A Stochastic Portfolio Optimization Model with Bounded Memory

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1 Introduction.

- Stochastic optimal control problems:

\[ dX(s) = f(s, X(s), u(s))ds + g(s, X(s), u(s))dW(s), \quad \forall s \in [t, T] \]
\[ X(t) = x. \]

Value function:

\[ V(t, x) = \sup_u \mathbb{E}_{t,x} \left[ \int_t^T L(s, X(s), u(s))ds + \Psi(T, X(T), u(T)) \right]. \]

- Typically, we can derive a HJB equation for the value function. By solving the associated HJB equation (explicitly or numerically), we can obtain the value function and optimal control policies.

- The HJB equation is usually a second order partial differential equation of parabolic type.

- In many cases, the value function is just a viscosity solution of the HJB equation.
• In many real world applications, some physical systems can only be modeled by stochastic dynamical systems whose evolutions depend on the past history of the states.

• General form:

\[
\begin{align*}
\frac{dX(s)}{ds} &= f(s, X_s, u(s)) ds + g(s, X_s, u(s)) dW(s), \quad s \in [t, T], \\
X(s) &= \psi(s), \quad s \in [t - h, t],
\end{align*}
\]

where \(X_s : [-h, 0] \rightarrow \mathbb{R}^n\) is defined by

\[
X_s(\theta) = X(s + \theta).
\]

- We will derive the associated HJB equation.
- The HJB equation is in infinite dimensional space.
- The HJB equation involves Fréchet derivatives.
• A model with memory:

\[
dX(s) = \alpha(s, X(s), Y(s), Z(s), u(s))ds \\
+ \beta(s, X(s), Y(s), Z(s), u(s))dW(s), \quad s \in [t, T],
\]

\[
X(s) = \psi(s), \quad \forall s \in [t - h, t],
\]

where

\[
Y(s) = \int_{-h}^{0} e^{\lambda \theta} X(s + \theta)d\theta, \quad Z(s) = X(s - h).
\]

– The value function \( V(t, \psi) \) will be defined on \([0, T] \times C[-h, 0] \), which is an infinite dimensional space.

– Under certain conditions, the associated HJB equation can be turned into a PDE in a finite dimensional space.
• Another model

\[
dX(t) = b(t, X(t), Y(t), Z(t)u(t))dt \\
+ \sigma(t, X(t), Y(t), Z(t), u(t))dW(t), \quad t \in (s, T]
\]

(5)

\[
X(t) = \eta(t - s), \quad t \in (s - \delta, s], \quad \eta \in C((−\delta, 0]; \mathbb{R}),
\]

(6)

where \(W(t)\) is a standard 1-dimensional Brownian motion, \(u(t)\) is the control variable, and \(Y(t)\) is given by

\[
Y(t) = \int_{-\delta}^{0} \psi(r)X(t + r)dr, \quad Z(t) = X(t - h), \quad t \in (s, T],
\]

(7)

where \(\psi(r)\) is a function of the form

\[
\psi(r) = a_0 + a_1r + a_2r^2 + \cdots + a_nr^n.
\]

(8)
2 A Portfolio Optimization Model with Memory

2.1 Problem Formulation

- One risky asset and one riskless asset with interest rate $r$.
- $K(t)$: the amount invested on the risky asset; $L(t)$: the amount invested on the riskless asset. Total wealth: $X(t) = K(t) + L(t)$.
- We consider the situation in which the performance of the risky asset depends on the history (memory) through the following delay variables $Y(t)$ and $Z(t)$:

$$
Y(t) \equiv \int_{-h}^{0} e^{\lambda s} X(t + s) ds, \quad Z(t) \equiv X(t - h).
$$

(9)
• Assume that $K(t) > 0$ almost surely. Instead of $Y(t), Z(t)$, we first consider

$$
\tilde{Y}(t) \equiv \frac{Y(t)}{K(t)} = \frac{1}{K(t)} \int_{-h}^{0} e^{\lambda s} X(t + s) ds,
$$

$$
\tilde{Z}(t) \equiv \frac{Z(t)}{K(t)} = \frac{X(t - h)}{K(t)}.
$$

• We model that $K(t)$ and $L(t)$ with the stochastic differential equations:

$$
dK(t) = \left[ (\mu_1 + \mu_2 \tilde{Y}(t) + \mu_3 \tilde{Z}(t))K(t) + I(t) \right] dt
$$

$$
+ \sigma K(t) dB(t),
$$

$$
dL(t) = [r L(t) - C(t) - I(t)] dt,
$$

where $I(t)$ is the investment rate and $C(t)$ is the consumption rate.
• Using the definition of $\tilde{Y}(t)$, $\tilde{Z}(t)$, we can get that $K(t)$ and $L(t)$ follow the stochastic differential equations:

$$
\begin{align*}
    dK(t) &= [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + I(t)]dt \\
          &\quad + \sigma K(t)dB(t), \\
    dL(t) &= [r L(t) - C(t) - I(t)]dt.
\end{align*}
\tag{14}
\tag{15}
$$

• Assume that $X(t) > 0$ almost surely. Then we can use $c(t) \equiv \frac{C(t)}{X(t)}$, $k(t) = \frac{K(t)}{X(t)}$ as our controls. It is easy to see that $L(t) = X(t)(1 - k(t))$. Now we can get the equation for $X(t)$ as

$$
\begin{align*}
    dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)]dt \\
          &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T].
\end{align*}
\tag{16}
$$

• Remark: It can showed that $X(t) > 0$ almost surely.
• The initial condition is the information about $X(s)$ for $s \in [-h, 0]$: 

$$X(s + t) = \varphi(t), \quad \forall t \in [-h, 0],$$

where $\varphi \in \mathbb{J}$ where $\mathbb{J} \equiv C[-h, 0]$ is the space for all continuous function defined on $[-h, 0]$ equipped with sup-norm:

$$\|\varphi\| = \sup_{s \in [-h, 0]} |\varphi(s)|.$$  

(17)

• Let $U(C)$ be the utility function and $\Psi$ be the terminal utility function. The objective function is

$$J(s, \varphi, k, c) = \mathbb{E}_{s,\varphi} \left[ \int_{s}^{T} e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right]$$

(19)

• The value function is given by

$$V(s, \varphi) = \sup_{k, c \geq 0} \mathbb{E}_{s,\varphi} \left[ \int_{s}^{T} e^{-\beta(t-s)} U(c(t)X(t)) dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].$$

(20)
• Under certain conditions, we have

\[ V(s, \varphi) = V(s, x, y, z), \quad (21) \]

where

\[ x = x(\varphi) \equiv \varphi(0), \quad (22) \]
\[ y = y(\varphi) \equiv \int_{-h}^{0} e^{\lambda s} \varphi(s) ds, \quad (23) \]
\[ z = z(\varphi) \equiv \varphi(-h). \quad (24) \]

• Further we will give the conditions that \( V \) only depends on \( s, x, y \), i.e.,

\[ V(s, \varphi) = V(s, x, y, z) = V(x, y). \quad (25) \]
2.2 Hamilton-Jacobi-Bellman Equation.

Let \( f \in C^{1,2,2}([0, T] \times \mathbb{R}^2) \) and define

\[
G(t) = f(t, X^\varphi(t), y(X^\varphi_t)),
\]

where

\[
y(\eta) = \int_{-h}^{0} e^{\lambda u} \eta(u) du, \quad \forall \eta \in J, \quad X_t(u) \equiv X(t + u), \quad \forall u \in [-h, 0].
\]

Then we have the following Ito’s formula:

**Lemma 2.1 (Ito’s formula)** Let the system be given by (16)-(17), and \( Y(t), Z(t) \) be given by (11). The Ito’s formula is

\[
dG(t) = \mathcal{L}f dt + (\sigma k x) f_x dB(t),
\]

where

\[
\mathcal{L}f = \mathcal{L}^{k,c}f(t, x, y, z) = f_t + (((\mu_1 - r)k - c + r)x + \mu_2 y + \mu_3 z) f_x \\
+ \frac{1}{2}(\sigma k x)^2 f_{xx} + (x - \lambda y - e^{-\lambda h} z) f_y.
\]

11
We assume that the value function $V$ depends on the initial path $\varphi$ only through the functionals $x(\varphi), y(\varphi)$ defined by (22-23). That is,

$$V(s, \varphi) = V(s, x(\varphi), y(\varphi)) = V(s, x, y).$$  \hspace{1cm} (30)

Then we can obtain the following HJB equation:

**Lemma 2.2 (HJB equation)** Assume that (30) holds and $V(s, x, y) \in C^2(\mathbb{R}^2)$. Then the HJB equation for $V(s, x, y)$ is given by

$$\beta V - V_s = \max_k \left[ \frac{1}{2} (\sigma k x)^2 V_{xx} + ((\mu_1 - r)k)xV_x \right] + (rx + \mu_2 y + \mu_3 z)V_x$$

$$+ \max_{c \geq 0} [-cxV_x + U(cx)] + (x - \lambda y - e^{-\lambda h} z)V_y, \quad \forall z \in \mathbb{R},$$ \hspace{1cm} (31)

with the boundary condition

$$V(T, x, y) = \Psi(x, y).$$  \hspace{1cm} (32)
2.3 The Solution of the HJB Equation.

Assume that the utility function is of the HARA type:

\[ U(cX) = \frac{1}{\gamma}(cX)^\gamma, \]  

(33)

where \( \gamma \in (-\infty, 1], \gamma \neq 0 \) is a constant. Then we can get

\[ \beta V - V_s = \max_k \left[ \frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (r x + \mu_2 y + \mu_3 z) V_x \]

\[ + \max_{c \geq 0} \left[ -c x V_x + \frac{1}{\gamma}(c x)^\gamma \right] + (x - \lambda y - e^{-\lambda h} z) V_y. \]  

(34)

The candidate for the optimal control policy is

\[ k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \]  

(35)

\[ c^* = \frac{1}{x} V_x^{\frac{1}{\gamma - 1}}. \]  

(36)
Plug $k^*$, $c^*$ into the HJB equation, and we can get

$$\beta V - V_s = -\frac{1}{2} \frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma - 1}} + (x - \lambda y - e^{-\lambda h} z) V_y$$
$$+ (r x + \mu_2 y + \mu_3 z) V_x. \quad (37)$$

It can be rewritten as

$$\beta V - V_s = -\frac{1}{2} \frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma - 1}} + (r x + \mu_2 y) V_x$$
$$+ (x - \lambda y) V_y + (\mu_3 V_x - e^{-\lambda h} V_y) z. \quad (38)$$

Suppose the terminal utility function $\Psi(x, y)$ is given in a form

$$\Psi(x, y) = \frac{1}{\gamma} (x + \mu_3 e^{\lambda h} y)^{\gamma}. \quad (39)$$
We look for a solution of the form

\[ V(s, x, y) = Q(s)\psi(x, y). \]  \hspace{1cm} (40)

Now we can get

\[
\left[ \beta Q(s) - Q'(s) \right] \psi(x, y) = -\frac{1}{2} \frac{(\mu_1 - r)^2 Q(s)\psi_x^2}{\sigma^2\psi_{xx}} + \left( \frac{1}{\gamma} - 1 \right) \left[ Q(s)\psi_x \right]^{\gamma-1} + (rx + \mu_2 y) Q(s)\psi_x \\
+ (x - \lambda y) Q(s)\psi_y + (\mu_3 \psi_x - e^{-\lambda h} \psi_y) Q(s) \zeta. \]  \hspace{1cm} (41)

Apparently, equation (41) has a solution which does not depend on \( \zeta \) if we have the following condition:

\[
(\mu_3 \psi_x - e^{-\lambda h} \psi_y) Q(s) \zeta = 0, \quad \forall \zeta \in \mathbb{R}. \]  \hspace{1cm} (42)

Define \( u \equiv x + \mu_3 e^{\lambda h} y \) and we look for a solution of the form

\[
\psi(x, y) = \frac{1}{\gamma} (x + \mu_3 e^{\lambda h} y)^{\gamma} = \frac{1}{\gamma} u^{\gamma}, \]  \hspace{1cm} (43)
Plug them into (41), and we can get

\[
\frac{1}{\gamma} [\beta Q(s) - Q'(s)] u^\gamma
\]

\[
= -\frac{1}{2\sigma^2(\gamma - 1)} Q(s) u^\gamma + \left(\frac{1}{\gamma} - 1\right) [Q(s)]^{\frac{\gamma}{\gamma - 1}} u^\gamma
\]

\[
+ \left[(r + \mu_3 e^{\lambda h}) x + (\mu_2 - \lambda \mu_3 e^{\lambda h}) y\right] Q(s) u^{\gamma - 1}.
\] (44)

Assume that

\[
\mu_2 - \lambda \mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h}.
\] (45)

(Some discussions on this assumption are given in Section 2.4.) Then it is easy to verify that

\[
\left[(r + \mu_3 e^{\lambda h}) x + (\mu_2 - \lambda \mu_3 e^{\lambda h}) y\right] Q(s) u^{\gamma - 1}
\]

\[
= (r + \mu_3 e^{\lambda h}) Q(s)(x + \mu_3 e^{\lambda h} y) u^{\gamma - 1}
\] (46)

\[
= (r + \mu_3 e^{\lambda h}) Q(s) u^\gamma
\] (47)
Canceling the term $u^\gamma$ on both sides, we can get

$$Q'(s) = (\gamma - 1) [Q(s)]^{\gamma-1} + \Lambda Q(s),$$

where

$$\Lambda \equiv \beta + \frac{(\mu_1 - r)^2\gamma}{2\sigma^2(\gamma - 1)} - \gamma(r + \mu_3 e^{\lambda h}).$$

We assume that the all parameters involved here satisfy

$$\Lambda > 0,$$

to guarantee that we have a well-defined solution.
At the point \( t = T \), we have

\[
V(T, x, y) = Q(T)\frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^\gamma = \Psi(x, y) = \frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^\gamma.
\] (51)

Therefore, the boundary condition for \( Q(s) \) at \( s = T \) is given by

\[
Q(T) = 1.
\] (52)

By solving (48) – (52), we can get the solution

\[
Q(s) = \left[ \left( 1 - \frac{1 - \gamma}{\Lambda} \right) e^{-\frac{\Lambda}{1-\gamma}(T-s)} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma}.
\] (53)

It is easy to verify that, if \( \Lambda > 0 \), we have

\[
Q(s) > 0, \quad \forall s \in [0, T].
\] (54)

Therefore, the solution of the HJB equation (31) – (32) is given by
\[
V(s, x, y) = \frac{1}{\gamma}Q(s)(x + \mu_3e^{\lambda h}y)^\gamma.
\] (55)

and the optimal investment ratio and the optimal consumption rate control are

\[
k^*(s) = \frac{(\mu_1 - r)(x + \mu_3e^{\lambda h}y)}{(1 - \gamma)\sigma^2 x},
\] (56)

\[
c^*(s) = \frac{x + \mu_3e^{\lambda h}y}{x}Q(s)^{\frac{1}{\gamma-1}},
\] (57)

where \(Q(s)\) is given by (53) and \(x, y\) are estimated at time \(s\) as the following:

\[
x = X(s), \quad y = Y(s) = \int_{-h}^{0} e^{\lambda \theta} X(s + \theta) d\theta.
\] (58)

A verification theorem is needed to ensure that the solution is actually equal to the value function defined by (20).
Theorem 2.1 (Verification Theorem) Let $V(s, x, y) \in C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R})$ be a solution of the HJB equation (31) – (32) such that

$$
E \left[ \int_0^T [k(t)X(t)V_x(t, X(t), Y(t))]^2 dt \right] < \infty, \quad \forall k \in \Pi. \quad (59)
$$

Then we have

$$
V(s, x, y) = \sup_{k,c \geq 0} E_{x,\phi} \left[ \int_s^T e^{-\beta(t-s)} U(c(t)X(t)) dt 
+ e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right]. \quad (60)
$$

In addition, if the utility function is given by

$$
U(x) = \frac{1}{\gamma} x^\gamma, \quad \gamma \in (-\infty, 1] \text{ and } \gamma \neq 0, \quad (61)
$$

then the optimal control policy is given by

$$
k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x} V_x^{-\frac{1}{\gamma-1}}. \quad (62)
$$
2.4 Some Examples.

In this section, we discuss some examples. For convenience, we rewrite the dynamic equation for the wealth process $X(t)$ here:

$$
\begin{align*}
\frac{dX(t)}{dt} &= \left[ (\mu_1 - r)k(t) - c(t) + r \right] X(t) + \mu_2 Y(t) + \mu_3 Z(t) \right] dt \\
&\quad + \left[ \sigma k(t)X(t)(t) \right] dB(t), \quad \forall t \in [s, T].
\end{align*}
$$

(63)

The initial condition is given by

$$
X(s + t) = \varphi(t), \quad \forall t \in [-h, 0].
$$

(64)

In last section, we have obtained an explicit solution given the assumption (equation (45)):

$$
\mu_2 - \lambda \mu_3 e^{\lambda h} = (\mu_3 e^{\lambda h} + r)\mu_3 e^{\lambda h}.
$$

(65)

We discuss some interest cases here.
Case 1. Let $\mu_3 = 0$ Then we must have $\mu_2 = 0$

\[
dX(t) = [((\mu_1 - r)k(t) - c(t) + r)X(t)] \, dt + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T],
\]

\[X(s + \theta) = \varphi(\theta), \quad \forall \theta \in [-h, 0].\] (67)

The optimal control policy is

\[
k_s^* = \frac{(\mu_1 - r)}{\sigma^2(1 - \gamma)}, \quad (68) \]

\[
c_s^* = [Q(s)]^{\frac{1}{\gamma - 1}}. \quad (69) \]

where $Q(s)$ is given by

\[
Q(s) = \left[ \left(1 - \frac{1 - \gamma}{\Lambda} \right) e^{-\frac{\Lambda (T-s)}{1-\gamma}} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma} \quad (70)\]

and $\Lambda$ is given by

\[
\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2 \sigma^2 (\gamma - 1)} - \gamma r. \quad (71)\]
The value function is given by

\[ V(s, x, y) = \frac{1}{\gamma}Q(s)x^\gamma. \] (72)

The value function and the optimal consumption control policy is the same with the optimal consumption control policy of the classical Merton’s problem on a finite time horizon with objective function

\[ J(s, x) = \max_{k,c} \mathbb{E}_{s,x} \left[ \int_s^T e^{-\beta(t-s)}\frac{1}{\gamma}(c(t)X(t))^\gamma dt + e^{-\beta(T-s)}\frac{1}{\gamma}[X(T)]^\gamma \right] , \] (73)

with dynamic equations for \( X(t) \) being

\[ dX(t) = [(\mu_1 - r)k(t) - c(t) + r]X(t)dt + \sigma k(t)X(t)dB(t), \] (74)

\[ X(0) = x. \] (75)

where \( x = x(\varphi) = \varphi(0) \).
Case 2. Let $\mu_3 = \nu e^{-\lambda h}$ for a constant $\nu > 0$ and let $\mu_2 = \nu^2 + \nu(r + \lambda)$. Now the equations of $X(t)$ become

$$dX(t) = \left[\left((\mu_1 - r)k(t) - c(t) + r\right)X(t) + (\nu^2 + \nu(r + \lambda))Y(t)
+ \nu e^{-\lambda h}Z(t)\right]dt + \sigma k(t)X(t)dB(t),$$

$$X(s + \theta) = \varphi(\theta), \quad \forall \theta \in [-h, 0].$$

(76) (77)

The optimal control is now given by

$$k^*_s = \frac{(\mu_1 - r)(x + \nu y)}{(1 - \gamma)\sigma^2 x} = \frac{\mu_1 - r}{\sigma^2(1 - \gamma)} + \left[\frac{(\mu_1 - r)\nu}{\sigma^2(1 - \gamma)}\right] \frac{y}{x};$$

$$c^*_s = \frac{x + \nu y}{x}Q(s)^\frac{1}{\gamma - 1} = \left(1 + \nu\frac{y}{x}\right)Q(s)^\frac{1}{\gamma - 1},$$

(78) (79)

where $Q(s)$ is given by

$$Q(s) = \left[\left(1 - \frac{1 - \gamma}{\Lambda}\right)e^{-\frac{\Lambda(T-s)}{\gamma - 1}} + \frac{1 - \gamma}{\Lambda}\right]^{1-\gamma}$$

(80)
and $\Lambda$ is given by

\[ \Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(\nu + r). \]  

(81)

The value function is given by

\[ V(s, x, y) = \frac{1}{\gamma}Q(s)(x + \nu y)^\gamma. \]  

(82)

As we can see, both the control policy and the value function now depend on the parameter $\nu$. 
Case 3. Now we assume $\mu_3 = e^{-\lambda h}$. Then we have
$$r + \mu_3 e^{\lambda h} = r + 1.$$ 
If $\mu_2$ satisfies
$$\mu_2 = r + 1 + \lambda,$$ 
then it is easy to verify that (65) holds. Then the dynamic equation for $X(t)$ is now given by
$$dX(t) = \left[\left((\mu_1 - r)k(t) - c(t) + r\right)X(t) + (\lambda + r + 1)Y(t) + e^{-\lambda h}Z(t)\right] dt$$
$$+ \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T],$$
$$X(s + \theta) = \varphi(\theta), \quad \forall \theta \in [-h, 0].$$

The optimal control policy is
$$k_s^* = \frac{(\mu_1 - r)(x + y)}{\sigma^2(1 - \gamma)x} = \frac{\mu_1 - r}{\sigma^2(1 - \gamma)} + \left[\frac{(\mu_1 - r)}{\sigma^2(1 - \gamma)}\right] \frac{y}{x},$$
$$c_s^* = \frac{x + y}{x}Q(s)^{\frac{1}{\gamma - 1}} = \left(1 + \frac{y}{x}\right) Q(s)^{\frac{1}{\gamma - 1}},$$
$$\forall t \in [s, T],$$
$$\forall \theta \in [-h, 0].$$
where $Q(s)$ is given by

$$Q(s) = \left[ \left( 1 - \frac{1 - \gamma}{\Lambda} \right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1 - \gamma}{\Lambda} \right]^{1-\gamma}$$

(88)

and $\Lambda$ is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(1 + r).$$

(89)

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + y)^\gamma.$$  

(90)

As we can see, both the control policy and the value function now depend on the delay variable $y$. Actually, this is a special case of Case 2 with $\nu = 1$. 

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**Final Remark.** From the condition (65), we can get

\[
\mu_2 = \lambda \mu_3 e^{\lambda h} + (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h} \\
= \mu_3 e^{\lambda h} (\lambda + r + \mu_3 e^{\lambda h}).
\]  

(91)

So it is easy to see that \( \mu_2 = 0 \) if and only if \( \mu_3 = 0 \), provided that \( \mu_3 \geq 0 \), and

\[
\lim_{\mu_3 \to \infty} \mu_2 = \infty.
\]

In other words, the price change of \( X(t) \) must depend on both \( Y(t) \) and \( Z(t) \) at the same time with similar manner in order to obtain a explicit solution \( V(s, x, y) \).
3 Stochastic Control Problems with Memory: General Framework

3.1 Problem Formulation

• We study the finite time horizon optimal control problem for a general system of stochastic functional differential equations on the interval $[t, T]$.

• Let $h > 0$ be a fixed constant, and let $J = [-h, 0]$ denote the duration of the bounded memory of the equations considered in this paper. For the sake of simplicity, we denote $C(J; \mathbb{R}^n)$, the space of continuous functions $\phi : J \to \mathbb{R}^n$, by $C$. Note that $C$ is a real separable Banach space under the supremum norm defined by

$$
\| \phi \| = \sup_{t \in J} |\phi(t)|, \quad \phi \in C,
$$

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^n$. 

• Denote by \(( \cdot | \cdot )\) the inner product in \(L^2(\mathbb{J}, \mathbb{R}^n)\) as the following
\[
(\phi|\psi) = \int_{-r}^{0} \langle \phi(s), \psi(s) \rangle ds, \quad \text{and} \quad \|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}}, \quad \forall \phi, \psi \in \mathbb{C},
\]
where \(\langle \cdot, \cdot \rangle\) is the inner product in \(\mathbb{R}^n\).

• Notation: If \(\psi \in C([-r, \infty); \mathbb{R}^n)\) and \(t \in \mathbb{R}_+\), let \(\psi_t \in \mathbb{C}\) be defined by
\[\psi_t(\theta) = \psi(t + \theta), \quad \theta \in \mathbb{J}.\]

• Let \(\{W(t), t \geq 0\}\) be a certain \(m\)-dimensional standard Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}, P; \mathcal{F})\).

• Let \(L^2(\Omega, \mathbb{C})\) be the space of \(\mathbb{C}\)-valued random variables \(\Xi : \Omega \to \mathbb{C}\) such that
\[
\|\Xi\|_{L^2} = \left\{ \int_{\Omega} \|\Xi(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.
\]
In addition, let \(L^2(\Omega, \mathbb{C}; \mathcal{F}(t))\) be those \(\Xi \in L^2(\Omega, \mathbb{C})\) which are \(\mathcal{F}(t)\)-measurable.
We consider the following system of controlled stochastic functional differential equations with a bounded memory:

\[ dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad s \in [t, T], \tag{92} \]

with the initial condition

\[ X_t = \psi_t, \quad \forall \psi_t \in L^2(\Omega, \mathbb{C}; \mathcal{F}(t)) \]

The functions, \( f : [0, T] \times \mathbb{C} \times U \to \mathbb{R}^n \) and \( g : [0, T] \times \mathbb{C} \times U \to \mathbb{R}^{n \times m} \) are given deterministic functions and they satisfy the following linear growth and Lipschitz conditions (See also Mohammed [?, ?]).

Assumption 1 There exists a constant \( \Lambda > 0 \) such that

\[
|f(t, \varphi, u) - f(t, \phi, u)| + |g(t, \varphi, u) - g(t, \phi, u)| \leq \Lambda \|\varphi - \phi\|,
\]

\[ \forall (t, \varphi, u), (t, \phi, u) \in [0, T] \times \mathbb{C} \times U. \]

Assumption 2 There exists a constant \( K > 0 \) such that

\[
|f(t, \phi, u)| + |g(t, \phi, u)| \leq K(1 + \|\phi\|), \quad \forall (t, \phi, u) \in [0, T] \times \mathbb{C} \times U.
\]
• Let $L$ and $\Psi$ be two continuous real-valued functions on $[0, T] \times C \times U$ and $[0, T] \times C$, with at most polynomial growth in $L^2(J; \mathcal{R})$. In other words, there exist a constant $\Lambda > 0$ and an integer $k > 0$ such that

$$|L(t, \phi, u)| \leq \Lambda(1 + \|\phi\|_2)^k,$$

and

$$|\Psi(t, \phi)| \leq \Lambda(1 + \|\phi\|_2)^k.$$  

• The objective function is

$$J(t, \psi; u(\cdot)) \equiv \mathbb{E}\left[ \int_t^T e^{-\rho(s-t)}L(s, X_s(t, \psi, u(\cdot)), u(s))ds + e^{-\rho(T-t)}\Psi(X_T(t, \psi, u(\cdot))) \right],$$

where $\rho > 0$ denotes a discount factor.

• The value function $V : [0, T] \times C \to \mathcal{R}$ is defined as

$$V(t, \psi) = \sup_{u(\cdot) \in U[t, T]} J(t, \psi; u(\cdot)).$$  

(94)
3.2 The Hamilton-Jacobi-Bellman Equation

- Let $C^*$ and $C^\dagger$ be the space of bounded linear functionals $\Phi : C \rightarrow \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : C \times C \rightarrow \mathbb{R}$, of the space $C$, respectively.

- Let $B = \{ v1_{\{0\}}, v \in \mathbb{R}^n \}$, where $1_{\{0\}} : [-r, 0] \rightarrow \mathbb{R}$ is defined by
  
  $$1_{\{0\}}(\theta) = \begin{cases} 
  0 & \text{for } \theta \in [-r, 0), \\
  1 & \text{for } \theta = 0.
  \end{cases}$$

  We form the direct sum
  
  $$C \oplus B = \{ \phi + v1_{\{0\}} : \phi \in C, \ v \in \mathbb{R}^n \}$$

  and equip it with the norm $\| \cdot \|$ defined by
  
  $$\| \phi + v1_{\{0\}} \| = \sup_{\theta \in [-r,0]} |\phi(\theta)| + |v|, \ \phi \in C, \ v \in \mathbb{R}^n.$$  

- Fréchet derivative: $D\Phi(\varphi) \in C^*$.

  It has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (C \oplus B)^*$.
• The Second order Fréchet derivative, $D^2\Phi(\varphi) \in C^\dagger$, has a unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbb{C} \oplus \mathbb{B})^\dagger$.

• $S$-operator: For a Borel measurable function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$, we also define

$$S(\Phi)(\phi) = \lim_{h \to 0^+} \frac{1}{h} \left[ \Phi(\tilde{\phi}_h) - \Phi(\phi) \right]$$  \hspace{1cm} (95)

for all $\phi \in \mathbb{C}$, where $\tilde{\phi} : [-r, T] \rightarrow \mathbb{R}^n$ is an extension of $\phi$ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and $\tilde{\phi}_t \in \mathbb{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

• Let $C_{1,2}^{lip}([0, T] \times \mathbb{C})$ be the space of functions $\Phi : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}$ such that $\frac{\partial \Phi}{\partial t} : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}$ and $D^2\Phi : [0, T] \times \mathbb{C} \rightarrow C^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\|^\dagger \leq K\|\phi - \varphi\| \quad \forall t \in [0, T], \ \phi, \varphi \in \mathbb{C}.$$
Theorem 3.2 Suppose that $\Phi \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap D(S)$. Let $u(\cdot) \in U[t, T]$ and $\{X_s, s \in [t, T]\}$ be the $\mathbb{C}$-valued Markov solution process of equation (92) with the initial data $(t, \varphi_t) \in [0, T] \times \mathbb{C}$. Then

$$\lim_{\epsilon \downarrow 0} \frac{E[\Phi(t + \epsilon, X_{t+\epsilon})] - \Phi(t, \varphi_t)}{\epsilon} = \frac{\partial}{\partial t} \Phi(t, \varphi_t) + S(\Phi)(t, \varphi_t) + \overline{D \Phi(t, \varphi_t)(f(t, \varphi_t, u(t))1_{\{0\})}
$$

$$+ \frac{1}{2} \sum_{j=1}^{m} D^2 \Phi(t, \varphi_t)(g(t, \varphi_t, u(t))e_j1_{\{0\}}, g(t, \varphi_t, u(t))e_j1_{\{0\}}),$$

where $e_j, j = 1, 2, \cdots, m,$ is the $j$th unit vector of the standard basis in $\mathbb{R}^m$. 
**Theorem (Larssen)** Let Assumptions 1-2 hold. Then for any \((t, \psi) \in [0, T] \times C\) and \(F(t)\)-stopping time \(\tau \in [t, T]\),

\[
V(t, \psi) = \sup_{u(\cdot) \in U[t,T]} \mathbb{E} \left[ \int_t^\tau e^{-\rho(s-t)} L(s, X_s(t, \psi, u(\cdot)), u(s)) ds 
+ e^{-\rho(\tau-t)} V(\tau, X_\tau(t, \psi, u(\cdot))) \right].
\]  

(97)

Let \(v \in U\). We define:

\[
\mathcal{A}^v V(t, \psi) \equiv S(V)(t, \psi) + \overline{DV(t, \psi)}(f(t, \psi, v)1_{\{0\}})
+ \frac{1}{2} \sum_{i=1}^m D^2 V(t, \psi)(g(t, \psi, v)e_i1_{\{0\}}, g(t, \psi, v)e_i1_{\{0\}}).
\]
Theorem 3.3 Suppose $V$ is the value function defined by (94) and atisfies $V \in C^{1,2}_{lip}([0,T] \times \mathbb{C}) \cap \mathcal{D}(\mathcal{S})$. Then the value function $V$ satisfies the following HJB equation:

$$
\rho V(t, \psi) - \frac{\partial V}{\partial t}(t, \psi) - \max_{v \in U} [A^v V(t, \psi) + L(t, \psi, v)] = 0 \quad (98)
$$
on $[0, T] \times \mathbb{C}$, and $V(T, \psi) = \Psi(\psi), \ \forall \psi \in \mathbb{C}$.

- The value function $V$ satisfies the necessary smoothness condition $V \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(\mathcal{S})$.
- In general we need to consider viscosity solution instead of a classical solution for HJB equation (98).
- Actually, the value function is a unique viscosity solution of the HJB equation (98).
3.3 Viscosity Solution of the HJB Equation

**Definition 1** Let $w \in C([0, T] \times \mathbb{C})$. We say that $w$ is a viscosity subsolution of (98) if, for every $\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap D(S)$, for $(t, \psi) \in [0, T] \times \mathbb{C}$ satisfying $\Gamma \geq w$ on $[0, T] \times \mathbb{C}$ and $\Gamma(t, \psi) = w(t, \psi)$, we have

$$
\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \max_{v \in U} \left[ A^v \Gamma(t, \psi) + L(t, \psi, v) \right] \leq 0.
$$

We say that $w$ is a viscosity super solution of (98) if, for every $\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap D(S)$, and for $(t, \psi) \in [0, T] \times \mathbb{C}$ satisfying $\Gamma \leq w$ on $[0, T] \times \mathbb{C}$ and $\Gamma(t, \psi) = w(t, \psi)$, we have

$$
\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \max_{v \in U} \left[ A^v \Gamma(t, \psi) + L(t, \psi, v) \right] \geq 0.
$$

We say that $w$ is a viscosity solution of (98) if it is both a viscosity supersolution and a viscosity subsolution of (98).

For our value function $V$ defined by (94), we now show that it has the following property.
Lemma 3.3 The value function $V$ defined in (94) is continuous and there exists a constant $\Lambda > 0$ and a positive integer $k$ such that, for every $(t, \phi) \in [0, T] \times \mathcal{C}$,

$$|V(t, \phi)| \leq \Lambda (1 + \|\phi\|_2)^k.$$  \hspace{1cm} (99)

and there exists a constant $K > 0$ such that

$$|V(s, \phi) - V(s, \varphi)| \leq K \|\phi - \varphi\|, \quad \forall (s, \phi), (s, \varphi) \in [0, T] \times \mathcal{C}. \hspace{1cm} (100)$$

We have the following result:

**Theorem 3.4** The value function $V$ is a viscosity solution of the HJB equation

$$\rho V(t, \psi) - \frac{\partial V}{\partial t}(t, \psi) - \max_{v \in \mathcal{U}} [\mathcal{A}^v V(t, \psi) + L(t, \psi, v)] = 0$$  \hspace{1cm} (101)

on $[0, T] \times \mathcal{C}$, and $V(T, \psi) = \Psi(\psi), \forall \psi \in \mathcal{C}$. 

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3.4 Uniqueness

Since a viscosity solution is both a subsolution and a supersolution, the uniqueness result will follow immediately after we establish the following comparison principle:

**Theorem 3.5 (Comparison Principle)** Assume that $V_1(t, \psi)$ and $V_2(t, \psi)$ are both continuous with respect to the argument $(t, \psi)$ and are respectively viscosity subsolution and supersolution of (98) with at most a polynomial growth. In other terms, there exists a real number $\Lambda > 0$ and a positive integer $k > 0$ such that,

$$|V_i(t, \psi)| \leq \Lambda(1 + \|\psi\|_2)^k, \text{ for } (t, \psi) \in [0, T] \times \mathbb{C}, \ i = 1, 2.$$  

Then

$$V_1(t, \psi) \leq V_2(t, \psi) \text{ for all } (t, \psi) \in [0, T] \times \mathbb{C}. \quad (102)$$
THANK YOU!