Stationary Solutions and Random Periodic Solutions of Stochastic Equations

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at

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Fixed Point of ODE

Consider ODE

\[
\begin{align*}
\frac{dv(t)}{dt} &= -v(t) \\
v(0) &= y \in \mathbb{R}^1.
\end{align*}
\]

For fixed \( t \geq 0 \), regard \( v \) as a mapping

\[v^y(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^1.\]

Then a fixed point is an initial value of the ODE satisfying

\[v^y(t) = y \text{ for all } t \geq 0.\]

It is easy to check that \( y = 0 \) satisfies the requirement.
A Nontrivial Example: Ornstein-Uhlenbeck Process

Consider Ornstein-Uhlenbeck process:

\[
\begin{cases}
  dv(t) = -v(t)\,dt + dW_t \\
  v(0) = Y(\omega) \in L^2(\Omega).
\end{cases}
\]

For fixed \( t \geq 0 \), regard \( v \) as a mapping

\[ v^{Y(\omega)}(t) : L^2(\Omega) \rightarrow L^2(\Omega). \]

Almost impossible to find a fixed point like

\[ v^{Y(\omega)}(t) = Y(\omega) \quad \text{for } t \geq 0 \text{ a.s.} \]
A Nontrivial Example: Ornstein-Uhlenbeck Process (Continued)

Define the “stochastic fixed point” like

\[ v^Y(\omega)(t) = Y(\theta_t \omega) \quad \text{for } t \geq 0 \ a.s., \]

where

\[ (\theta_t W)(s) = W(t + s) - W(t) \quad \text{for any } s \in (-\infty, +\infty). \]

You can verify that the “stochastic fixed point” is

\[ Y(\omega) = \int_{-\infty}^{0} e^s dW_s. \]
Definition for Stationary Solution (Stochastic Fixed Point)

A measurable space: \((V, \mathcal{B}(V))\).

A metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})\), 

\((\theta_t)_{t \geq 0} : \Omega \to \Omega\) satisfies:

- \(P \cdot \theta_t^{-1} = P\);
- \(\theta_0 = I\), where \(I\) is the identity transformation on \(\Omega\);
- \(\theta_s \circ \theta_t = \theta_{s+t}\) for all \(s, t \geq 0\).

For a measurable random dynamical system 
v : \([0, \infty) \times V \times \Omega \to V\), the stationary solution is a \(\mathcal{F}\) measurable r.v. \(Y : \Omega \to V\) such that (Arnold 1998)

\[ v^{Y(\omega)}(t, \omega) = Y(\theta_t \omega) \quad \text{for } t \geq 0 \text{ a.s.} \]
Some Existing Results

- **Sinai 1991, 1996** Stochastic Burgers equations with $C^3$ noise under strong smooth conditions

- **Mattingly 1999, CMP** 2D Stochastic Navier-Stokes equation with additive noise

- **E & Khanin & Mazel & Sinai 2000, AM** Stochastic inviscid Burgers equations with additive $C^3$ noise

- **Caraballo & Kloeden & Schmalfuss 2004, AMO** Stochastic evolution equations with small Lipschitz constant and linear noise
A Basic Assumption in Invariant Manifold Theory: There Exists Stationary Solution

- Arnold 1998
- Duan & Lu & Schmalfuss 2003, AP
- Mohammed & T. Zhang & Zhao 2008, Memoirs of AMS
- Lian & Lu 2010, Memoirs of AMS
Stationary Solutions of Parabolic SPDEs

We use the correspondence between SPDEs and BDSDEs to construct the stationary solutions of SPDEs:

\[ v(t, x) = v(0, x) + \int_0^t \left[ \mathcal{L} v(s, x) + f(x, v(s, x)) \right] ds \]

\[ + \int_0^t g(x, v(s, x)) dB_s. \]

\( f : \mathbb{R}^d \times \mathbb{R}^1 \rightarrow \mathbb{R}^1, \ g : \mathbb{R}^d \times \mathbb{R}^1 \rightarrow \mathcal{L}^2_{U_0}(\mathbb{R}^1); \)

\( B: \) Wiener process with values in a Hilbert space;

\( \mathcal{L}: \) a second order differential operator given by

\[ \mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} \]

with \( (a_{ij}(x)) = \sigma \sigma^*(x). \)
Results

- **Zhang & Zhao 2007, JFA** Lipschitz coefficients
- **Zhang & Zhao 2010, JDE** linear growth coefficients
- **Zhang & Zhao 2013, SPA** polynomial growth coefficients
- **Zhang 2011, SD** stationary stochastic viscosity solutions

Using BDSDEs, we construct the stationary solutions of non-linear SPDEs with non-additive noise.
For arbitrary $T > 0$, define $\hat{B}_s = B_{T-s} - B_T$.

By the integral transformation, $u(t, x) \triangleq v(T-t, x)$ satisfies terminal-value SPDE

$$
u(t, x) = u(T, x) + \int_t^T \left[ \mathcal{L}u(s, x) + f(x, u(s, x)) \right] ds$$

$$- \int_t^T g(x, u(s, x)) d^\dagger \hat{B}_s.$$
Infinite horizon BDSDE:

\[ e^{-K_s} Y_{t,x}^s = \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr \]

\[ - \int_s^\infty e^{-Kr} g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{B}_r \]

\[ - \int_s^\infty e^{-Kr} \langle Z_r^{t,x}, dW_r \rangle, \]

where

\[ X_{t,x}^s = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r. \]

Infinite horizon BDSDE has a unique solution \( (Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2p,-K}_{\mathcal{F}B,W} \cap L^{2p,-K}_{\mathcal{F}B,W} ([t, \infty]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \times L^{2,-K}_{\mathcal{F}B,W} ([t, \infty]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d)) \)
The Correspondence between SPDEs and BDSDEs on Finite Time Interval

BDSDE: \( Y_{s}^{t,x} = h(X_{T}^{t,x}) + \int_{s}^{T} f(r, X_{r}^{t,x}, Y_{r}^{t,x})dr \)

\[- \int_{s}^{T} \langle g(r, X_{r}^{t,x}, Y_{r}^{t,x}), d\hat{B}_{r} \rangle - \int_{s}^{T} \langle Z_{r}^{t,x}, dW_{r} \rangle.\]

SPDE: \( u(t, x) = h(x) + \int_{t}^{T} \{ L u(s, x) + f_{n}(s, x, u(s, x)) \} ds \)

\[- \int_{t}^{T} \langle g(s, x, u(s, x)), d\hat{B}_{s} \rangle, \quad 0 \leq t \leq T.\]

Correspondence: \( u(t, x) \triangleq Y_{t}^{t,x} \) is a solution of SPDE, and

\( (Y_{s}^{t,x}, Z_{s}^{t,x}) = (u(s, X_{s}^{t,x}), (\sigma \nabla u)(s, X_{s}^{t,x})). \)
Based on different smooth requirements for coefficients, this correspondence was established for differential types of solutions of SPDEs.

- **Pardoux & Peng 1994, PTRF smooth solution**
- **Buckdahn & Ma 2001, SPA stochastic viscosity solution**
- **Bally & Matoussi 2001, JTP weak solution**
Define $\hat{\theta}_t : \Omega \longrightarrow \Omega$, $t \geq 0$, by

$$
\hat{\theta}_t \left( \begin{array}{c} \hat{B}_s \\ W_s \end{array} \right) = \left( \begin{array}{c} \hat{B}_{s+t} - \hat{B}_t \\ W_{s+t} - W_t \end{array} \right)
$$

Then for any $s, t \geq 0$,

- $P \cdot \hat{\theta}_t^{-1} = P$;
- $\hat{\theta}_0 = I$, where $I$ is the identity transformation on $\Omega$;
- $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$.

Also for an arbitrary $\mathcal{F}$ measurable $\phi$, set

$$
\hat{\theta} \circ \phi(\omega) = \phi(\hat{\theta}(\omega)).
$$
By uniqueness of solution, the solution of infinite horizon BDSDE \((Y^{t,\cdot}, Z^{t,\cdot})\) satisfies the stationary property: for any \(t \geq 0\),

\[
\hat{\theta}_r \circ Y^{s,\cdot}_s = Y^{t+r,\cdot}_{s+r} \quad \hat{\theta}_r \circ Z^{s,\cdot}_s = Z^{t+r,\cdot}_{s+r} \quad \text{for } r \geq 0, \ s \geq t \ a.s.
\]

In particular, for any \(t \geq 0\),

\[
\hat{\theta}_r \circ Y^{t,\cdot}_t = Y^{t+r,\cdot}_{t+r} \quad \text{for } r \geq 0 \ a.s.
\]
So, for any $t \geq 0$,

$$\hat{\theta}_r \circ u(t, \cdot) = u(t + r, \cdot) \quad \text{for } r \geq 0 \ a.s.$$

By Kolmogorov’s continuity lemma, $u(t, \cdot)$ is continuous w.r.t. $t$. Thus

$$\hat{\theta}_r \circ u(t, \cdot) = u(t + r, \cdot) \quad \text{for } t, r \geq 0 \ a.s.$$
Time Reverse Transformation

For arbitrary $T > 0$, choose $\hat{B}$ in terminal-value SPDEs as $\hat{B}_s = B_{T-s} - B_T$. We see that $v(t, x) \triangleq u(T - t, x)$ satisfies initial-value SPDE

$$v(t, x) = v(0, x) + \int_0^t \left[ L v(s, x) + f(x, v(s, x)) \right] ds + \int_0^t g(x, v(s, x)) d B_s, \quad t \geq 0.$$ 

In fact, we can prove that $v(t, x, \omega) \triangleq Y^{T-t, x}_{T-t} (\hat{\omega}) = Y^{0, x}_0 (\hat{\theta}_{T-t} \hat{\omega})$ is independent of the choice of $T$ as follows:

$$\hat{\theta}_{T-t} \hat{\omega} = \hat{\omega}(T - t + s) - \hat{\omega}(T - t) = (B_{T-(T-t+s)} - B_T) - (B_{T-(T-t)} - B_T) = B_{t-s} - B_t.$$
Transferring the Stationary Property from Terminal-Value to Initial-Value SPDEs

Define \( \theta_t = (\hat{\theta}_t)^{-1} \), \( t \geq 0 \), then \( \theta_t \) is a shift w.r.t. \( B \) satisfying

\[
\theta_t \circ B_s = B_{s+t} - B_t.
\]

So

\[
\theta_r v(t, \cdot, \omega) = v(t + r, \cdot, \omega) \quad \text{for } r \geq 0 \text{ a.s.}
\]

In particular, let \( Y(\cdot, \omega) = v(0, \cdot, \omega) = Y_T^{T, \cdot}(\hat{\omega}). \)

Then the above implies that \( Y(\cdot, \omega) \) satisfies the definition of stationary solution:

\[
v^{Y(\cdot, \omega)}(t, \cdot, \omega) = Y(\cdot, \theta_t \omega) \quad \text{for } t \geq 0 \text{ a.s.}
\]
Assume that we had known that

- the correspondence between SPDE and BDSDE in some reasonable sense
- the existence and uniqueness of solution of infinite horizon BDSDE

**Theorem**

For arbitrary $T$ and $t \in [0, T]$, let $\nu(t, x) \triangleq Y^{T-t,x}_{T-t}$, where $(Y^{t, \cdot}_{\cdot}, Z^{t, \cdot}_{\cdot})$ is the solution of the infinite horizon BDSDE with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $\nu(t, \cdot)$ is a “perfect” stationary solution of SPDE.
Assumptions

(H.1). \( \exists \ p \geq 2 \) and \( f_0 \) with \( \int_0^\infty \int_{\mathbb{R}^d} |f_0(s, x)|^{8p} \rho^{-1}(x)\,dx\,ds < \infty \) s.t.
\[
|f(s, x, y)| \leq L(|f_0(s, x)| + |y|^p);
\]
\[
|\partial_y f(s, x, y)| \leq L(1 + |y|^{p-1}).
\]

(H.2). \[
|f(s, x_1, y) - f(s, x_2, y)| \leq L(1 + |y|^p)|x_1 - x_2|,
\]
\[
|\partial_y f(s, x_1, y) - \partial_y f(s, x_2, y)| \leq L(1 + |y|^{p-1})|x_1 - x_2|,
\]
\[
|\partial_y f(s, x, y_1) - \partial_y f(s, x, y_2)| \leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|,
\]

\( g(s, x, y) \): Lipschitz condition on \( (s, x, y) \),
\( \partial_y g(s, x, y) \): bounded and Lipschitz condition on \( (x, y) \).

(H.3). \( \exists \ \mu > 0 \) with \( 2\mu - K - p(2p - 1) \sum_{j=1}^{\infty} L_j > 0 \) s.t.
\[
(y_1 - y_2)(f(s, x, y_1) - f(s, x, y_2)) \leq -\mu |y_1 - y_2|^2.
\]
(H.4). Diffusion coefficients $b \in C^2_{l,b}(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C^3_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$.

(H.5). Matrix $\sigma(x)$ is uniformly elliptic, i.e. $\exists \varepsilon > 0$ s.t.

$$\sigma \sigma^*(x) \geq \varepsilon I_d.$$
Approximating Sequences

Step 1. To approximate infinite horizon BDSDEs:

\[
Y_{s}^{t,x,m} = \int_{s}^{m} f(r, X_{r}^{t,x}, Y_{r}^{t,x,m})dr - \int_{s}^{m} g(r, X_{r}^{t,x}, Y_{r}^{t,x,m})d\hat{B}_{r}
- \int_{s}^{m} \langle Z_{r}^{t,x,m}, dW_{r} \rangle.
\]

Step 2. To approximate the polynomial growth generator:

\[
f_{n}(s, x, y) = f(s, x, y)I_{\{|y|\leq n\}} + \partial_{y}f(s, x, \frac{n}{|y|}y)(y - \frac{n}{|y|}y)I_{\{|y|>n\}}.
\]

\[
f_{n}(s, x, y) \to f(s, x, y), \quad \text{as } n \to \infty.
\]

We need

(i) strongly convergent subsequence in \(L^{2}(\Omega \times [0, T]; L_{\rho}^{2}(\mathbb{R}^{d}; \mathbb{R}^{1}))\)

(ii) \(L^{p}(\Omega \times [0, T]; L_{\rho}^{p}(\mathbb{R}^{d}; \mathbb{R}^{1})), p \geq 1\), estimate.
Define \((U^{t,\cdot,n}, V^{t,\cdot,n}) \triangleq (f_n(r, X^{t,x}_r, Y^{t,x}_r), g_n(r, X^{t,x}_r, Y^{t,x}_r))\).

By Alaoglu lemma, a subsequence \((Y^{t,\cdot,n}, Z^{t,\cdot,n}, U^{t,\cdot,n}, V^{t,\cdot,n})\) converges weakly to a limit \((Y^{t,\cdot}, Z^{t,\cdot}, U^{t,\cdot}, V^{t,\cdot})\) in \(L^2(\Omega \times [t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1) \times L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d) \times L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1) \times L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^l))\).

\[
Y^{t,x}_s = h(X^{t,x}_{T_s}) + \int_s^T U^{t,x}_r dr - \int_s^T \langle V^{t,x}_r, d\hat{B}_r \rangle - \int_s^T \langle Z^{t,x}_r, dW_r \rangle.
\]

**Key:** finding a strongly convergent subsequence of \((Y^{t,\cdot,n}, Z^{t,\cdot,n})\) to get \((U^{t,x}_r, V^{t,x}_r) = (f(r, X^{t,x}_r, Y^{t,x}_r), g(r, X^{t,x}_r, Y^{t,x}_r))\).
The Correspondence between SPDEs and BDSDEs with Coefficients $f_n$

BDSDEs:

$$Y_{s}^{t,x,n} = h(X_{T}^{t,x}) + \int_{s}^{T} f_{n}(r, X_{r}^{t,x}, Y_{r}^{t,x,n}) dr$$

$$- \int_{s}^{T} \langle g(r, X_{r}^{t,x}, Y_{r}^{t,x,n}), d\dot{\hat{B}}_{r} \rangle - \int_{s}^{T} \langle Z_{r}^{t,x,n}, dW_{r} \rangle.$$

SPDEs:

$$u_{n}(t, x) = h(x) + \int_{t}^{T} \{ \mathcal{L} u_{n}(s, x) + f_{n}(s, x, u_{n}(s, x)) \} ds$$

$$- \int_{t}^{T} \langle g(s, x, u_{n}(s, x)), d\dot{\hat{B}}_{s} \rangle, \quad 0 \leq t \leq T.$$

Correspondence:

$$u_{n}(t, x) \triangleq Y_{t}^{t,x,n}, u_{n}(s, X_{s}^{t,x}) = Y_{s}^{t,x,n}, \ (\sigma \nabla u_{n})(s, X_{s}^{t,x}) = Z_{s}^{t,x,n}.$$
We apply Rellich-Kondrachov Compactness Theorem to approximating PDEs to derive a strongly convergent subsequence of $u_n$ in Zhang & Zhao 2012, JTP.

**Theorem**

Let $X \subset\subset H \subset Y$ be Banach spaces, with $X$ reflexive. Here $X \subset\subset H$ means $X$ is compactly embedded in $H$. Suppose that $u_n$ is a sequence which is uniformly bounded in $L^2([0,T]; X)$, and $d u_n/dt$ is uniformly bounded in $L^p([0,T]; Y)$, for some $p > 1$. Then there is a subsequence which converges strongly in $L^2([0,T]; H)$.

But this method does not work for the SPDE/BDSDE as Rellich-Kondrachov Compactness Theorem stands for PDEs and for fixed $\omega \in \Omega$ the subsequence choice may depend on $\omega$. 
Instead, we use **Sobolev-Wiener Compactness Theorem**, which is an extension of Rellich-Kondrachov compactness theorem to stochastic case with the help of Malliavin derivatives, proved in **Bally & Saussereau 2004, JFA**.

The time and space independent case was considered by **Da Prato & Malliavin & Nualart 1992** and **Peszat 1993**
Sobolev-Wiener Compactness Theorem

Let \((u_n)_{n \in \mathbb{N}}\) be a sequence in \(L^2([0, T] \times \Omega; H^1(\mathcal{O}))\). Define
\[ u^\varphi_n(s, \omega) \triangleq \int_\mathcal{O} u_n(s, x, \omega) \varphi(x) dx. \]
Suppose that

1. \[ \sup_n E[\int_0^T \| u_n(s, \cdot) \|_{H^1(\mathcal{O})}^2 ds] < \infty. \]
2. For all \( \varphi \in C^k_c(\mathcal{O}) \) and \( t \in [0, T] \), \( u^\varphi_n(s) \in \mathbb{D}^{1,2} \) and
   \[ \sup_n \int_0^T \| u^\varphi_n(s) \|_{\mathbb{D}^{1,2}}^2 ds < \infty. \]
3. For all \( \varphi \in C^k_c(\mathcal{O}) \), \( (E[u^\varphi_n])_{n \in \mathbb{N}} \) of \( L^2([0, T]) \) satisfies
   3i. For any \( \varepsilon > 0 \), there exists \( 0 < \alpha < \beta < T \) s.t.
   \[ \sup_n \int_{[0, T] \setminus (\alpha, \beta)} \| E[u^\varphi_n(s)] \|^2 ds < \varepsilon. \]
   3ii. For any \( 0 < \alpha < \beta < T \) and \( h \in \mathbb{R}^1 \) s.t. \( |h| < \min(\alpha, T - \beta) \),
   \[ \sup_n \int_{\alpha}^{\beta} \| E[u^\varphi_n(s + h)] - E[u^\varphi_n(s)] \|^2 ds < C_p \| h \|. \]
Sobolev-Wiener Compactness Theorem

(4) For all $\varphi \in C^k_c(O)$, the following conditions are satisfied:

(4i) For any $\varepsilon > 0$, $\exists \ 0 < \alpha < \beta < T$ and $0 < \alpha' < \beta' < T$ s.t.

$$\sup_n E\left[ \int_{[0,T]^2 \setminus (\alpha,\beta) \times (\alpha',\beta')} |D_\theta u_n^\varphi(s)|^2 d\theta ds \right] < \varepsilon.$$ 

(4ii) For any $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ and $h, h' \in \mathbb{R}^1$ s.t.

$max(|h|, |h'|) < min(\alpha, \alpha', T - \beta, T - \beta'),$

$$\sup_n E\left[ \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} |D_\theta + h u_n^\varphi(s + h') - D_\theta u_n^\varphi(s)|^2 d\theta ds \right] < C_p(|h| + |h'|).$$

Then $(u_n)_{n \in \mathbb{N}}$ is relatively compact in $L^2(\Omega \times [0, T] \times O; \mathbb{R}^1)$. 
Generalized Equivalence of Norm Principle

The generalized equivalence of norm principle (based on Barles & Lesigne 1997, Bally & Matoussi 2001, JTP) is used to establish the equivalence of norm between the solutions of terminal-value SPDEs and the solutions of BDSDEs. Consider stochastic flows:

$$X_{s,x}^t = x + \int_t^s b(X_{r,x}^t)dr + \int_t^s \sigma(X_{r,x}^t)dW_r \quad s \geq t,$$

where $b \in C^2_{l,b}(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$.

**Lemma**

If $s \in [t,T]$, $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^1$ is independent of $\mathcal{F}_{t,s}^W$ and $\varphi \rho^{-1} \in L^1(\Omega \times \mathbb{R}^d; \mathbb{R}^1)$, then $\exists c, C > 0$ s.t.

\[
cE \left[ \int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx \right] \leq E \left[ \int_{\mathbb{R}^d} |\varphi(X_{s,x}^t)| \rho^{-1}(x) dx \right] \leq CE \left[ \int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx \right].\]
A measurable space: \((V, \mathcal{B}(V))\).

A metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})\).

For a measurable random dynamical system
\[ v : \mathbb{R}^1 \times \mathbb{R}^1 \times V \times \Omega \rightarrow V, \]
the random periodic solution with period \(\tau > 0\) is an \(\mathcal{F}\) measurable r.v. \(Y : \mathbb{R}^1 \times \Omega \rightarrow V\) such that
\[ v^{t,Y(t,\omega)}(t + \tau, \omega) = Y(t + \tau, \omega) = Y(t, \theta_{\tau} \omega), \quad t \geq 0 \text{ a.s.} \]
Some Existing Results

- **Zhao & Zheng 2009, JDE** Random periodic solutions for $C^1$-cocycles

- **Feng & Zhao & Zhou 2011, JDE** Random periodic solutions of SDEs with additive noise

- **Feng & Zhao 2012, JFA** Random periodic solutions of SPDEs with additive noise
We study the following SDE valued in $\mathbb{R}^d$:

$$u^{t,\xi}(s) = \xi + \int_t^s \left[ -Au^{t,\xi}(r) + b(r, u^{t,\xi}(r)) \right] dr + \int_t^s \sigma(r, u^{t,\xi}(r)) dB_r.$$ 

$A$ is an invertible matrix satisfying

$$\delta \triangleq \inf \{ \text{Re}(\lambda) : (\lambda_i)_{i=1,...,d} \text{ are the eigenvalues of } A \} > 0;$$

$b : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$;

$b(t, x) = b(t + \tau, x)$, $\sigma(t, x) = \sigma(t + \tau, x)$. 
By Duhamel’s formula,

\[ u^{t,\xi}(s) = e^{-A(s-t)}\xi + \int_t^s e^{-A(s-r)}b(r, u^{t,\xi}(r))\,dr + \int_t^s e^{-A(s-r)}\sigma(r, u^{t,\xi}(r))\,dB_r. \]

Introduce the infinite horizon integral equation:

\[ X_s = \int_{-\infty}^s e^{-A(s-r)}b(r, X_r)\,dr + \int_{-\infty}^s e^{-A(s-r)}\sigma(r, X_r)\,dB_r. \]

Then

\[ X_s = e^{-A(s-t)}X_t + \int_t^s e^{-A(s-r)}b(r, X_r)\,dr + \int_t^s e^{-A(s-r)}\sigma(r, X_r)\,dB_r. \]
Random Periodic Property of $X_s$

If the original SDE admits a unique solution $u^{t,\xi}(s)$, then

$$u^{t,X_t}(s) = X_s.$$ 

$X_s$ is a random periodic solution of the original SDE if

$$X_{s+\tau}(\omega) = X_s(\theta_{\tau}\omega).$$

Recursive sequence (Qiao & Zhang & X. Zhang, Preprint):

$$X_{s+1} = \int_{-\infty}^{s} e^{-A(s-r)} b(r, X_r^n) dr + \int_{-\infty}^{s} e^{-A(s-r)} \sigma(r, X_r^n) dB_r.$$ 

By recursion, for all $n$,

$$X_{s+\tau}(\omega) = X_s(\theta_{\tau}\omega).$$
Theorem

Assume $b, \sigma, \nabla b, \nabla \sigma$ are bounded and

$$2 \| \nabla b \|_\infty^2 \delta^{-2} + 2 \| \nabla \sigma \|_\infty^2 (2\delta)^{-1} < 1.$$  

Then the infinite horizon integral equation has a unique solution $X_s$ which is a random periodic solution of SDE.

Sketch of Proof: 1. $X^n$ is a Cauchy sequence in $C(\mathbb{R}^1; L^2(\Omega))$.

2. Take $X$ such that

$$\lim_{n \to \infty} \sup_{s \in \mathbb{R}^1} E|X^n_s - X_s|^2 = 0.$$  

3. As $n \to \infty$, it appears that $X$ satisfies the infinite horizon integral equation and

$$X_{s+\tau}(\omega) = X_s(\theta_\tau \omega).$$
Thank You