1. Some general methods for solving control problems
2. Linear-quadratic control and generalizations
3. Fractional Brownian motions and other noise processes for controlled linear systems
4. Linear-exponential-quadratic Gaussian control and games
5. Nash equilibria for some stochastic differential games
6. Discrete time LQ control with correlated noise
7. Control and differential games for nonlinear stochastic systems in spheres, projective spaces and hyperbolic spaces
8. Infinite time horizon linear-quadratic control for distributed parameter (SPDEs) systems with fractional Brownian motions
9. Linear-quadratic control for SPDEs with multiplicative Gaussian noise
10. Some potential generalizations

Some of this is joint work with B. Maslowski and B. Pasik-Duncan.
Solution Methods for Stochastic Control Problems

1. Hamilton-Jacobi-Bellman equations

2. Stochastic maximum principle or dynamic programming and backward stochastic differential equations
Hamilton-Jacobi Equation for a deterministic problem

\[
\frac{\partial V(t, x(t))}{\partial t} + \min_{u \in \mathbb{R}^m} H(x(t), u(t), \nabla V, t) = 0
\]

\(H(x(t), u(t), p(t), t)\) is the Hamiltonian.

Hamilton-Jacobi-Bellman Equation for a stochastic problem

\[
\frac{\partial V}{\partial t} + \min_{u \in \mathbb{R}^m} \left[ \frac{1}{2} \text{tr}(D^2 V) + H(x(t), u(t), p(t), t) \right] = 0
\]
Maximize the Hamiltonian for the problem.

This method provides a necessary condition for optimality. With some convexity conditions, the necessary condition is also sufficient. Since a condition is at the final time it is necessary to solve a backward stochastic differential equation which means solving backward in time but having a forward measurability for the solution.
Linear-Quadratic and Linear-Quadratic Gaussian Control

\[
\frac{dx}{dt} = Ax + Bu 
\]

\[ x(0) = x_0 \]

\[
J(u) = \frac{1}{2} \left[ \int_0^T (\langle Qx(t), x(t) \rangle + \langle Ru(t), u(t) \rangle) dt + \langle Mx(T), x(T) \rangle \right] 
\]

Admissible controls \( \mathcal{U} = \{ u : u \in L^2([0, T]) \} \)

\[
dX(t) = (AX(t) + BU(t))dt + dW(t) 
\]

\[ X(0) = X_0 \]

\[
J(U) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (\langle QX(t), X(t) \rangle + \langle RU(t), U(t) \rangle) dt + \langle MX(T), X(T) \rangle \right] 
\]
Solution Methods

Hamilton-Jacobi Equation for the deterministic problem

\[ \frac{\partial V(t, x(t))}{\partial t} + \min_{u \in \mathbb{R}^m} H(x(t), u(t), \nabla V, t) = 0 \]

\[ H(x(t), u(t), p(t), t) = \frac{1}{2} < Qx(t), x(t) > + \frac{1}{2} < Ru(t), u(t) > + < p(t), Ax(t) > + < p(t), Bu(t) > \]

Hamilton-Jacobi-Bellman Equation for the stochastic problem

\[ \frac{\partial V}{\partial t} + \min_{u \in \mathbb{R}^m} \left[ \frac{1}{2} \text{tr}(D^2 V) + < D V, Ax + Bu > + \frac{1}{2} < Q, x > + \frac{1}{2} < Ru, u > \right] = 0 \]
An optimal control for both problems is

\[ U^*(t) = -R^{-1}BP(t)X(t) \]

where \( P \) is the unique, symmetric, positive solution of the following Riccati equation

\[
\frac{dP}{dt} = -PA - A^T P + PBR^{-1}B^T P - Q
\]

\[ P(T) = M \]

The value function for the deterministic problem is

\[ V(s, y) = \frac{1}{2} < P(s)y, y > \]

and the value function for the stochastic problem is

\[
V(s, y) = \frac{1}{2} (< P(s)y, y > + q(s))
\]

\[
q(s) = \int_s^T tr(P(r))dr
\]
Let $H \in (0, 1)$ be fixed. The process $(B(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with the Hurst parameter index $H \in (0, 1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$
\mathbb{E}[B(t)] = 0
$$

$$
\mathbb{E}[B(s)B(t)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right)
$$

for all $s, t \in \mathbb{R}_+$. 

The formal derivative $\frac{dB}{dt}$ is called fractional Gaussian noise.
Some properties

1. Self-similarity

\[(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)\]

for \(\alpha > 0\)

2. Long range dependence for \(H \in (\frac{1}{2}, 1)\)

\[r(n) = \mathbb{E}[B^H(1)(B^H(n + 1) - B^H(n))]\]

\[\sum_{n=0}^{\infty} r(n) = \infty\]
3. A sample path property

\((B^H(t), t \geq 0)\) is of unbounded variation so the sample paths are not differentiable a.s.

\[
\sum_i |B^H(t^{(n)}_{i+1}) - B^H(t^{(n)}_i)|^p \rightarrow \begin{cases} 
0 & \text{pH} > 1 \\
c(p) & \text{pH} = 1 \\
+\infty & \text{pH} < 1 
\end{cases}
\]

\(c(p) = \mathbb{E}|B^H(1)|^p\)

\((t^{(n)}_i, i = 0, 1, \ldots, n; n \in \mathbb{N})\) is a sequence of nested partitions of \([0, 1]\) such that \(t^{(n)}_0 = 0\) and \(t^{(n)}_n = 1\) for all \(n \in \mathbb{N}\) and the sequence of partitions becomes dense in \([0, 1]\).

4. For \(H \neq \frac{1}{2}\) a FBM is neither Markov nor semimartingale.
Some Applications of FBMs

1. Turbulence
2. Hydrology
3. Economic Data
4. Telecommunications
5. Earthquakes
6. Epilepsy
7. Cognition
8. Biology
**Theorem.** For the control problem given above where $W$ is an arbitrary square integrable process with continuous sample paths and filtration $(\mathcal{F}(t), t \in [0, T])$ and the family of admissible controls, $\mathcal{U}$, there is an optimal control $U^*$ that can be expressed as

$$U^*(t) = -R^{-1}B^T(P(t)X(t) + V(t))$$

where $(P(t), t \in [0, T])$ is the unique symmetric positive definite solution of the Riccati equation

$$\frac{dP}{dt} = -PA - A^TP + PBR^{-1}B^TP - Q$$

$$P(T) = M$$
\[ V(t) = \mathbb{E}\left[ \int_t^T \Phi_P(s, t)P(s)dW(s) | \mathcal{F}(t) \right] \]

and \( \Phi_P \) is the fundamental solution for the matrix equation

\[
\frac{d\Phi_P(s, t)}{dt} = -(A^T - P(t)BR^{-1}B^T)\Phi_P(s, t)
\]

\[
\Phi_P(s, s) = I
\]
Corollary. If $W$ is an arbitrary standard fractional Brownian motion then

$$V(t) = \int_0^t u_{1/2-H} I_{t-}^{1/2-H} (I_{T-}^{H-1/2} u_{H-1/2} \Phi_P(\cdot, t) P) dW$$

and $I_b^a$ is a fractional integral if $a > 0$ and a fractional derivative if $a < 0$. 
A Method of Proof

\[ J^0_n(U) = \frac{1}{2} < P(0)X_0, X_0 > - < \phi_n(0), X_0 > \]

\[ = \frac{1}{2} \left( \int_0^T \left[ (|R^{-1/2}[RU + B^T PX_n + B^T \phi_n]|^2 - |R^{-1/2} B^T \phi_n|^2) dt + 2 < \phi_n, dB_n > \right] \right) \]

\[ U^*_n(t) = -R^{-1}(B^T P(t)X_n(t) + B^T \phi_n(t)) \]
A Hilbert Space for a FBM

Let $L^2_H$ be the Hilbert space whose inner product $< \cdot, \cdot >_H$ is given by

$$< f, g >_H = \rho(H) \int_0^T u_{\frac{1}{2}-H}(r)(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}f)(r)(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}g)(r) dr$$

where $\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}$. The term $I_{T-}^{H-\frac{1}{2}}$ is a fractional integral for $H \in (\frac{1}{2}, 1)$ and a fractional derivative for $H \in (0, \frac{1}{2})$. This Hilbert space is naturally associated with a fractional Brownian motion with Hurst parameter $H$ by the covariance factorization.

$$(I_{T-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s) ds$$

$$(D_{T-}^\alpha \psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{\psi(t)}{(T-t)^\alpha} + \alpha \int_t^T \frac{\psi(s) - \psi(t)}{(s-t)^{\alpha+1}} ds \right)$$

Tyrone E. Duncan

Solvable Stochastic Control and Stochastic Differential Games
A General Linear Stochastic Control System

\[ X(t-) = x + \int_0^t (A(s)X(s) + B(s)U(s))ds + \int_0^t \sum_{i=1}^d (C(s)^i X(s) + D(s)^i U(s))dW^i(s) + \int_0^t F(s)dW(s) + Y(t-) \]
Linear Exponential Quadratic Gaussian Control

Stochastic system

\[ dX(t) = AX(t)dt + BU(t)dt + FdW(t) \]
\[ X(0) = X_0 \]

where \( X_0 \in \mathbb{R}^n \) is not random, \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \), \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \), \( F \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n) \), \( U(t) \in \mathbb{R}^m \), \( U \in \mathcal{U} \), \( (W(t), t \in [0, T]) \) is an \( \mathbb{R}^p \)-valued standard Brownian motion.

The family of admissible controls is
\[ \mathcal{U} = \{ U : U \text{ is an } \mathbb{R}^m \text{-valued process adapted to } (\mathcal{F}(t), t \in [0, T]) \text{ such that } U \in L^2([0, T]) \text{ a.s.} \} . \]

Risk sensitive cost functional

\[ J(U) = \mu \mathbb{E} \exp[\frac{\mu}{2} \int_0^T (\langle QX(s), X(s) \rangle + \langle RU(s), U(s) \rangle) ds] + \frac{\mu}{2} \langle M X(T), X(T) \rangle \]

where \( \mu > 0 \) is fixed.
**Theorem.** For the LEQG control problem given above there is an optimal control \((U^*(t), t \in [0, T])\) in \(U\) given by

\[
U^*(t) = -R^{-1}B^TP(t)X(t)
\]

where \((P(t), t \in [0, T])\) is assumed to be the unique, symmetric, positive solution of the following Riccati equation

\[
\begin{align*}
-\frac{dP}{dt} &= PA + A^TP - P(BR^{-1}B^T - \mu FF^T)P + Q \\
P(T) &= M
\end{align*}
\]

and the optimal cost is

\[
J(U^*) = \mu G(0) \exp\left[\frac{\mu}{2} < P(0)X_0, X_0 > \right]
\]

and \((G(t), t \in [0, T])\) satisfies

\[
\begin{align*}
-\frac{dG}{dt} &= \frac{\mu}{2} G \text{ tr}(PFF^T) \\
G(T) &= 1
\end{align*}
\]
Sketch of Proof

\[ J(U) = \mu \mathbb{E} \exp[L(U)] \]

\[ L(U) - \frac{\mu}{2} < P(0)X_0, X_0 > \]

\[ = \frac{\mu}{2} \left[ \int_0^T \left( < RU, U > + < PCR^{-1}B^T PX, X > + 2 < B^T PX, U > \right) dt \right. \]

\[ + 2 \int_0^T < PX,FdW > - \mu \int_0^T < PFF^T PX, X > dt + \left. \int_0^T \text{tr}(PFF^T) dt \right] \]

\[ = \frac{\mu}{2} \int_0^T \left| R^{-\frac{1}{2}}[RU + B^T PX]\right|^2 dt \]

\[ + \mu \int_0^T < PX, FdW > - \frac{\mu^2}{2} \int_0^T < PFF^T PX, X > dt \]

\[ + \frac{\mu}{2} \int_0^T \text{tr}(PFF^T) dt \]
\[ dX(t) = AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \]

\[ X(0) = X_0 \]

where \( X_0 \in \mathbb{R}^n \) is not random, \( X(t) \in \mathbb{R}^n \), \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \), \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \), \( U(t) \in \mathbb{R}^m \), \( U \in \mathcal{U} \), \( C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n) \), \( V(t) \in \mathbb{R}^p \), \( V \in \mathcal{V} \), and \( F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n) \). The positive integers \( (m, n, p, q) \) are arbitrary. The process \( (W(t), t \geq 0) \) is an \( \mathbb{R}^q \) valued standard Brownian motion.
\[ J^0_\mu(U, V) = \mu \exp\left( \frac{\mu}{2} \int_0^T \langle QX(s), X(s) \rangle \right.
+ \left. \langle RU(s), U(s) \rangle - \langle SV(s), V(s) \rangle \right) ds
+ \frac{\mu}{2} \langle MX(T), X(T) \rangle \right] \]

\[ J_\mu(U, V) = \mathbb{E}[J^0_\mu(U, V)] \]

where

\[ Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \quad R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), \quad S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p), \quad M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \]

and \( Q > 0, \ R > 0, \ S > 0, \) and \( M \geq 0 \) are symmetric linear transformations and \( \mu \neq 0 \) is fixed. An assumption on the possible values for \( \mu \) is given in the following theorem. The player with control \( U \) seeks to minimize the payoff \( J_\mu \) while the player with control \( V \) seeks to maximize the payoff \( J_\mu \).
Theorem. The two person zero sum stochastic differential game described above has a Nash equilibrium using the optimal admissible control strategies for the two players, denoted $U^*$ and $V^*$, given by

$$
U^*(t) = -R^{-1}B^TP(t)X(t)
$$
$$
V^*(t) = S^{-1}C^TP(t)X(t)
$$

where $(P(t), t \in [0, T])$ is the unique positive symmetric solution of the following Riccati equation

$$
-\frac{dP}{dt} = Q + PA + A^TP
$$

$$
- P(BR^{-1}B^T - CS^{-1}C^T - \mu FF^T)P
$$

$$
P(T) = M
$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T - \mu FF^T > 0$. The optimal payoff is

$$
J_\mu(U^*, V^*) = \mu \exp[\frac{\mu}{2}(< P(0)X_0, X_0 > + \int_0^T tr(PFF^T)dt)]
$$
\[ L_\mu(U, V) - \frac{\mu}{2} < P(0)X_0, X_0 > \]
\[ = \frac{\mu}{2} \left[ \int_0^T \left( < RU, U > - < SV, V > + 2 < PBU, X > + 2 < PCV, X > + < PBR^{-1} B^T PX, X > - < PCS^{-1} C^T PX, X > + 2 < FdW, PX > - \mu < PFF^T PX, X > + \text{tr}(PFF^T) \right) dt \right] \]
\[ = \frac{\mu}{2} \int_0^T \left( |R^{-\frac{1}{2}}[RU + B^T PX]|^2 - |S^{-\frac{1}{2}}[SV - C^T PX]|^2 \right) dt \]
\[ + \mu \int_0^T < PX, FdW > - \frac{\mu^2}{2} \int_0^T < PFF^T PX, X > dt \]
\[ + \frac{\mu}{2} \int_0^T \text{tr}(PFF^T) dt \]
\[ dX(t) = AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \]

\[ X(0) = X_0 \]

where \( X_0 \in \mathbb{R}^n \) is not random,
\( X(t) \in \mathbb{R}^n \), \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \), \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \), \( U(t) \in \mathbb{R}^m \), \( U \in \mathcal{U} \),
\( C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n) \), \( V(t) \in \mathbb{R}^p \), \( V \in \mathcal{V} \), and \( F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n) \). The process \( W \) is square integrable with continuous sample paths with the filtration \( (\mathcal{F}(t), t \in [0, T]) \). The terms \( U \) and \( V \) are the strategies of the two players.

A Nash equilibrium occurs if the optimal strategy of one player is not influenced by knowledge of the strategy of the other player.
The payoff, $J$, is

$$J^0(U, V) = \frac{1}{2} \left[ \int_0^T (\langle QX(s), X(s) \rangle + \langle RU(s), U(s) \rangle \right.$$ 
$$\left. - \langle SV(s), V(s) \rangle)ds + \langle MX(T), X(T) \rangle \right]$$

$$J(U, V) = \mathbb{E}[J^0(U, V)]$$

where

- $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,
- $R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$,
- $S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$,
- $M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,

and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations.
The family of admissible strategies for $U$ is $\mathcal{U}$ and for $V$ is $\mathcal{V}$ where

$$\mathcal{U} = \{ U : U \text{ is an } \mathbb{R}^m\text{-valued process that is progressively measurable with respect to } (\mathcal{F}(t), t \in [0, T]) \text{ such that } U \in L^2([0, T]) \text{ a.s.} \}$$

and

$$\mathcal{V} = \{ V : V \text{ is an } \mathbb{R}^p\text{-valued process that is progressively measurable with respect to } (\mathcal{F}(t), t \in [0, T]) \text{ such that } V \in L^2([0, T]) \text{ a.s.} \}$$

$$J^+ = \inf_{U \in \mathcal{U}} \sup_{V \in \mathcal{V}} J(U, V)$$ is the upper value of the game.

$$J^- = \sup_{V \in \mathcal{V}} \inf_{U \in \mathcal{U}} J(U, V)$$ is the lower value of the game.

If these two values are equal then the game is said to have a value.
Theorem. The two person zero sum stochastic differential game has optimal admissible strategies for the two players, denoted $U^*$ and $V^*$, given by

$$U^*(t) = -R^{-1}(B^TP(t)X(t) + B^T\hat{\phi}(t))$$
$$V^*(t) = S^{-1}(C^TP(t)X(t) + C^T\hat{\phi}(t))$$

where $(P(t), t \in [0, T])$ is the unique positive solution of the following equation

$$-\frac{dP}{dt} = Q + PA + A^TP - P(BR^{-1}B^T - CS^{-1}C^T)P$$
$$P(T) = M$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T > 0$. The optimal strategies form a Nash equilibrium. $\hat{\phi}$ is the best estimate (conditional expectation) of the response of the optimal system to the future noise.
The discrete time controlled linear stochastic system is described as follows:

\[ X(k + 1) = A(k)X(k) + B(k)U(k) + W(k) \]
\[ X(0) = x_0 \]

where \( x_0 \in \mathbb{R}^n \), \( A(k) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \), \( B(k) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \), \( k \in \{0, 1, \ldots, T - 1\} = \mathbb{T} \), and \((W(k), k \in \mathbb{T})\) is an \( \mathbb{R}^n \)-valued family of square integrable, zero mean random variables that are defined on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The term \( \mathcal{L}(\mathbb{R}^k, \mathbb{R}^l) \) denotes the family of linear transformations from \( \mathbb{R}^k \) to \( \mathbb{R}^l \). The filtration for \((X(k), k \in \mathbb{T})\) is denoted \((\mathcal{G}(k), k \in \mathbb{T})\).
Quadratic Cost Functional

\begin{align*}
J^0_T(U) & = \sum_{k=0}^{T-1} [\langle Q(k)X(k), X(k) \rangle + \langle R(k)U(k), U(k) \rangle ] \\
& + \langle Q(T)X(T), X(T) \rangle & (4) \\
J_T(U) & = \mathbb{E}[J^0_T(U)] & (5)
\end{align*}

where \((Q(k), k \in \{0, \ldots, T\})\) and \((R(k), k \in \mathbb{T})\) are two families of symmetric, nonnegative definite linear transformations and \(\langle \cdot, \cdot \rangle\) is the standard inner product in the appropriate Euclidean space.
**Theorem.** For the optimal control problem with the linear system (3), the cost functional (5) and the family of admissible controls, $\mathcal{U}$, there is an optimal control $(U^*(k), k \in \mathbb{T})$ that is given by

$$U^*(k) = - P^{-1}(k + 1)B^T(k)S(k + 1)A(k)X(k) - E[P^{-1}(k + 1)B^T(k)\phi(k + 1)|G(k)] - E[P^{-1}(k + 1)B^T(k)S(k + 1)W(k)|G(k)]$$
\( S(k) = A^T(k)S(k+1)A(k) \)
\[- A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k)S(k+1)A(k) \]
\[+ Q(k) \]
\( S(T) = Q(T) \)
\( \phi(k) = (A^T(k) - A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k))\phi(k+1) \)
\[+ A^T(k)S(k+1)W(k) \]
\[+ A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k)S(k+1)W(k) \]
\( \phi(T) = 0 \)
\( P(k+1) = B^T(k)S(k+1)B(k) + R(k) \)

It is assumed that \((P(k), k \in \{1, \ldots, T\})\) is a family of positive definite linear transformations.
Linear-quadratic control of SPDEs with Fractional Brownian Motions

\[ dX(t) = (AX(t) + Bu(t))dt + dB_H(t) \]

\[ X(0) = x \]

where \( x \in V, X(t) \in V, V \) is an infinite dimensional real separable Hilbert space with inner product \( < \cdot, \cdot > \) and norm \( | \cdot | \). The process \((B_H(t), t \geq 0)\) is a \( V \)-valued fractional Brownian motion with the Hurst parameter \( H \in (\frac{1}{2}, 1) \) and having the incremental covariance \( \tilde{Q} \) where \( \tilde{Q} \) is trace class \( (Tr(\tilde{Q}) < \infty) \) so that

\[ \mathbb{E} < B_H(t), x > < B_H(s), y > = \frac{1}{2} < \tilde{Q}x, y > (t^{2H} + s^{2H} - |t - s|^{2H}). \]

for \( x, y \in V \). The operator \( A : Dom(A) \to V \) with \( Dom(A) \subset V \) is a linear, densely defined operator on \( V \) which is the infinitesimal generator of a strongly continuous semigroup \((S(t), t \geq 0)\).
\[ \mathcal{U} = \{ u : \mathbb{R}_+ \times \Omega \rightarrow U, u \text{ is progressively measurable,} \]
\[ \mathbb{E} \int_0^T |u(t)|_U^2 \, dt < \infty \text{ for all } T > 0 \} \]
Ergodic Quadratic Cost Functional

\[ J_T(x, u) := \frac{1}{2} \int_0^T (|LX(s)|^2 + \langle Ru(s), u(s) \rangle_U) ds \]

where \( L \in \mathcal{L}(V) \), \( R \in \mathcal{L}(U) \), \( R \) is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

\[ \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} J_T(x, u). \]
(A1) There are $K \in \mathcal{L}(V), M_K > 0, \text{ and } \omega_K > 0$ such that

$$|e^{(A+KL)t}|_{\mathcal{L}(V)} \leq M_K e^{-\omega_K t}$$

for all $t > 0$ (detectability).

(A2) There are $F \in \mathcal{L}(V, U), M_F > 0, \text{ and } \omega_F > 0$ such that

$$|e^{(A+BF)t}|_{\mathcal{L}(V)} \leq M_F e^{-\omega_F t}$$

for all $t > 0$ (stabilizability).
The stationary Riccati equation has a weak solution as follows

\[ < Px, Ay > + < Ax, Py > + < L^* Lx, y > - < R^{-1} B^* Px, B^* Py > = 0 \]

for all \( x, y \in \text{Dom}(A) \). Moreover the strongly continuous semigroup \( (\Phi(t), t \geq 0) \) generated by \( A_P = A - BR^{-1}B^*P \) is exponentially stable, that is

\[ |\Phi(t)|_{\mathcal{L}(V)} \leq M_P e^{-\tilde{\omega}t} \]

for some constants \( M_P > 0 \) and \( \tilde{\omega} > 0 \).
Let (A1)-(A2) be satisfied and let \( u \in \mathcal{U} \) be a control satisfying

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E} < PX^u(T), X^u(T) > = 0
\]  

(6)

where \((X^u(T), T \in [0, \infty))\) is the solution to the system equation with the control \( u \in \mathcal{U} \). Then

\[
\lim sup_{T \to \infty} \frac{1}{T} \mathbb{E} J_T(x, u) \geq J_\infty
\]  

where

\[
J_\infty := \lim sup_{T \to \infty} \frac{-1}{2T} \mathbb{E} \int_0^T |R^{\frac{1}{2}} B^* W(s)|^2_U ds
\]

\[+ \int_0^\infty Tr(\tilde{Q} P \Phi(t)) \phi_H(r) dr\]

for each \( x \in V \) where \( \phi_H(r) = H(2H - 1)|r|^{2H-2}, r \in \mathbb{R} \), \( W(t) = \mathbb{E}[\varphi(t)|\mathcal{F}(t)] \). Moreover, the feedback control \( \hat{u}(t) = -R^{-1} B^* (PX^\hat{u}(s) + V(s)) \) is admissible, satisfies the condition (6).
An optimal control $\hat{u}$ with the measurability condition is

$$\hat{u}(t) = -R^{-1}B^*P(t)X(t) + \psi(t)$$

$$\psi(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)]$$

$$= \int_0^t s^{-(H-\frac{1}{2})}(l_{t^-}(l_{T^-}^{(H-\frac{1}{2})} u_{H-\frac{1}{2} U} P(\cdot, t) P(\cdot)C)))(s)dB_H(s)$$

$$(l_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt$$

$$u_a(s) = s^a$$
Stochastic Parabolic Equation

\[
\frac{\partial y}{\partial t}(t, \xi) = (L_{2m} y)(t, \xi) + (Bu_t)(\xi) + \eta^H(t, \xi)
\]

for \((t, \xi) \in \mathbb{R}_+ \times D\) with the initial condition

\[
y(0, \xi) = x(\xi)
\]

for \(\xi \in D\) and the Neumann boundary conditions

\[
\frac{\partial^k}{\partial \nu^k} y(t, \xi) = 0
\]

for \((t, \xi) \in \mathbb{R}_+ \times \partial D, k = 0, 1, \ldots, m - 1\), where \(D \subset \mathbb{R}^d\) is a bounded domain with a smooth boundary, \(\frac{\partial}{\partial \nu}\) stands for conormal derivative, \(x \in L^2(D)\), \(\eta^H\) is a space dependent fractional noise and \(L_{2m}\) is a \(2m\)th order uniformly elliptic operator of the form

\[
L_{2m} = \sum_{|\alpha| \leq 2m} a_\alpha(\xi) D^\alpha
\]

with \(a_\alpha \in C_0^\infty(D)\).
For stochastic heat equation $\text{Dom}(A) = H^1_0(D) \cap H^2(D)$ for Dirichlet boundary conditions

$\text{Dom}(A) = \{ \varphi \in H^2(D) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial D \}$ for Neumann boundary conditions
\[
\frac{\partial^2 w}{\partial t^2} (t, \xi) = \frac{\partial^2 w}{\partial \xi^2} (t, \xi) + u_t(\xi) + \eta^H(t, \xi)
\]
for \((t, \xi) \in \mathbb{R}_+ \times (0, 1)\) with the boundary condition
\[
w(t, 0) = w(t, 1) = 0, \quad t > 0,
\]
and initial condition
\[
w(0, \xi) = x_1(\xi), \quad \frac{\partial w}{\partial t} (0, \xi) = x_2(\xi), \quad \xi \in (0, 1),
\]
where \(u_t \in L^2(0, 1)\) and \(\eta^H\) is a fractional noise on \(L^2(0, 1)\).
Stochastic Evolution Equations with a Multiplicative Gaussian Noise

\[ dX(t) = (A(t)X(t) + B(t)K(t)X(t))dt + \sigma(t)X(t)dB(t) \]
\[ X(0) = x_0 \]

where \( X(t) \in V \) a real, separable Hilbert space, \((B(t), t \geq 0)\) is a real-valued Volterra-type Gaussian process, \((A(t), t \geq 0)\) is a family of closed, unbounded operators on \( V \) such that \( \text{Dom}(A(t)) = \text{Dom}(A(0)) \) and \( \text{Dom}(A^*(t)) = \text{Dom}(A^*(0)) \) for each \( t \in \mathbb{R}_+ \) and the family generates a strongly continuous evolution operator, \( B \in C_s(\mathbb{R}_+, \mathcal{L}(U, V)) \) and \( K \in C_s(\mathbb{R}_+, \mathcal{L}(V, U)) \), \( \sigma \) is a real-valued continuous function. The control is

\[ u(t) = K(t)X(t) \]

where \( K \) is to be determined. This can be described as a Markov type control.
The noise $B$ is generated from a Wiener process $W$ as follows

\[(R2)\quad B(t) = \int_0^t K(t, r)dW(r) \quad t \in \mathbb{R}_+\]

There is a continuous version of the process $B$.

Assume that $K(\cdot, s)$ has bounded variation on $(s, T)$ and

\[(R3)\quad \int_0^T |K|^2((s, T], s)ds < \infty\]

This family of noise processes includes the family of FBM$s$ for $H \in (\frac{1}{2}, 1)$.
The \( V \)-valued process \((X(t), t \geq 0)\) is a strong solution to the equation if

\[
X(t) = x + \int_0^t (A(s)X(s) + B(s)K(s)X(s))\,ds + \int_0^t \sigma(s)X(s)\,dB(s)
\]

and a weak solution to the equation exists if for each \( z \in \mathcal{D}, z \in \text{Dom}(A^*(0)) \) the following equality is satisfied

\[
< X(t), z > = < x, z > + \int_0^t < X(s), A^*(s)z > \,ds
\]

\[
+ < B(s)K(s)X(s), z > \,ds + \int_0^t \sigma(s) < X(s), z > \,dB(s)
\]
\[ \tilde{A}(t) = A(t) + B(t)K(t) - \alpha(t)I \quad t \geq 0 \]
\[ \tilde{A}_\lambda(t) = A(t) + B_\lambda(t)K(t) - \alpha(t)I \quad t \geq 0 \]

\( \tilde{A} \) and \( \tilde{A}_\lambda \) generate mild and strong evolution operators respectively on \( V \) denoted \((U(t, s))\) and \((U_\lambda(t, s))\), \( B_\lambda = \lambda(\lambda I - A)^{-1}B \) and

\[ U(t, s) = \exp\left[- \int_s^t \alpha(r)dr\right] U_K(t, s) \]
\[ U_\lambda(t, s) = \exp\left[- \int_s^t \alpha(r)dr\right] U_\lambda^K(t, 0) \]
\[ X_\lambda(t) = \exp[Z(t)] U_\lambda(t, s) \quad t \geq 0 \]
\[ X(t) = \exp[Z(t)] U(t, 0) \quad t \geq 0 \]

where
\[ Z(t) = \int_0^t \sigma(r)dB(r) \]
\[ \alpha(t) = \sigma^2 \frac{\partial}{\partial t} \left( \int_0^t K(t, r) dr \right)^2 = \sigma^2 \frac{\partial}{\partial t} R(t, t) = \sigma^2 \frac{\partial}{\partial t} (\mathbb{E} B^2(t)) \]

For fractional Brownian motion or Liouville fractional Brownian motion \( \alpha(t) = c_H t^{2H-1} \) where the constant depends on whether it is FBM or LFBM.

For FBM if \( \sigma \) is not a constant
\[ \alpha(t) = \sigma(t) \int_0^t \sigma(s) \phi_H(t - s) ds \]
where \( \phi(t) = H(2H - 1) t^{2H-2} \) so that for continuous \( \sigma \) the condition \((K3)\) is satisfied.
The cost functional, $J_T$, is the following

$$J_T(K) = \mathbb{E} \int_0^T \left( |L(t)X(t)|^2 + \langle R(t)K(t)X(t), K(t)X(t) \rangle_U \right) dt + \mathbb{E} \langle GX(T), X(T) \rangle$$

where $L \in C_s([0, T], \mathcal{L}(V))$, $G = G^*$, $G \in \mathcal{L}(V)$, $G > 0$, $R \in C_s([0, T], \mathcal{L}(U))$, $R(t) = R^*(t)$, $\langle R(t)u, u \rangle \geq \lambda_0 |u|^2_U$, $u \in U$, $t \in [0, T]$.

The family of admissible controls is $K \in C_s([0, T], \mathcal{L}(V, U))$. 
The Riccati differential equation associated with the control problem is

\[
\frac{dP}{dt} + A^*P + PA - PBR^{-1}B^*P + LL - 2\alpha(t)P = 0 \quad t \in [0, T]
\]

\[
P(T) = G
\]

**Lemma.** With the assumptions given above, there is a unique weak solution \((P(t), t \in [0, T])\) to the Riccati equation that satisfies \(P \in C_s([0, T], \mathcal{L}(V)), P(t) \geq 0, P(t) = P^*(t) \quad t \in [0, T]\) such that

\[
\frac{d\langle P(t)x, y \rangle}{dt} + \langle A(t)x, P(t)y \rangle + \langle P(t)x, A(t)y \rangle \\
- \langle R^{-1}(t)B^*(t)P(t)x, B^*P(t)y \rangle \\
+ \langle L(t)x, L(t)y \rangle - 2\alpha(t) \langle P(t)x, y \rangle = 0 \quad P(T) = G
\]

for \(t \in [0, T], x, y \in D.\)
**Theorem.** Let (A1), (K1)-(K3) be satisfied. The feedback control

\[ u(t) = -R^{-1}(t)B^*(t)P(t)X(t) \]

\[ K(t) = -R^{-1}(t)B^*(t)P(t) \]

is an optimal control for the control problem. The optimal cost is

\[ J_T(K) = < P(0)x_0, x_0 > \]

**Proof.** Apply an Itô formula to \(< P(t)X_\lambda(t), X_\lambda(t) >, t \in [0, T]\) and then let \( \lambda \to \infty \).
1. Some structure for the nonlinear stochastic systems—symmetric spaces (quotients of Lie groups)
2. Some symmetries for these nonlinear systems to facilitate explicit solutions
3. Some rank one symmetric spaces (compact—spheres and projective spaces; noncompact—hyperbolic spaces)
$\mathbb{H}^2(\mathbb{R})$ is the real hyperbolic space of dimension two that is the (noncompact) symmetric space

$$SO_o(2,1)/SO(2) \times SO(1) = G/K$$

Let $o \in G/K$ be chosen and denoted as the origin. $G/K$ can be modeled as the open unit disk in $T_o G/K$. with the metric

$$ds^2 = 4(1 - |y|^2)^{-2}(dy_1^2 + dy_2^2).$$

The controlled stochastic system for the distance from the origin $o$ is

$$dX(t) = \frac{1}{2} \coth \frac{X(t)}{2} dt + U(t) dt + dB(t)$$

where $(B(t), t \in [0, T])$ is a standard Brownian motion.

$$J^0(U) = \int_0^T (a \sinh^2 \frac{X(t)}{4} + U^2(t) \cosh^2 \frac{X(t)}{4}) dt$$

$$J(U) = \mathbb{E} J^0(U)$$
**Theorem.** An optimal control, $U^*$, is given by

$$U^*(t) = -\frac{1}{2}g(t)\tanh\frac{X(t)}{4}$$

where $t \in [0, T]$ and $g$ satisfies the Riccati equation given below. The optimal cost is

$$J(U^*) = g(0)\sinh^2\frac{X(0)}{4} + h(0)$$

\[
\begin{align*}
dg(t) &= -\frac{3}{8}g + \frac{1}{4}g^2 - a \\
g(T) &= 0 \\
dh(t) &= -\frac{3}{16}g \\
h(T) &= 0
\end{align*}
\]
\[ f(t, x) = g(t) \sinh^2 \frac{x}{2} + h(t) \text{ and } Y(t) = f(t, X(t)) \text{ where } X \text{ is the solution of the system equation. Apply the Ito formula to } (Y(t), t \in [0, T]). \]
Some generalizations of the above example of $H^2(\mathbb{R})$. Let $R(\Delta_{G/K})$ be the radial part of the Laplacian.

$$R(\Delta_{G/K}) = \frac{d^2}{dr^2} + (\gamma p \coth \gamma r + 2\gamma q \coth 2\gamma r) \frac{d}{dr}$$

Eigenvalue-eigenfunction problem

$$z(z - 1) \frac{d^2\phi_{\lambda}}{dz^2} + [(a + b + 1)z - c] \frac{d\phi_{\lambda}}{dz} + ab\phi_{\lambda} = 0$$

A solution is given in terms of a hypergeometric function, $F(-m, b, c, z)$. If $m$ is a positive integer, then $F$ is a polynomial in $z$.

Let $z = - (\sinh \gamma r)^2$ so $F(-m, b, c, -(\sinh \gamma r)^2)$ is an eigenfunction for the radial part of the Laplacian.
The sphere $S^2$ is diffeomorphic to the rank one symmetric space $SO(3)/SO(2)$ and is a simply connected compact Riemannian manifold of constant positive sectional curvature. A metric is obtained by restricting the standard metric in $\mathbb{R}^3$. The maximal distance between any two points in $S^2$ using this metric is $L = \pi$. The mapping $\exp_o : T_o S^2 \to S^2$ is a diffeomorphism of the open ball $B_L(o) = \{x \in T_o S^2 : |x| < L\}$ onto the open set $S^2 \setminus A_o$. 

$$\exp_o Y \to (r, \theta)$$

where $Y \in B_L(o)$, $r = |Y|$ and $\theta$ is the local coordinate of the unit vector $Y/|Y|$. In these coordinates the Laplace-Beltrami operator $\Delta_{S^2}$ is

$$\Delta_{S^2} = \frac{\partial^2}{\partial r^2} + \cot\left(\frac{r}{2}\right) \frac{\partial}{\partial r} + \Delta_{S_r}$$
Control System in $S^2$

\[ dX(t) = \frac{1}{2} \cot\left( \frac{X(t)}{2} \right) dt + U(t) dt + dB(t) \]

\[ X(0) = X_0 \]

where $X(t) \in S^2 \setminus A_o$, $(B(t), t \in [0, T])$ is a real-valued standard Brownian motion for a fixed $T > 0$, and $X_0 \in (0, L)$ is a constant. The Brownian motion is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for the Brownian motion $B$. If $U(t)$ is a smooth function of $X(t)$ then $(X(t), t \in [0, T])$ is a Markov process with the infinitesimal generator

\[ \frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{1}{2} \cot\left( \frac{r}{2} \right) \frac{\partial}{\partial r} + U(r) \frac{\partial}{\partial r} \]
\[ J^0(U) = \int_0^T \left( a \sin^2 \frac{X(t)}{4} + U^2(t) \cos^2 \frac{X(t)}{4} \right) dt \]  \tag{7} \\
\[ J(U) = \mathbb{E} J^0(U) \]  \tag{8} 

The cost functional only depends on the radial distance from the origin \( o \), that is, \( X(t) = |Y(t)| \) for \( Y(t) \in S^2 \), so the control only appears in the radial component of the process. Note that \( \sin^2 \frac{x}{4} \) is an increasing function for \( x \in (0, \pi) \). The family of admissible controls, \( \mathcal{U} \) is

\[ \mathcal{U} = \{ U | U : [0, T] \times \Omega \rightarrow \mathbb{R} \text{ is jointly measurable, } (U(t), t \in [0, T]) \text{ is adapted to } (\mathcal{F}(t), t \in [0, T]) \text{ and } \int_0^T |U(t)|^2 dt < \infty \text{ a.s. } \} \]
\[
\frac{dg(t)}{dt} = \frac{3}{8}g + \frac{1}{4}g^2 - a \quad (9)
\]
\[
g(T) = 0
\]
\[
\frac{dh(t)}{dt} = -\frac{3}{16}g \quad (10)
\]
\[
h(T) = 0
\]
The stochastic control problem described above has an optimal admissible control, $U^*$, that is

$$U^*(t) = -\frac{1}{2} g(t) \tan \frac{X(t)}{4}$$

where $t \in [0, T]$ and $g$ satisfies (9). The optimal cost is

$$J(U^*) = g(0) \sin^2 \frac{X(0)}{4} + h(0)$$

where $h$ satisfies (10).
1. Stochastic control and differential games in spheres of higher dimension and in projective spaces
2. Stochastic control and differential games in hyperbolic spaces of higher dimension
3. Future work:
   a) Control and differential games in higher rank symmetric spaces, e.g. positive definite matrices, Grassmannians
   b) Random matrices, e.g. asymptotic behavior of eigenvalues
References


Thank You