Approximate Hedging and BSDEs with weak boundaries

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based on joint works with
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Motivation

- **Stock price**: (with large investor’s strategy $\pi$)
  \[
  \frac{dS^\pi_u}{S^\pi_u} = \mu(u, S^\pi_u, \pi_u) \, du + \sigma(u, S^\pi_u, \pi_u) \, dW_u
  \]

- **Wealth process**: (risk free interest rate $r = 0$)
  \[
  dX^\pi_u = \pi_u \frac{dS^\pi_u}{S^\pi_u} = \pi_u [\mu(u, S^\pi_u, \pi_u) \, du + \sigma(u, S^\pi_u, \pi_u) \, dW_u]
  \]

- **Super Hedging** problem of claim $h(S^\pi_T)$:
  \[
  \inf \left\{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } X^{0,x,\pi}_T \geq h(S^\pi_T) \, \mathbb{P} - \text{ps} \right\}.
  \]
  \[
  \implies \text{Prudential approach which leads to expensive prices}
  \]

- **Quantile Hedging** of the claim $h(S^\pi_T)$: Given $p \in (0, 1)$, find
  \[
  \inf \left\{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[ X^{t,x,\pi}_T \geq h(S^\pi_T) \right] \geq p \right\}.
  \]

How decreases the price when one accepts to keep some hedging risk?
Agenda

1. Dual approach of Föllmer and Leukert
2. A stochastic target approach
3. Non Markovian BSDE representation
Explicit solution in a complete market

- Restriction to a complete market (super-replication ⇔ replication)

- Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:

$$
\frac{dS_u}{S_u} = \sigma(u, S_u) \, dW_u \quad \text{(independent on } \pi)\n$$

- Wealth process:

$$
dX^\pi_u = \pi_u \sigma(u, S_u) \, dW_u
$$

- Dual problem reformulation:

Maximize the probability of hedge for a given starting wealth $x$

$$
\max_{\pi \in \mathcal{A}} \mathbb{P} \left[ X^{0,x,\pi}_T \geq h(S_T) \right]
$$
Föllmer and Leukert approach to quantile hedging

Maximize the probability of hedge for a given initial wealth $x$

\[
\max_{\pi \in \mathcal{A}} \mathbb{P}[X_{T}^{0,x,\pi} \geq h(S_{T})] 
\]

\[
\max \mathbb{P}[X \geq h(S_{T})] \quad \text{under the constraint} \quad \mathbb{E}^{Q}[X] \leq x
\]

$A = \{X \geq h(S_{T})\}$

\[
\max_{A \in \mathcal{F}_{T}} \mathbb{P}[A] \quad \text{under the constraint} \quad \mathbb{E}^{Q}[h(S_{T})1_{A}] \leq x
\]

\[
dQ^{h} := \frac{h(S_{T})}{\mathbb{E}^{Q}[h(S_{T})]}dQ
\]

\[
\max_{A \in \mathcal{F}_{T}} \mathbb{P}[A] \quad \text{under the constraint} \quad \mathbb{Q}^{h}[A] \leq \frac{x}{\mathbb{E}^{Q}[h(S_{T})]} ,
\]

$A$ interprets as the critical region while testing $\mathbb{Q}^{h}$ against $\mathbb{P}$.

Neyman-Pearson lemma $\implies$ optimal critical region $A^{*}(x)$

Optimal strategy $\pi^{*}(x)$ : the one which replicates $h(S_{T})1_{A^{*}(x)}$

Quantile replication price : $x^{*}(p)$ such that $\mathbb{P}[A^{*}(x^{*}(p))] = p$
Solution in General Case

- **Pros:**
  - Explicit solution in some simple (but important) cases.
  - Generic solution of the form: \( X_T^{0,x,\pi} = h(S_T) 1_A \)
  - Similar structure in incomplete markets.

- **Cons:**
  - Resolution of the dual problem
  - Explicit solution not known in general (numerics)
  - In incomplete markets, the dual problem is a control problem: how to solve it?
  - Relies heavily on the duality between super-hedgeable claims and risk neutral measures.

\[ \rightarrow \text{Alternative dynamic approach} \]
The particular case of super-hedging

- The super hedging price at time 0

\[ \inf \{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} [ X_{T}^{0,x,\pi} \geq h(S_{T}^{\pi}) ] = 1 \} \]

- Dynamic version of the super-hedging problem

\[ v(t, s, 1) = \inf \{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} [ X_{T}^{t,x,\pi} \geq h(S_{T}^{t,s,\pi}) ] = 1 \} \]

- Dual approach:  
  \[ v(t, s; 1) = \sup_{Q} \mathbb{E}^{Q} [ h(S_{T}^{t,s}) ] \]

- Direct approach of Soner and Touzi:
  - (DP1): \( x > v(t, s, 1) \Rightarrow \exists \pi \in \mathcal{A} \text{ s.t. for all stopping time } \tau \leq T \)
    \[ X_{T}^{t,x,\pi} \geq v(\tau, S_{T}^{t,s,\pi}, 1) \]
  - (DP2): \( x < v(t, s, 1) \Rightarrow \text{for all stopping time } \tau \leq T \text{ and } \pi \in \mathcal{A} \)
    \[ \mathbb{P} [ X_{T}^{t,x,\pi} > v(\tau, S_{T}^{t,s,\pi}, 1) ] < 1 \]

\( \Rightarrow \text{Allows to derive PDEs associated to } v(\cdot, 1). \)
A stochastic target approach to quantile hedging

- The quantile hedging price at time 0

\[ \inf \{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} [ X_T^{0,x,\pi} \geq h(S_T^\pi) ] \geq p \} \]

- Dynamic version of the super-hedging problem

\[ v(t, s, p) = \inf \{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} [ X_T^{t,x,\pi} \geq h(S_T^{t,s,\pi}) ] \geq p \} \]

- Non consistent dynamic problem

- Idea: consider the "probability of super-hedging" as a process \((P_s)_{s \leq t \leq T}\)

- This process must be a martingale and therefore of the form

\[ P_s^{t,p,\alpha} = p + \int_t^s \alpha_u dW_u , \quad t \leq s \leq T , \quad \text{with } \alpha \in L^2 \]

- The quantile hedging price rewrites

\[ v(t, s, p) = \inf \left\{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ and } \alpha \in L^2 \text{ s.t. } 1_{X_T^{t,x,\pi} \geq h(S_T^{t,s,\pi})} \geq P_T^{t,p,\alpha} \right\} \]
Dynamic programming for quantile replication

- **Dynamic** version of the quantile hedging price:
\[
\nu(t, s, p) := \inf \left\{ x \in \mathbb{R} , \quad \exists (\pi, \alpha) \in A \times L^2 \quad \text{s.t.} \quad 1_{X_T^{t, x, \pi} \geq h(S_T^{t, s, \pi})} \geq P^t_{\tau, P^t_{\alpha}} \right\}
\]

- Dynamic programming principle:

  (DP1) : Starting with a wealth at time \(t\) greater than \(\nu(t, s, p)\), one can at any time \(\tau \geq t\) be able to \((P_{\tau})\)-quantile replicate:

\[
x > \nu(t, s, p) \Rightarrow \exists (\pi^*, \alpha^*) \quad \text{s.t.} \quad X_T^{t, x, \pi^*} \geq \nu(\tau, S_T^{t, s, \pi^*}, P^t_{\tau, P^t_{\alpha^*}}) , \quad \forall \tau \in [t, T]
\]

  (DP2) : Starting with a wealth at time \(t\) lower than \(\nu(t, s, p)\), it is impossible to quantile replicate:

\[
x < \nu(t, s, p) \Rightarrow \forall (\pi, \alpha) \quad P \left[ X_T^{t, x, \pi} > \nu(\tau, S_T^{t, s, \pi}, P^t_{\tau, P^t_{\alpha}}) \right] < 1 , \quad \forall \tau \in [t, T]
\]
Formal derivation of the Hamilton Jacobi Bellman equation

- Portfolio dynamics: 
  \[ dX^\pi_r = \mu(r, S^\pi_r, \pi_r) \pi_r dr + \sigma(r, S^\pi_r, \pi_r) \pi_r dW_r \]

- Dynamics of \( v(., S^\pi_r, P^\alpha_r) \):
  \[
  dv(r, S^\pi_r, P^\alpha_r) = \left[ v_t + \mu S^\pi_r v_x + \frac{\sigma^2 S^\pi_r}{2} v_{xx} + \frac{\alpha^2}{2} v_{pp} + 2\alpha \sigma S^\pi_r v_{xp} \right] (r, S^\pi_r, P^\alpha_r) dr \\
  + \left[ \sigma S^\pi_r v_x + \alpha_r v_p \right] (r, S^\pi_r, P^\alpha_r) dW_r
  \]

- Take \( x \sim v(t, s, p) \):

  \[
  (DP1) \quad \Rightarrow \quad \exists (\bar{\pi}^*, \bar{\alpha}^*) \text{ s.t. } X^{t,x,\bar{\pi}^*,\bar{\alpha}^*}_t \geq v(t, S^t_{\bar{\pi}^*,\bar{\alpha}^*}) \text{, } \forall \tau \in [t, T] \\
  (DP2) \quad \Rightarrow \quad \forall (\pi, \alpha) \quad \mathbb{P} \left[ X^{t,x,\pi}_t > v(t, S^t_{\pi}, P^t_{\pi,\alpha}) \right] < 1, \quad \forall \tau \in [t, T]
  \]

- Formally, we deduce the following HJB equation

  \[
  \sup_{(\alpha, \pi)} \mu \pi - \left[ v_t + \mu S v_s + \frac{\sigma^2 S}{2} v_{ss} + \frac{\alpha^2}{2} v_{pp} + 2\alpha \sigma S v_{sp} \right] (t, s, p) = 0
  \]

  under the constraint \( \sigma \pi = [\sigma s v_s + \alpha v_p](t, s, p) \)
Rigorous derivation

• PDE dynamics in the domain :

$$\sup_{(\alpha, \pi)} \mu \pi - \left[ v_t + \mu s v_s + \frac{\sigma^2 s}{2} v_{ss} + \frac{\alpha^2}{2} v_{pp} + 2\alpha \sigma s v_{sp} \right] (t, s, p) = 0$$

under the constraint $$\sigma \pi = [\sigma s v_s + \alpha v_p](t, s, p)$$

• Main technical difficulty : the auxiliary control $$\alpha$$ is not bounded.

• The auxiliary control $$\alpha$$ is directly related to the primal control $$\pi$$.

• Boundary conditions :

at $$p = 0+$$: $$v(t, s, 0) = 0$$

at $$p = 1-$$: $$v(t, s, 1)$$ is the super-replication price

at $$t = T-$$: $$v(T, s, p) = pg(s)$$

• Possible numerical approximation of the solution via PDE scheme
Explicit resolution in the Black Scholes model

- PDE in the Black Scholes model:
  \[
  v_t + \frac{\sigma^2 s^2}{2} v_{ss} - \frac{\sigma^2 s^2}{2} \left| v_{sp} \right|^2 - \frac{\mu^2}{2\sigma^2} \frac{v_p^2}{v_{pp}} + \mu S \frac{v_p v_{sp}}{v_{pp}} = 0 \quad \text{with} \quad v(T, s, p) = pg(s)
  \]

- Introduction of the Fenchel-Legendre transform \( \tilde{v}(t, s, \cdot) \) of \( v(t, s, \cdot) \):
  \[
  \tilde{v}(t, s, y) := \sup_{p \in [0,1]} pq - v(t, s, p)
  \]

- The Fenchel Legendre transform \( \tilde{v} \) "solves" the following linear PDE:
  \[
  \tilde{v}_t + \frac{\sigma^2 s^2}{2} \tilde{v}_{ss} + \mu s q \tilde{v}_{sq} + \frac{\mu^2}{2\sigma^2} q^2 \tilde{v}_{qq} = 0 \quad \text{with} \quad \tilde{v}(T, s, q) = (q - g(s))^+
  \]

- We deduce the probabilistic representation:
  \[
  \tilde{v}(t, s, q) = \mathbb{E}[(Q^{t,q}_T - h(S^{t,s}_T))^+] \quad \text{with} \quad Q^{t,q}_t := q + \int_t^T \frac{\mu}{\sigma} Q^{t,q}_{s} dW_s
  \]

- We retrieve \( v \) by re-applying the Fenchel transform.
Extensioins

- **On the Dynamics:**
  \[
  S^\pi = s + \int_t^T \mu(S^\pi_u, \pi_u) \, du + \int_t^T \sigma(S^\pi_u, \pi_u) \, dW_u \\
  X^\pi = x + \int_t^T \rho(S^\pi_u, X^\pi_u, \pi_u) \, du + \int_t^T \beta(S^\pi_u, X^\pi_u, \pi_u) \, dW_u
  \]

- **On the Problems:** Given \( \ell : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) and \( p \in \text{Im}(\ell) \),
  \[
  \nu(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[ \ell(S^{t,s,\pi}_T, X^{t,x,\pi}_T) \right] \geq p \right\}
  \]

- **Possible range of applications**
  \[
  \ell(s, x) = 1 \{x \geq g(s)\} \quad \Rightarrow \quad \text{Quantile Hedging}
  \]
  \[
  \ell(s, x) = U([x - g(s)]^+) \text{ with } U \nearrow \text{ concave} \quad \Rightarrow \quad \text{Loss function}
  \]
  \[
  \ell(s, x) = U(x - g(s)) \text{ with } U \searrow \text{ concave} \quad \Rightarrow \quad \text{Indifference pricing}
  \]

- **Dynamic programming** based on the reformulation
  \[
  \nu(t, s; p) = \inf \left\{ x \in \mathbb{R}_+ : \exists (\pi, \alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell(S^{t,s,\pi}_T, X^{t,x,\pi}_T) \geq P^{t,p,\alpha}_T \right\}
  \]
Utility maximization under quantile hedging type constraint:
\[ \implies \text{PDE characterization but no numerics (at that point)} \]

Combination of several constraints:
Given \( \ell_1, \ell_2, \ldots, \ell_m \) and \( p_i \in \text{Im}(\ell_i) \) for \( i \leq m \),
\[ \nu(t, s; p) := \inf \{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}[\ell_i(S_{T}^{t,s}, X_{T}^{t,x,\pi})] \geq p_i, \forall i \leq m \} \]
\[ \implies \text{leads to high dimensional PDE, impossible to solve numerically} \]

Robust quantile hedging under model uncertainty
Given a class of model \((\mathbb{P}^\lambda)_\lambda\), try to quantile hedge in any model
\[ \inf \{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}^{\mathbb{P}^\lambda}[\ell(\lambda, S_{T}^{t,s}, X_{T}^{t,x,\pi})] \geq p_\lambda, \forall \lambda \} . \]
\[ \implies \text{consider dynamic games} \]

One day ahead constraint:
Given a time delay \( \delta > 0 \), try to find
\[ \inf \{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}_s[\ell(X_{s+\delta}^{t,x,\pi})] \geq p, \forall s \leq T \} . \]
\[ \implies \text{hard to get a dynamic programming principle} \]
Consideration of non markovian terminal claim $\xi$.

In a complete market, the replication price identifies as the solution of the BSDE (with no driver)

$$Y_t = Y_T - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T$$

with $Y_T = \xi$

$\implies$ Y price process and Z investment strategy (up to the volatility)

In case of imperfections (e.g. portfolio constraints), the super-replication price of $\xi$ identifies to the minimal solution to the BSDE

$$Y_t = Y_T - \int_t^T Z_s dW_s + \int_t^T dL_s, \quad 0 \leq t \leq T$$

with $Y_T \geq \xi$

where $L$ is an increasing process.

For the quantile replication price, we expect

$$Y_T \geq \xi$$

to be replaced by

$$\mathbb{P}(Y_T \geq \xi) \geq p$$
Hence, this formally leads to a (no driver) new type of BSDE of the form

\[ Y_t = Y_T - \int_t^T Z_s dW_s + \int_t^T dL_s \quad 0 \leq t \leq T \quad \text{with} \quad \mathbb{P}(Y_T \geq \xi) \geq p \]

More generally, for an increasing loss function \( \ell \), we get

\[ dY_t = Z_t dW_t - dL_t \quad \text{with} \quad \mathbb{E}[\ell(Y_T - \xi)] \geq p \]

For a random increasing function \( \psi \), we look towards the minimal solution to the new type of BSDE

\[ dY_t = -g(t, Y_t, Z_t)dt + Z_t dW_t - dL_t \quad \text{with} \quad \mathbb{E}[\psi(Y_T)] \geq p \]

Constraint on the terminal condition distribution

\[ \Rightarrow "\text{BSDE with weak terminal condition}" \]
For a random increasing function $\psi$ and Lipschitz driver $g$, we look towards the minimal solution to the BSDE

$$dY_t = -g(t, Y_t, Z_t)dt + Z_t dW_t - dL_t,$$

with $\mathbb{E}[\psi(Y_T)] \geq p$

- Introduction of a supplementary control $\alpha \in L^2$ and $P_{p,\alpha} := p + \int_0^T \alpha_s dW_s$

- Set of all possible terminal conditions: $(\psi^{-1}(P_{T,\alpha}))_{\alpha \in L^2}$

- We suppose for simplicity $\psi : [0, 1] \to [0, 1]$

- Let $(Y^\alpha, Z^\alpha)_{\alpha \in L^2}$ be the set of solutions to the classical BSDEs

$$dY_t^\alpha = -g(t, Y_t^\alpha, Z_t^\alpha)dt + Z_t^\alpha dW_t,$$

with $Y_T^\alpha = \psi^{-1}(P_{T,\alpha})$

- At any time $t$, we can rewrite $Y_t^\alpha = \mathcal{E}_{t, T}^g [\psi^{-1}(P_{T,\alpha})]$
Representation of the solution

- For $\alpha \in L^2$, $(Y^\alpha = E^g_{.,T} [\psi^{-1}(P^p_{T,\alpha})], Z^\alpha)$ solves the classical BSDE

$$dY^\alpha_t = -g(t, Y^\alpha_t, Z^\alpha_t) dt + Z^\alpha_t dW_t,$$

with $Y^\alpha_T = \psi^{-1}(P^p_{T,\alpha})$

- Any $Y$-component of a super-solution to the BSDE with weak terminal condition is of the form $Y^\alpha$.

- For any path $\alpha$, in order to pay the cheapest price, we define:

$$\bar{Y}^\alpha_t := \text{essinf} \left\{ E^g_{t, T} [\psi^{-1}(P^p_{T,\alpha'})], \alpha' \in L^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0, t] \right\}, \quad \forall t$$

- Obtention of Dynamic Programming Principle for the family $(\bar{Y}^\alpha)_\alpha$

$$\bar{Y}^\alpha_t = \text{essinf} \left\{ E^g_{t, t'} [\bar{Y}^\alpha_{t'}], \alpha' \in L^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0, t] \right\}, \quad 0 \leq t \leq t' \leq T.$$

- $\bar{Y}^\alpha$ is indistinguishable from a ladlag $g$-submartingale
Representation of the solution

- For $\alpha \in L^2$, $(Y^\alpha = \mathcal{E}_{\cdot,T}^g [\psi^{-1}(P^p_T,\alpha)] , Z^\alpha)$ solves the classical BSDE
  \[ dY^\alpha_t = -g(t, Y^\alpha_t, Z^\alpha_t)dt + Z^\alpha_t dW_t , \quad \text{with} \quad Y^\alpha_T = \psi^{-1}(P^p_T,\alpha) \]

- For any path $\alpha$, in order to pay the cheapest price, we define :
  \[ \bar{Y}^\alpha_t := \text{essinf} \left\{ \mathcal{E}_{t,T}^g [\psi^{-1}(P^p_T,\alpha')] , \alpha' \in L^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0, t] \right\} , \quad \forall t \]

- Additional assumption : $\psi^{-1}(\omega,.)$ is continuous for $\mathbb{P}$-a.e $\omega$

  $\implies$ $\bar{Y}^\alpha$ is indistinguishable from a cadlag $g$-submartingale

- Characterization of the family $(\bar{Y}^\alpha)_{\alpha \in L^2}$ of solutions :

  \begin{align*}
  \text{Dynamics} : \quad \bar{Y}^\alpha_t &= \psi^{-1}(P^p_T,\alpha) + \int_0^T g(s, \bar{Y}^\alpha_s, \bar{Z}^\alpha_s)ds - \int_0^T \bar{Z}^\alpha_s dW_s + \bar{L}^\alpha_T - \bar{L}^\alpha_T \text{ on } [0, T] \\
  \text{Minimality} : \quad \bar{L}^\alpha_{\tau_1} &= \text{essinf} \left\{ E \left[ \bar{L}^\alpha'_{\tau_2} | \mathcal{F}_{\tau_1} \right] , \alpha' \in L^2 \text{ s.t. } \alpha' = \alpha \text{ on } [0, \tau_1] \right\} , \quad \forall \tau_1 \leq \tau_2 \\
  \text{Futur indep.} : \quad \alpha' = \alpha \text{ on } [0, \tau] \quad \implies \quad (\bar{Y}^\alpha', \bar{Z}^\alpha', \bar{L}^\alpha')_{[0,\tau]} = (\bar{Y}^\alpha, \bar{Z}^\alpha, \bar{L}^\alpha)_{[0,\tau]} .
  \end{align*}
Regularity at time $t$ of $P^\alpha_t \mapsto \tilde{Y}^\alpha_t$?

Introduction of a modulus of continuity:

$Err_t(\eta) := \text{ess sup}\{ |\mathcal{E}_{t,T}^g[\psi^{-1}(M)] - \mathcal{E}_{t,T}^g[\psi^{-1}(M')]|, M, M' \text{ s.t. } E_t[|M - M'|^2] \leq \eta \}$

For any $t < T$, we get

$$|\tilde{Y}^\alpha_t - \tilde{Y}^\alpha_t'| \leq Err_t(\Delta(P^\alpha_t, P^\alpha_t')) + Err_t(\Delta(P^\alpha_t', P^\alpha_t)),$$

where

$$\Delta : (\mu_1, \mu_2) \mapsto \frac{\mu_2 - \mu_1}{\mu_2} 1_{\{\mu_1 < \mu_2\}} + \frac{\mu_1 - \mu_2}{1 - \mu_1} 1_{\{\mu_1 > \mu_2\}}$$

Similar properties on $\{P^\alpha_t = 0\}$ or $\{P^\alpha_t = 1\}$.

For a Lipschitz map $\psi^{-1}$, stability results on classical BSDEs

$\Rightarrow \tilde{Y}^\alpha_t$ is $L^2$-continuous with respect to $P^\alpha_t$. 
Convexity of the solution

- Whenever $g(., .)$ and $\psi^{-1}$ are convex, there exists $\hat{\alpha}$ such that $\bar{Y}^{\hat{\alpha}} = Y^{\hat{\alpha}}$
  \(\implies\) a BSDE with weak terminal condition boils down to a classical BSDE

- For any $t < T$, the solution $\bar{Y}_t^\alpha$ is $\mathcal{F}_t$-convex with respect to $P_t^\alpha$.
  (need to consider the l.s.c. envelope of the solution)

- Probabilistic proof of the property.

- "Facelift"
  \(\implies\) if $\psi$ deterministic, one can replace $\psi^{-1}$ by its convex envelope
  \(\implies\) similar solutions on $[0, T)$

- In a Markovian framework, natural link with the previous PDEs.
Duality for the solution

- Suppose that $g$ and $\psi^{-1}$ are convex + technical conditions
- Introduce $\tilde{g}$ the Fenchel transform of $g$ w.r.t. $(y, z)$.
- Introduce $\tilde{\psi}^{-1}$ the Fenchel transform of $\psi^{-1}$ w.r.t. $p$.
- Consider the following dual control problem:

$$\tilde{Y}_0(\ell) := \inf_{(\nu, \lambda) \in \text{Dom}(\tilde{g})} E \left[ \int_0^T L_{s, \lambda}^{\nu} \tilde{g}(s, \nu_s, \lambda_s) ds + L_T^{\nu, \lambda} \tilde{\psi}^{-1}(\ell / L_T^{\nu, \lambda}) \right]$$

where $L_t^{\nu, \lambda} = 1 + \int_0^t L_s^{\nu, \lambda}(\nu_s ds + \lambda_s dW_s)$

- We have the following correspondence

$$\tilde{Y}_0(p) = \sup_{\ell > 0} (p \ell - \tilde{Y}_0(\ell)) \quad \text{and} \quad \tilde{Y}_0(\ell) = \sup_{p > 0} (p \ell - \tilde{Y}_0(p))$$

- Standard explicit relation between the optimizers
Formal link with second order BSDEs

- Particular case of deterministic coefficients and driver independent of $z$

- For $\alpha \in L^2$ (with $\alpha > 0$), recall that $(Y^\alpha, Z^\alpha)$ is solution to the classical BSDE with driver $g$ and terminal condition $\psi^{-1}(P^\alpha_T)$

- Denoting $B^\alpha := \int_0^\cdot \alpha_s dW_s$; $\hat{Y}^\alpha := -Y^\alpha$ and $\hat{Z}^\alpha := -Z^\alpha / \alpha$, we get

\[
\hat{Y}^\alpha = \psi^{-1}(p + B^\alpha_T) + \int_0^T -g(s, -\hat{Y}_s^\alpha) ds - \int_0^T \hat{Z}_s^\alpha dB_s^\alpha, \quad P - \text{a.s.}
\]

- $B^\alpha$ behaves under the canonical meas. $P^o$ as $B$ under the pullback one $P^\alpha$

\[\implies \hat{Y}^\alpha \text{ under } P^o \text{ looks like } \hat{Y}^{P^\alpha} \text{ under } P^\alpha \text{ where } (\hat{Y}^{P^\alpha}, \hat{Z}^{P^\alpha}) \text{ solves}
\]

\[
\hat{Y}^{P^\alpha} = \psi^{-1}(p + B_T) + \int_0^T -g(s, -\hat{Y}^{P^\alpha}_s) ds - \int_0^T \hat{Z}^{P^\alpha}_s dB_s, \quad P^\alpha - \text{a.s.}
\]

- Therefore, we get: $-\bar{Y}_0 = \text{ess sup}_\alpha \hat{Y}^\alpha_0 = \text{ess sup}_\alpha \hat{Y}^{P^\alpha}_0$.

\[\implies \text{Link with 2BSDE solution but no aggregation procedure.}\]
One possible extension: BSDE with mean reflexion

- Consider a **time running constraint** on the distribution of $Y$:
  \[
  \mathbb{E}[\psi(Y_t)] \geq 0, \quad 0 \leq t \leq T.
  \]
- For any date $t$, the reflection is related to the **law of** $Y_t$
- Consider the BSDE dynamics
  \[
  Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dB_s + K_T - K_t
  \]
  with the previous constraint and the analogous **new Skorokhod condition**:
  \[
  \int_0^T \mathbb{E}[\psi(Y_t)] dK_t = 0.
  \]
- **Dynamically non consistent problem** but we derive the well posed-ness of the BSDE
BSDE with mean reflexion

\[ Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s + K_T - K_t \]

\[ \mathbb{E}[\psi(Y_t)] \geq 0, \quad 0 \leq t \leq T \]

\[ \int_0^T \mathbb{E}[\psi(Y_t)]dK_t = 0. \]

- First observation: **K must be deterministic** or there is no continuous minimal solution.

  The classical penalization procedure is a priori non monotonic.

- **Existence and uniqueness** of the solution under the bi-Lipschitz condition:

  \[ c_1 |x - y| \leq |h(x) - h(y)| \leq c_2 |x - y| \]

- Use of a fixed point argumentation.

- The Skorokhod condition implies the **minimality** of the solution (at least when the driver does not depend on \( Y \)).
Question: BSDE with mean reflexion $\leftrightarrow$ mean field BSDE?

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \cdot dB_s + K_T - K_t
\]

\[\mathbb{E}[\psi(Y_t)] \geq 0, \quad 0 \leq t \leq T\]

\[\int_0^T \mathbb{E}[\psi(Y_t)] \, dK_t = 0.\]

- Can we approximate the solution of this mean-field reflected BSDE by a reflected BSDEs?

- If $B = (B^1, \ldots, B^N)$ are independent BM, can we solve the coupled system?

\[
Y_{t,i}^{i,N} = \xi + \int_t^T g(s, Y_{s,i}^{i,N}, Z_{s,i}^{i,N}) \, ds - \int_t^T Z_{s,i}^{i,N} \cdot dB_s + K_{T,i}^{i,N} - K_{t,i}^{i,N}
\]

with \[\frac{1}{N} \sum_{i=1}^N \psi(Y_{t,i}^{i,N}) \geq 0\]

- What are the asymptotics when $N \to \infty$?
other possible extensions...

...which are on tracks:

- Addition of jumps
- Consideration of a constraint in non linear expectation
- Consideration of weak reflections in a dynamically consistent manner:
  \[ \mathbb{E} \psi(Y_{\tau}) \geq 0, \quad \text{for any stopping time } \tau \leq T. \]

... which should be reasonable:

- Extension to a quadratic driver
- BSDE for utility maximization with quantile hedging constraint

... which seem more challenging:

- Case of coupled FBSDE for insider models
- BSDE for quantile hedging under portfolio constraints
- Consideration of one day ahead constraints:
  \[ \mathbb{E}_t \psi(Y_{t+\delta}) \geq 0. \]
- 2BSDE with weak terminal condition for robust quantile hedging
- Numerics for BSDE with weak terminal condition?