (au - 1)e^{au} + C.$$  

This doesn’t excite much admiration among your loving professors, but if you wrote out this general form, it would have been accepted.
(c) Substitute \( u = 2x - 1 \), so that \( x = \frac{1}{2} + \frac{1}{2}u \) and \( du = 2\,dx \), i.e. \( dx = \frac{1}{2}\,du \). Thus
\[
\int \frac{x}{2x - 1}\,dx = \int \frac{\frac{1}{2} + \frac{1}{2}u}{u} \cdot \frac{1}{2}\,du = \int \frac{1}{4u + 1}\,du = \frac{1}{4} \ln |u| + \frac{1}{4}u = \frac{1}{4} \ln |2x - 1| + \frac{1}{4}(2x - 1) + \frac{1}{2}x.
\]
where the \(-\frac{1}{4}\) has been absorbed into the constant:

\[
\text{Answer (c): } \frac{1}{4}x + \frac{1}{4} \ln |2x - 1| + C.
\]

**Problem 8.** Local environmentalists studying Lake Plentiful estimate that there are presently 2500 trout in the lake. However, because of an insufficient oxygen supply, they expect the trout to die at the rate of \(125\) per day, where \( t \) denotes the number of days that have passed. Find the population of trout in Lake Plentiful 15 days from now.

**Solution.** If \( P(t) \) denotes the population of the trout at time \( t \) (in days that have passed), then we are told that
\[
P'(t) = 125e^{-t/20}, \quad P(0) = 2500.
\]
Therefore
\[
P(t) - P(0) = \int_0^t 125e^{-s/20}\,ds = \left. -2500e^{-s/20} \right|_{s=0}^{s=t} = -2500e^{-t/20} + 2500.
\]
We have used the fact that \( \int e^{ks}\,dx = \frac{1}{k}e^{ks} \), and that when \( t = 0 \) is substituted into \(-2500e^{-t/20}\) we get \(-2500\).
Adding \( P(0) \) to both sides, therefore
\[
P(t) = 5000 - 2500e^{-t/20}.
\]
We check that
\[
P(0) = 5000 - 2500 = 2500,
\]
which is as it should be; and that
\[
P'(t) = \frac{2500}{20}e^{-t/20} = 125e^{-t/20},
\]
which is also as it should be. (Big sigh of relief.)

Now, what was the question? Oh, yeah, what’s the population after 15 days. That’s
\[
P(15) = 5000 - 2500e^{-15/20} = 5000 - 2500e^{-3/4},
\]
and that’s as far as we can take it. (If you wrote 3819, which is the numerical value, we’d be inclined to think you used a calculator…)

**Answer:** \( 5000 - 2500e^{-3/4} \)

**Problem 9.** Let \( f(x, y) = x^3 + \frac{1}{2}y^2 - 3xy - 4y + 23 \). Find all the critical points of \( f(x, y) \), and classify them as to whether they yield relative maxima, relative minima, or saddle points.

**Solution.** We compute
\[
\begin{align*}
    f_x &= 3x^2 - 3y \\
    f_y &= y - 3x - 4 \\
    f_{xx} &= 6x \\
    f_{xy} &= -3 \\
    f_{yx} &= -3 \\
    f_{yy} &= 1.
\end{align*}
\]
(As of course, I know that \( f_{xy} = f_{yx} \), but it’s always a good idea to compute both anyway, as a partial check on your work.)

The critical points are therefore solutions of
\[
\begin{align*}
    3x^2 - 3y &= 0 \\
    y - 3x - 4 &= 0.
\end{align*}
\]
We can take the second of these, \( y = 3x + 4 \), and substitute it into the first, so as to eliminate \( y \):
\[
3x^2 - 3(3x + 4) = 0.
\]
This simplifies to
\[
x^2 - 3x - 4 = 0,
\]
which has roots $x = 4, x = -1$. We compute the corresponding values of $y$
from the equation we used to eliminate $y$,

$$y = 3x + 4,$$

to find that $x = 4, y = 16$ is one critical point and $x = -1, y = 1$ is the
other.

To classify these we must compute the discriminant,

$$D = f_{xx}f_{yy} - f_{xy}^2 = 6x - (-3)^2 = 6x - 9,$$

at the two critical points. At $x = 4, y = 16$ we get $D = 15$, which is positive;
checking $f_{xx}$ or $f_{yy}$ at these values, they’re both positive; thus $(4, 16)$ is a
local minimum point of $f$.

At the second critical point $(-1, 1)$ we compute $D = -15$, which is negative.
The second derivative test immediately tells us that $(-1, 1)$ is a saddle point.

There can be no local maximum point, because it would have to be a critical
point, and we’ve already used up all the critical points. Thus the answers to
the problem are:

| Critical points: $x = 4, y = 4$ and $x = -1, y = 1$ |
| Relative maxima: None |
| Relative minima: $x = 4, y = 4$ |
| Saddle points: $x = -1, y = 1$ |

**Problem 10.** Evaluate the double integrals below over the specified rectan-
gular region $R$. Choose the order of integration carefully, where appropriate.

(a) We compute the double integral as an iterated integral:

$$\int \int_R x^2 y + 1 \, dA = \int_{x=0}^{x=1} \left( \int_{y=0}^{y=1} x^2 y + 1 \, dy \right) dx$$

and begin working on the innermost integral first:

$$\int_{y=0}^{y=1} x^2 y + 1 \, dy = \left( \frac{1}{2} x^2 y^2 + y \right) \bigg|_{y=0}^{y=1}$$

$$= \left( \frac{1}{2} x^2 + 1 \right) - \left( \frac{1}{2} x^2 \cdot 0 + 0 \right)$$

$$= \frac{1}{2} x^2 + 1.$$

Substituting into equation (??), we get

$$\int \int_R x^2 y + 1 \, dA = \int_{x=0}^{x=1} \frac{1}{2} x^2 + 1 \, dx$$

$$= \left( \frac{1}{6} x^3 + x \right) \bigg|_{x=0}^{x=1}$$

$$= \frac{7}{6} - 0$$

$$= \frac{7}{6}.$$

And that’s the answer:

Answer (a): $\frac{7}{6}$

(b) We will give two solutions. The first utilizes the fact that $\ln(x y) = \ln x + \ln y$, a fact which a good student might utilize.

This leads to

(1) $$\int \int_R \frac{\ln(x y)}{y} \, dA = \int \int_R \frac{\ln x}{y} \, dA + \int \int_R \frac{\ln y}{y} \, dA.$$ 

Now, each of the integrands in (1) is what is called “separable”: it is
the product of a function $f(x)$ and a function $g(y)$. It’s easy to find
the double integral of a separable function $f(x)g(y)$ on a rectangle
\( R : a \leq x \leq b, \ c \leq y \leq d, \) since
\[
\iint_R f(x)g(y) \, dA = \int_a^b \left( \int_c^d f(x)g(y) \, dy \right) \, dx
\]
\[
= \int_a^b f(x) \left( \int_c^d g(y) \, dy \right) \, dx
\]
\[
= \left( \int_c^d g(y) \, dy \right) \left( \int_a^b f(x) \, dx \right).
\]
(The \( f(x) \) factors out of the inner integral because it’s a constant as far as the variable of integration \( y \) is concerned; in the next line, the \( \int_c^d g(y) \, dy \) factors out of the integral with respect to \( x \), because it doesn’t depend on \( x \).) In other words, the double integral of \( f(x)g(y) \) is simply the product of the two single integrals \( \int f(x) \, dx \) and \( \int g(y) \, dy \), evaluated on the corresponding intervals.

With reference to (1), we see that the first integral is separable,
\[
\iint_R \frac{\ln x}{y} \, dA = \left( \int_1^e \ln x \, dx \right) \left( \int_1^e \frac{1}{y} \, dy \right)
\]
\[
= (y \ln y - y) \bigg|_1^e (\ln y) \bigg|_1^e
\]
\[
= ((e \ln e - e) - (1 \ln 1 - 1)) (\ln e - \ln 1)
\]
\[
= 1
\]
since \( \ln e = 1 \) and \( \ln 1 = 0 \). We have done the default integration by parts,
\[
\int \ln y \, dy = y \ln y - \int y \, d(\ln y)
\]
\[
= y \ln y - \int y \frac{1}{y} \, dy
\]
\[
= y \ln y - \int 1 \, dy
\]
\[
= y \ln y - y.
\]
The second integral in (1) is also separable,
\[
\iint_R \frac{\ln y}{y} \, dA = \left( \int_1^e \frac{\ln y}{y} \, dy \right) \left( \int_1^e \frac{1}{x} \, dx \right)
\]
\[
= \left( \frac{1}{2} (\ln y)^2 \right) \bigg|_1^e (e - 1)
\]
\[
= e - 1
\]
where we have used the substitution \( v = \ln y \) to compute
\[
\int \frac{\ln y}{y} \, dy = \int v \, dv = \frac{1}{2} v^2 = \frac{1}{2} (\ln y)^2.
\]
Substituting (2) and (3) into (1), we find
\[
\iint_R \frac{\ln(xy)}{y} \, dA = 1 + \frac{e - 1}{2} = \frac{e + 1}{2}.
\]

Answer (b): \( \frac{e + 1}{2} \)

The more direct way of doing this problem doesn’t utilize separability, but requires computing a slightly more complicated integral. (On the other hand, it’s only one integral, not two.) We begin by writing the double integral as an iterated integral:
\[
\iint_R \frac{\ln(xy)}{y} \, dA = \int_{x=1}^{x=e} \left( \int_1^e \ln(xy) \, dy \right) \, dx.
\]
We work on the inner integral first. In indefinite form it is
\[
\int \frac{\ln(xy)}{y} \, dy,
\]
which suggests a change of variable \( u = \ln(xy) \) (where \( x \) is fixed and \( y \) is variable). We have
\[
du = \frac{1}{y} \, dy
\]
(compare with Problem 2(d)). We obtain
\[
\int \frac{\ln(xy)}{y} \, dy = \int u \, du
\]
\[
= \frac{1}{2} u^2
\]
\[
= \frac{1}{2} (\ln(xy))^2.
\]
Integrating between \( y = 1 \) and \( y = e \) therefore results in
\[
\int_1^e \frac{\ln(xy)}{y} \, dy = \frac{1}{2} \left( (\ln(ex))^2 - (\ln x)^2 \right)
\]
\[
= \frac{1}{2} \left( (1 + \ln x)^2 - (\ln x)^2 \right)
\]
\[
= \frac{1}{2} (1 + 2 \ln x)
\]
\[
= \frac{1}{2} + \ln x.
\]
(It’s almost miraculous that the \((\ln x)^2\) term disappears—at least very fortunate, since it would be nasty to integrate.)

Substituting into the original integral, we obtain

\[
\begin{align*}
\iint_R \frac{\ln(xy)}{y} dA &= \int_1^e \int_1^x \frac{1}{2} + \ln x \, dx \\
&= \left[ \frac{1}{2}(e - 1) + (x \ln x - x) \right]_1^e \\
&= \frac{1}{2}(e - 1) + 1 \\
&= \frac{e + 1}{2}.
\end{align*}
\]

What would have happened if we had integrated first with respect to \(x\), then with respect to \(y\)? The result is slightly more complicated. It turns out that

\[
\int \frac{\ln(xy)}{y} \, dx = \frac{x \ln(xy) - x}{y}.
\]

Whichever way we do the order of integration,

\[
\int \left( \int \frac{\ln(xy)}{y} \, dy \right) \, dx = \int \left( \int \frac{\ln(xy)}{y} \, dx \right) \, dy
\]

\[
= xy \ln(xy) - 2xy.
\]

You’ve seen both solutions; which do you think is easier?