QUIVER ALGEBRAS, PATH COALGEBRAS AND CO-REFLEXIVITY

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Abstract. We study the connection between two combinatorial notions associated to a quiver: the quiver algebra and the path coalgebra. We show that the quiver coalgebra can be recovered from the quiver algebra as a certain type of finite dual, and we show precisely when the path coalgebra is the classical finite dual of the quiver algebra, and when all finite dimensional quiver representations arise as comodules over the path coalgebra. We also study connections to the notion of coreflexive (co)algebras, and give an application to our considerations to a partial answer to an open problem concerning tensor products of coreflexive coalgebras.

1. Introduction and Preliminaries

Let \( K \) be a field and \( \Gamma \) be a quiver. The associated quiver algebra \( K[\Gamma] \) is an important object studied extensively in representation theory, and one theme in the field is to relate and understand combinatorial properties of the quiver via the properties of the category of representations of the quiver, and vice versa. Quiver algebras also play a role in general representation theory of algebras; for example, every finite dimensional pointed algebra is a quiver algebra "with relations". A closely related object is the path coalgebra \( K\Gamma \), introduced in [2], together its comodules (quiver corepresentations). Comodules over path coalgebras turn out to form a special kind of representations of the quiver, called locally nilpotent representations in [1]. A natural question arises of what is the precise connection between the two objects \( K[\Gamma] \) and \( K\Gamma \).

We aim to provide such connections, by finding out when one of these objects can be recovered from the other one. This is also important from the following viewpoint: one can ask when the finite dimensional locally nilpotent representations of the quiver (i.e. quiver corepresentations), provide all the finite dimensional quiver representations. This situation will be exactly the one in which the path coalgebra is recovered from the quiver algebra as a by a certain natural construction involving representative functions, which we recall below.

Given an coalgebra \( C \), its dual \( C^\ast \) is always an algebra. Given an algebra \( A \), one can associate a certain subspace \( A^0 \) of the dual \( A^\ast \), which has a coalgebra structure. This is called the finite dual of \( A \), and it has the following interpretation. Consider the finite dimensional representations (modules) of \( A \), and let \( \mathcal{C} \) be the category of all rational left \( A \)-modules, i.e. the category generated by the finite dimensional representations, equivalently, modules which are sums of their finite dimensional ones. These form a category which is equivalent to the category of right comodules over the coalgebra \( A^0 \). This coalgebra is sometimes also called the coalgebra of representative functions, and consists of all \( f : A \to K \) whose kernel contains a cofinite ideal. We show that the path coalgebra \( K\Gamma \) can be reconstructed from the quiver algebra \( K[\Gamma] \) as a certain type of "graded" finite dual, that is, \( K\Gamma \) embeds in the dual space \( K[\Gamma]^\ast \) as the subspace of linear functions \( f : K[\Gamma] \to K \) whose kernel contains a cofinite monomial ideal. This is an "elementwise" answer to the recovery problem. In order to give also a categorical answer, we first note that in general the quiver algebra does not have identity, but it has enough idempotents. Therefore, to answer the question, we first extend the construction of the finite dual to algebras with enough idempotents (Section 2). To such an algebra \( A \) we associate a coalgebra with counit \( A^0 \). In Section 3 we show that the path coalgebra \( K\Gamma \) embeds in \( K[\Gamma]^0 \), and we prove that this embedding is an isomorphism, i.e. the path coalgebra can be recovered as the finite dual of the

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quiver algebra, if and only if the quiver has no cycles and there are finitely many arrows between any two vertices. On the other hand, $K[\Gamma]$ embeds as an algebra without identity in the dual algebra $(KT)^*$ of the path coalgebra. We show that the image of this embedding is the rational (left or right) part of $(KT)^*$, i.e. the quiver algebra can be recovered as the rational part of the dual of the path coalgebra, if and only if for any vertex $v$ of $\Gamma$ there are finitely many paths starting at $v$ and finitely many paths ending at $v$. This is also equivalent to the fact that $KT$ is a left and right semiperfect coalgebra.

It is also interesting to know when can $KT$ be recovered from $(KT)^*$, and how this relates to the results of Section 3. This problem is related to an important notion in coalgebra theory, that of coreflexive coalgebra. A coalgebra $C$ over $K$ is coreflexive if the natural coalgebra embedding $C \rightarrow (C^*)0$ is an isomorphism. In other words, $C$ is coreflexive if it can be completely recovered from its dual. In Section 4 we aim to study this condition for path coalgebras and their subcoalgebras, and give the connection with the results of Section 3. We show that, in fact, a path coalgebra of a quiver with no loops and finitely many arrows between any two vertices is not necessarily coreflexive, and also, that the quivers of coreflexive path coalgebras can contain loops. We then prove a general result stating that under certain conditions a coalgebra $C$ is coreflexive if and only if its coradical is coreflexive. In particular, this result holds for subcoalgebras of a path coalgebra $KT$ with the property that there are finitely many paths between any two vertices of $\Gamma$. A particular class of coalgebras to which this result applies consists of incidence coalgebras, which are also objects of a great combinatorial interest, see for instance [6]. For both a path coalgebra and an incidence coalgebra the coradical is a grouplike coalgebra (over the set of vertices of the quiver for the first one, or the underlying ordered set for the second one). Thus the coreflexivity of such a coalgebra reduces to the coreflexivity of a grouplike coalgebra $K^X$. By the results of Heynemann and Radford [5, Theorem 3.7.3], if $K$ is an infinite field, it is reasonable to expect that $K^X$ is always coreflexive. More precisely, an ultrafilter $F$ on a set $X$ is called an Ulam ultrafilter if $F$ is closed under countable intersection. $X$ is called reasonable if every Ulam ultrafilter is principal (i.e. it equals the collection of all subsets of $X$ containing some fixed $x \in X$). The class of reasonable sets contains the countable sets and is closed under usual set-theoretic constructions, such as the power set, products and unions. Thus if a non-reasonable set exists, its cardinal has to be ”very large” (i.e. inaccessible in the sense of set theory).

We apply our results and as a consequence, we give some partial answers to a question of E.Taft and D.Radford asking whether the tensor product of two coreflexive coalgebras is coreflexive. In particular, we show that tensor products of coreflexive pointed coalgebras which embed in path coalgebras of quivers with only finitely many paths between any two vertices is coreflexive.

Throughout the paper $\Gamma = (\Gamma_0, \Gamma_1)$ will be a quiver. $\Gamma_0$ is the set of vertices, and $\Gamma_1$ is the set of arrows of $\Gamma$. If $a$ is an arrow from the vertex $u$ to the vertex $v$, we denote $s(a) = u$ and $t(a) = v$. A path in $\Gamma$ is a finite sequence of arrows $p = a_1a_2 \ldots a_n$, where $n \geq 1$, such that $t(a_i) = s(a_{i+1})$ for any $1 \leq i \leq n - 1$. We will write $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Also the length of such a $p$ is length$(p) = n$. Vertices $v$ in $\Gamma_0$ are also considered as paths of length zero, and we write $s(v) = t(v) = v$. If $q$ and $p$ are two paths such that $t(q) = s(p)$, we consider the path $qp$ by taking the arrows of $q$ followed by the arrows of $p$. We denote by $KT$ the path coalgebra, which is the vector space with a basis consisting of all paths in $\Gamma$, and comultiplication $\Delta$ defined by $\Delta(p) = \sum_{qr=p} q \otimes r$ for any path $p$, and counit $\epsilon$ defined by $\epsilon(v) = 1$ for any vertex $v$, and $\epsilon(p) = 0$ for any path of positive length. The underlying space of $KT$ can be also endowed with a structure of an algebra, not necessarily with identity, with the multiplication defined such that the product of two paths $p$ and $q$ is $pq$ if $t(p) = s(q)$, and 0 otherwise. This algebra is denoted by $K[\Gamma]$, and it is called the quiver algebra. It has identity if and only if $\Gamma_0$ is finite, and in this case the sum of all vertices is the identity.

Let $(X, \leq)$ be a partially ordered set which is locally finite, i.e. the set $\{z | x \leq z \leq y\}$ is finite for any $x \leq y$ in $X$. The incidence coalgebra of $X$, denoted by $KX$, is the vector space with basis
\{e_{x,y} | x, y \in X, x \leq y\}, and comultiplication and counit defined by $\Delta(e_{x,y}) = \sum_{x \leq z \leq y} e_{x,z} \otimes e_{z,y}$, $\epsilon(e_{x,y}) = \delta_{x,y}$ for any $x, y \in X$ with $x \leq y$. For such an $X$, we can consider the quiver $\Gamma$ with vertices the elements of $X$, and such that there is an arrow from $x$ to $y$ if and only if $x < y$ and there is no element $z$ with $x < z < y$. It was proved in [4] that the linear map $\phi : KX \to K\Gamma$, defined by

$$\phi(e_{x,y}) = \sum_{p \text{ path } x \in (p) = x, t(p) = y} p$$

for any $x, y \in X, x \leq y$, is an injective coalgebra morphism.

For basic terminology and notation about coalgebras and comodules we refer to [10], [7] and [3].

2. The finite dual of an algebra with enough idempotents

In this section we extend the construction of the finite dual of an algebra with identity to the case where $A$ does not necessarily have a unit, but it has enough idempotents. Throughout the section we consider a $K$-algebra $A$, not necessarily having a unit, but having a system $(e_\alpha)_{\alpha \in R}$ of pairwise orthogonal idempotents, such that $A = \oplus_{\alpha \in R} Ae_\alpha = \oplus_{\alpha \in R} e_\alpha A$. Let us note that $A$ has local units, i.e. if $a_1, \ldots, a_n \in A$, then there exists an idempotent $e \in A$ (which can be taken to be the sum of some $e_\alpha$’s) such that $ea_i = a_i e = a_i$ for any $1 \leq i \leq n$. Our aim is to show that there exists a natural structure of a coalgebra (with counit) on the space

$$A^0 = \{ f \in A^* | \text{Ker}(f) \text{ contains a coideal of } A \text{ of finite codimension} \}$$

We will call $A^0$ the finite dual of the algebra $A$.

Lemma 2.1. Let $I$ be an ideal of $A$ of finite codimension. Then the set $R' = \{ \alpha \in R | e_\alpha \notin I \}$ is finite.

Proof. Denote by $\hat{a}$ the class of an element $a \in A$ in the factor space $A/I$. We have that $(\hat{e}_\alpha)_{\alpha \in R'}$ is linearly independent in $A/I$. Indeed, if $\sum_{\alpha \in R'} \lambda_\alpha \hat{e}_\alpha = 0$, then $\sum_{\alpha \in R'} \lambda_\alpha e_\alpha \in I$. Multiplying by some $e_\alpha$ with $\alpha \in R'$, we find that $\lambda_\alpha e_\alpha \in I$, so then necessarily $\lambda_\alpha = 0$. Since $A/I$ is finite dimensional, the set $R'$ must be finite. \qed

Assume now that $B$ is another algebra with enough idempotents, say that $(f_\beta)_{\beta \in S}$ is a system of orthogonal idempotents in $B$ such that $B = \oplus_{\beta \in S} Bf_\beta = \oplus_{\beta \in S} f_\beta B$.

Lemma 2.2. Let $H$ be an ideal of $A \otimes B$ of finite codimension. Let $I = \{a \in A | a \otimes B \subseteq H\}$ and $J = \{ b \in B | A \otimes b \subseteq H \}$. Then $I$ is an ideal of $A$ of finite codimension, $J$ is an ideal of $B$ of finite codimension and $I \otimes B + A \otimes J \subseteq H$.

Proof. Let $a \in I$ and $a' \in A$. If $b \in B$ and $f$ is an idempotent in $B$ such that $fb = b$, we have that $a' a \otimes b = a' a \otimes fb = (a' \otimes f)(a \otimes b) \in H$. Thus $a' a \otimes B \subseteq H$, so $a' a \in I$. Similarly $aa' \in I$, showing that $I$ is an ideal of $A$. Similarly $J$ is an ideal of $B$.

It is clear that $(e_\alpha \otimes f_\beta)_{\alpha \in R, \beta \in S}$ is a set of orthogonal idempotents in $A \otimes B$ and $A \otimes B = \oplus_{\alpha \in R, \beta \in S} (A \otimes B)(e_\alpha \otimes f_\beta) = \oplus_{\alpha \in R, \beta \in S} (e_\alpha \otimes f_\beta)(A \otimes B)$. By Lemma 2.1 we have that there are finitely many elements, say $e_{\alpha_1} \otimes f_{\beta_1}, \ldots, e_{\alpha_n} \otimes f_{\beta_n}$ of the set $(e_\alpha \otimes f_\beta)_{\alpha \in R, \beta \in S}$ which lie outside $H$. If $\alpha \in R \setminus \{\alpha_1, \ldots, \alpha_n\}$, then for any $\beta \in S$ we have that $e_\alpha \otimes f_\beta \in H$, so $e_\alpha \otimes Bf_\beta = (e_\alpha \otimes Bf_\beta)(e_\alpha \otimes f_\beta) \subseteq H$. Then $e_\alpha \otimes B \subseteq H$, so $e_\alpha \in I$. Similarly for any $\beta \in S \setminus \{\beta_1, \ldots, \beta_n\}$ we have that $f_\beta \in J$.

For any $\beta \in S$ let $\phi_\beta : A \to A \otimes B$ be the linear map defined by $\phi_\beta(a) = a \otimes f_\beta$. If $a \in A$, then $a \in I$ if and only if for any $\beta \in S$ we have $a \otimes Bf_\beta \subseteq H$; because there is a local unit for $a$, this is further equivalent to $a \otimes f_\beta \in H$ for $\beta \in S$. This condition is obviously satisfied for $\beta \in S \setminus \{\beta_1, \ldots, \beta_n\}$ since $f_\beta \notin J$, so we obtain that $I = \cap_{1 \leq i \leq n} \phi^{-1}_i(H)$, a finite intersection of finite codimensional subspaces of $A$, thus a finite codimensional subspace itself. Similarly $J$ has finite codimension in $B$. The fact that $I \otimes B + A \otimes J \subseteq H$ is obvious. \qed
Now we essentially proceed as in [10, ] or [3, Section 1.5], with some arguments adapted to the case of enough idempotents.

**Lemma 2.3.** Let $A$ and $B$ be algebras with enough idempotents. The following assertions hold.

(i) If $f : A \to B$ is a morphism of algebras, then $f^*(B^0) \subseteq A^0$, where $f^*$ is the dual map of $f$.

(ii) If $\phi : A^* \otimes A^* \to (A \otimes A)^*$ is the natural linear injection, then $\phi(A^0 \otimes B^0) = (A \otimes B)^0$.

(iii) If $M : A \otimes A \to A$ is the multiplication of $A$, and $\psi : A^* \otimes B^* \to (A \otimes B)^*$ is the natural injection, then $M^*(A^0) \subseteq \psi(A^0 \otimes A^0)$.

**Proof.** It goes as the proof of [3, Lemma 1.5.2], with part of the argument in (ii) replaced by using the construction and the result of Lemma 2.2.

Lemma 2.3 shows that by restriction and corestriction we can regard the natural linear injection $\psi$ as an isomorphism $\psi : A^0 \otimes A^0 \to (A \otimes A)^0$. We consider the map $\Delta : A^0 \to A^0 \otimes A^0$, $\Delta = \psi^{-1} M^*$. Thus $\Delta(f) = \sum u_i \otimes v_i$ is equivalent to $f(xy) = \sum_i u_i(x)v_i(y)$ for any $x, y \in A$.

On the other hand, we define a linear map $\varepsilon : A^0 \to K$ as follows. If $f \in A^0$, then $\text{Ker}(f)$ contains a finite codimensional ideal $I$. By Lemma 2.1, there are finitely many $e_\alpha$’s outside $I$. Therefore only finitely many $e_\alpha$’s lie outside $\text{Ker}(f)$, so it makes sense to define $\varepsilon(f) = \sum_{\alpha \in R} f(e_\alpha)$ (only finitely many terms are non-zero).

**Proposition 2.4.** Let $A$ be an algebra with enough idempotents. Then $(A^0, \Delta, \varepsilon)$ is a coalgebra with counit.

**Proof.** The proof of the coassociativity works exactly as in the case where $A$ has a unit, see [3, Proposition 1.5.3]. To check the property of the counit, let $f \in A^0$ and $\Delta(f) = \sum u_i \otimes v_i$. Let $a \in A$ and $F$ a finite subset of $R$ such that $a = \sum_{\alpha \in F} e_\alpha A$. Then clearly $(\sum_{\alpha \in F} e_\alpha) a = a$. We have that

$$
(\sum_i \varepsilon(u_i)v_i)(a) = \sum_{i, \alpha} u_i(e_\alpha)v_i(a) \\
= \sum_{\alpha} f(e_\alpha a) \\
= \sum_{\alpha \in F} f(e_\alpha a) \\
= f(\sum_{\alpha \in F} e_\alpha a) \\
= f(a)
$$

so $\sum_i \varepsilon(u_i)v_i = f$. Similarly $\sum_i \varepsilon(v_i)u_i = f$, and this ends the proof.

Let us note that if $f : A \to B$ is a morphism of algebras with enough idempotents, then the map $f^0 : B^0 \to A^0$ induced by $f^*$ is compatible with the comultiplications of $A^0$ and $B^0$, but not necessarily with the counits (unless $f$ is compatible in some way to the systems of orthogonal idempotents in $A$ and $B$).

We denote by $\rightarrow$ (respectively $\leftarrow$) the usual left (respectively right) actions of $A$ on $A^*$. As in the unitary case, we have the following characterization of the elements of $A^0$.

**Proposition 2.5.** Let $f \in A^*$. With notation as above, the following assertions are equivalent.

(1) $f \in A^0$.

(2) $M^*(f) \in \psi(A^0 \otimes A^0)$.

(3) $M^*(f) \in \psi(A^* \otimes A^*)$.

(4) $A \to f$ is finite dimensional.

(5) $f \leftarrow A$ is finite dimensional.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) works exactly as in the case where $A$ has identity, see [3, Proposition 1.5.6]. We adapt the proof of (4) $\Rightarrow$ (1) to the case of enough idempotents. Since
A → f is a left A-submodule of A∗, there is a morphism of algebras (without unit) π : A → End(A → f) defined by π(a)(m) = a → m for any a ∈ A, m ∈ A → f. Since End(A → f) has finite dimension, we have that I = Ker(π) is an ideal of A of finite codimension. Let a ∈ I. Then a → (b → f) = (ab) → f = 0 for any b ∈ A, so f(xab) = 0 for any x, b ∈ A. Let e ∈ A such that ea = ae = a. Then f(a) = f(eae) = 0, so a ∈ Ker(f). Thus I ⊆ Ker(f), showing that f ∈ A0. The equivalence (1) ⇔ (5) is proved similarly.

□

3. Quiver algebras and path coalgebras

We examine the connection between the quiver algebra K[Γ] and the path coalgebra KΓ associated to a quiver Γ. The algebra K[Γ] has identity if and only if Γ has finitely many vertices. However K[Γ] has always enough idempotents (the set of all vertices). Thus by Section 2 we can consider the finite dual K[Γ]0, which is a coalgebra with counit. One has that K[Γ]0 ⊆ KΓ, i.e. the path coalgebra can be embedded in the finite dual of the path algebra. The embedding is given as follows: for each path p ∈ Γ, denote θ(p) ∈ K[Γ]∗ the function θ(p)(q) = δpq. We have that θ(p) ∈ K[Γ]0 since if we denote by S(p) the set of all subpaths of p, and by P the set of all paths in Γ, the span of P \ S(p) is a finite codimensional ideal of K[Γ] contained in Ker θ(p). It is easy to see that θ : KΓ ↪ K[Γ]0 is an embedding of coalgebras. In general, K[Γ] ↪ (KΓ)∗ is surjective if and only if the quiver Γ is finite. Also, in general, θ is not surjective. To see this, let A be the path algebra of a loop Γ, i.e. a quiver with one vertex and one arrow:

so A = K[X], the polynomial algebra in one indeterminate. The finite dual of this algebra is

\[ \lim_{f \text{ irreducible; } n \in \mathbb{N}} (K[X]/(f^n))^∗ = \bigoplus_{f \text{ irreducible } n \in \mathbb{N}} \lim_{n} (K[X]/(f^n))^∗, \]

while the path coalgebra is precisely the divided power coalgebra, which can be written as lim (K[X]/(X^n))^∗. These two coalgebras are not isomorphic, so the map θ above is not a surjection. Indeed, KΓ has just one grouplike element, the vertex of Γ, while the grouplike elements of A0, which are the algebra morphisms from A = K[X] to K, are in bijection to K.

The embedding of coalgebras θ : KΓ ↪ K[Γ]0 also gives rise to a functor Fθ : M^KΓ → M^K[Γ]0, associating to a right KΓ-comodule the right K[Γ]0-comodule structure obtained by extension of scalars via θ. We aim to provide a criterion for when is the above map θ bijective, that is, when the path coalgebra is recovered as the finite dual of the quiver algebra. Even though this is not always the case, we show that it is possible to interpret the quiver algebra as a certain kind of "graded" finite dual. We will think of KΓ as embedded into K[Γ]0 through θ, and sometimes write KΓ instead of θ(KΓ).

Recall that in a quiver algebra K[Γ], there is an important class of ideals, those which have a basis of paths, equivalently, those ideals generated by paths. Let us call such an ideal a monomial ideal. When I is a cofinite monomial left ideal, the quotient K[Γ]/I produces an interesting type of representation often considered in the representation theory of quivers. In fact, such a representation also becomes a KΓ-comodule, i.e. it is in the "image" of the functor Fθ. To see this, let B be basis of paths for I and let E be the set of paths not belonging to I; then E is finite, and because I is a left ideal, one sees that if p ∈ E and p = qr, then r ∈ E. This shows that KE, the span of E, is a left subcomodule of KΓ, and then (KE)∗ has a right KΓ comodule structure and a rational left (KΓ)∗-module structure (see [3, Lemma 2.2.12]). Hence (KE)∗ is a left K[Γ]-module via the algebra map K[Γ] ↪ (KΓ)∗. Now, it is straightforward to see that K[Γ]/I ∼= (KE)∗ as left K[Γ]-modules, and this proves the claim. Thus, Fθ(((KE)∗)KΓ) = (K[Γ]/I)K[Γ]0.

We can now state a characterization of the path coalgebra in terms of the quiver algebra, as a certain type of finite dual:
Proposition 3.1. The coalgebra $\theta(K\Gamma)$ consists of all $f \in K[\Gamma]^*$ such that $\ker(f)$ contains a two-sided cofinite (i.e. having finite codimension) monomial ideal.

Proof. Let $P$ be the set of paths in $\Gamma$. If $p$ is a path, and $S(p)$ is the set of subpaths of $p$, then the cofinite dimensional vector space with basis $P \setminus S(p)$ is an ideal, and it is obviously contained in $\ker(\theta(p))$. Then clearly $\ker(\theta(z))$ contains a cofinite monomial ideal for any $z \in K\Gamma$.

Let now $f \in K[\Gamma]^*$ such that $\ker(f)$ contains the cofinite monomial ideal $I$. Let $B$ be a basis of $I$ consisting of paths, and let $E = P \setminus B$, which is finite, since $I$ is cofinite. Then if $q \in E$, we have that $f(q) = 0 = \sum_{p \in E} f(p)\theta(p)(q)$, while if $q \in E$, we have that $(\sum_{p \in E} f(p)\theta(p))(q) = f(q)$. Therefore $f = \sum_{p \in E} f(p)\theta(p) \in \theta(K\Gamma)$. □

The following easy combinatorial condition will be the core of our characterization.

Proposition 3.2. Let $\Gamma$ be a quiver. The following conditions are equivalent:

(i) $\Gamma$ has no (oriented) cycles and between any two vertices there are only finitely many arrows.

(ii) For any finite set of vertices $E \subset \Gamma$, there are only finitely many paths passing only through vertices of $E$.

We can now characterize precisely when the path coalgebra can be recovered from the quiver algebra, that is, when the above mentioned embedding is an isomorphism.

Theorem 3.3. Let $\Gamma$ be a quiver. The following assertions are equivalent:

(i) $\Gamma$ has no (oriented) cycles and between every two vertices of $\Gamma$ there are only finitely many arrows.

(ii) $\theta(K\Gamma) = K[\Gamma]^0$.

(iii) Every cofinite ideal of $K[\Gamma]$ contains a cofinite monomial ideal.

(iv) The functor $F^\theta : M^{K\Gamma} \to M^{K[\Gamma]^0}$ is an equivalence.

Proof. The equivalence of (ii) and (iv) is a general coalgebra fact: if $C \subseteq D$ is an inclusion of coalgebras, then the corestriction of scalars $F : M^C \to M^D$ is an equivalence if and only if $C = D$. Indeed, if $F$ is an equivalence, pick an arbitrary $x \in D$ and let $N = D^*x \in M^D$ be the finite dimensional $D$-subcomodule of $D$ generated by $x$. Then $N \simeq F(M)$, $M \in M^C$, and considering the coalgebras of coefficients $C_N$ and $C_M$ of $N$ and $M$, we see that $C_N = C_M \subseteq C$ by the definition of $F$. Since $x \in C_N$, this ends the argument.

The equivalence of (ii) and (iii) follows immediately from Proposition 3.1.

(i) $\Rightarrow$ (iii) Let $I$ be an ideal of $K[\Gamma]$ of finite codimension. Note that $\{a | a \text{ vertex in } \Gamma\}$ is a set of orthogonal idempotents in $K[\Gamma]$. If $F$ is a set of idempotents (vertices in $\Gamma$) of $K[\Gamma]$ with $F \cap I = \emptyset$, then $F$ is linearly independent over $I$, i.e. $\pi(F)$ is linearly independent in $K[\Gamma]/I$, where $\pi : K[\Gamma] \to K[\Gamma]/I$ is the natural projection. Indeed, if $\sum_{a \in F} \lambda_a a \in I$, then for each $c \in F$ we have $c \cdot (\sum_{a \in F} \lambda_a a) = \lambda_c c \in I$, so $\lambda_c = 0$ since $c \notin I$. Since $K[\Gamma]/I$ is finite dimensional, the set $S' = \{a \text{ vertex in } \Gamma | a \notin I\}$ must be finite. Let $S = \{a \text{ vertex in } \Gamma | a \in I\}$. Note that any path $p$ starting or ending at a vertex in $S$ belongs to $I$, since $p = s(p)p = pt(p) \in I$ if either $s(p) \in I$ or $t(p) \in I$. Furthermore, this shows that if $p$ contains a vertex in $S$, then $p \in I$, since in that case $p = qr$ with $x = t(q) = s(r) \in S$. Denote the set of paths containing some vertex in $S$ by $M$. Let $H$ be the vector space spanned by $M$ and let $M'$ be the set of the rest of the paths in $\Gamma$. Obviously, $M'$ consists of the paths whose all vertices belong to $S'$. Since $S'$ is finite, we see that $M'$ is finite, by the conditions of (i) and Proposition 3.2. Therefore $H$ has finite codimension. Also, since $H$ is spanned by paths passing through some vertex in $S$, we see that $H$ is an ideal.

We conclude that $I$ contains the cofinite monomial ideal $H$ has finite codimension.

(iii) $\Rightarrow$ (i) We show first that there are no cycles in $\Gamma$. Assume $\Gamma$ has a cycle $C : v_0 \xrightarrow{x_0} v_1 \xrightarrow{x_1} \ldots \to v_{s-1} \xrightarrow{x_{s-1}} v_s = v_0$, and consider such a cycle that does not selfintersect. We can consider the vertices $v_0, \ldots, v_{s-1}$ modulo $s$. Denote by $q_{n,k}$ the path starting at the vertex $v_n$ ($0 \leq n \leq s - 1$), winding around the cycle $C$ and of length $k$. Denote again by $P$ the set of all
paths in \( \Gamma \), and by \( X = \{q_{n,k}|0 \leq n \leq s-1, k \geq 0\} \). Since the set \( X \) is closed under subpaths, it is easy to see that the vector space \( H \) spanned by the set \( P \setminus X \) is an ideal of \( K[\Gamma] \). Let \( E \) be the subspace spanned by \( S = \{q_{n,ks+i} - q_{n,i}|0 \leq n \leq s-1, i \geq 0, k \geq 1\} \), and let \( I = E + H \). We have that

\[
(q_{n,ks+i} - q_{n,i})q_{n+i,j} = q_{n,ks+i+j} - q_{n,i+j} \in S \\
(q_{n,ks+i} - q_{n,i})q_{m,j} = 0 \text{ for } m \neq n + i \\
q_{m,j}(q_{n,ks+i} - q_{n,i}) = q_{m,ks+i+j} - q_{m,i+j} \in S \text{ if } m + j = n \\
q_{m,j}(q_{n,ks+i} - q_{n,i}) = 0 \text{ if } m + j \neq n
\]

showing that if we multiply an element of \( S \) to the left (or right) by an element of \( X \), we obtain either an element of \( S \) or 0. Combined to the fact that \( H \) is an ideal, this shows that \( I \) is an ideal.

It is clear that \( I \) has finite codimension, since \( S \cup \{q_{n,i}|0 \leq n \leq s-1, 0 \leq i \leq s-1\} \) spans \( KC = \langle X \rangle \). Indeed, if \( 0 \leq n \leq s-1 \) and \( j \) is a non-negative integer, write \( j = ks + i \) with \( k \geq 0 \) and \( 0 \leq i \leq s-1 \), and we have that \( q_{n,j} = q_{n,ks+i} = (q_{n,ks+i} - q_{n,i}) + q_{n,i} \).

On the other hand, \( I \) does not contain a cofinite monomial ideal. Indeed, it is easy to see that an element of the form \( q_{m,j} \) cannot be in \( \langle S \cup (P \setminus X) \rangle = I \), so any monomial ideal contained in \( I \) must have infinite codimension.

Thus, we have found a cofinite ideal \( I \) which does not contain a cofinite monomial ideal. This contradicts \((iii)\), and we conclude that \( \Gamma \) cannot contain cycles.

We now show that in \( \Gamma \) there are no pair of vertices with infinitely many arrows between them. Assume such a situation exists between two vertices \( a, b \): \( a \xrightarrow{x_n} b, n \in \mathbb{N} \). We let \( X = \{x_n|n \in \mathbb{N}\} \cup \{a, b\} \), \( H \) be the span of \( P \setminus X \), which is an ideal since \( X \) is closed to subpaths. Let \( S = \{x_n - x_0|n \geq 1\} \) and \( I \) be the span of \( \langle P \setminus X \rangle \cup S \). As above, since \( x_n - x_0 \) multiplied by an element of \( X \) gives either \( x_n - x_0 \) or 0, we have that \( I \) is an ideal. \( I \) has finite codimension since \( \{a, b, x_0\} \cup S \cup (P \setminus X) \) spans \( KT \). Also, \( I \) does not contain a monomial ideal of finite codimension since no \( x_n \) lies in \( I \). Thus we contradict \((iii)\). In conclusion there are finitely many arrows between any two vertices, and this ends the proof. \( \square \)

Let \( \psi : K[\Gamma] \to (KT)^* \) be the linear map defined by \( \psi(p)(q) = \delta_{p,q} \) for any paths \( p \) and \( q \). In fact \( \psi \) is just \( \theta \) as a linear map, but we denote it differently since we regard it now as a morphism in the category of algebras without identity. Indeed, it is easy to check that \( \psi \) is multiplicative.

Thus the quiver algebra embeds in the dual of the path coalgebra. Our aim is to show that in certain situations \( K[\Gamma] \) can be recovered from \((KT)^*\) as the rational part. The next result characterizes completely these situations. We recall that if \( C \) is a coalgebra, the rational part of the left \( C^*\)-module \( C^* \), consisting of all elements \( f \in C^* \) such that there exist finite families \( (c_i)_i \) in \( C \) and \( (f_i)_i \) in \( C^* \) with \( c^*f = \sum_i c^*(c_i)f_i \) for any \( c^* \in C^* \), is denoted by \( C_{\text{rat}}^r \). This is the largest \( C^*\)-submodule which is rational, i.e. whose \( C^*\)-module structure comes from a right \( C \)-comodule structure. Similarly, \( C_{\text{rat}}^l \) denotes the rational part of the right \( C^*\)-module \( C^* \). A coalgebra \( C \) is called right (respectively left) semiperfect if the category of right (respectively left) \( C \)-comodules has enough projectives. This is equivalent to the fact that \( C_{\text{rat}}^r \) (respectively \( C_{\text{rat}}^l \)) is dense in \( C^* \) in the finite topology, see [3, Section 3.2].

**Theorem 3.4.** The following are equivalent.

(i) \( \text{Im}(\psi) = (KT)^{\text{rat}}_r \).

(ii) \( \text{Im}(\psi) = (KT)^{\text{rat}}_l \).

(iii) For any vertex \( v \) of \( \Gamma \) there are finitely many paths starting at \( v \) and finitely many paths ending at \( v \).

(iv) The path coalgebra \( KT \) is left and right semiperfect.
Proof. (iii) ⇒ (i) Let \( p \) be a path. We show that \( p^* = \psi(p) \in \text{Im}(\psi) \). If \( c^* \in (K\Gamma)^* \) and \( q \) is a path, we have that

\[
(c^*p^*)(q) = \sum_{r \ni q} c^*(r)p^*(v) = \begin{cases} c^*(r), & \text{if } q = rp \text{ for some path } r \\ 0, & \text{if } q \text{ does not end with } p \end{cases}
\]

Let \( q_1 = r_1p, \ldots, q_n = r_np \) be all the paths ending with \( p \). By the formula above, \((c^*p^*)(q_i) = c^*(r_i)\) for any \( 1 \leq i \leq n \), and \((c^*p^*)(q) = 0\) for any path \( q \neq q_1, \ldots, q_n \). This shows that \( c^*p^* = \sum_{1 \leq i \leq n} c^*(r_i)q_i^* \), thus \( p^* \in (K\Gamma)^\text{rat} \), and we have that \( \text{Im}(\psi) \subseteq (K\Gamma)^\text{rat} \).

Now let \( c^* \in (K\Gamma)^\text{rat} \), so there exist \((c_i)_{1 \leq i \leq n}\) in \( K\Gamma \) and \((c_i^*)_{1 \leq i \leq n}\) in \((K\Gamma)^*\) such that \( d^*c^* = \sum_{1 \leq i \leq n} d^*(c_i)c_i^* \) for any \( d^* \in (K\Gamma)^* \). Let \( p_1, \ldots, p_m \) all the paths that appear with non-zero coefficients in some of the \( c_i \)'s (represented as a linear combination of paths). Then for any \( p \neq p_1, \ldots, p_m \) we have that \( p^*(c_i) = 0 \), so then \( p^*c^* = 0 \). Let \( v \) be a vertex which is not on any of \( p_1, \ldots, p_m \). For any path \( p \) starting at \( v \) we have that \( 0 = (v^*c^*)(p) = v^*(v)c^*(p) = c^*(p) \). Therefore \( c^* \) may be non-zero on a path \( p \) only if \( s(p) \in \{p_1, \ldots, p_m\} \). By condition (iii), there are only finitely many such paths \( p \), denote them by \( q_1, \ldots, q_e \). Then \( c^* = \sum_{1 \leq i \leq e} c^*(q_i)q_i^* \in \text{Im}(\psi) \), and we also have that \((K\Gamma)^\text{rat} \subseteq \text{Im}(\psi) \).

(i) ⇒ (iii) Let \( v \) be a vertex. Then \( v^* = \psi(v) \in (K\Gamma)^\text{rat} \), so there exist finite families \((c_i) \subseteq K\Gamma \) and \((c_i^*) \subseteq (K\Gamma)^* \) such that \( c^*v^* = \sum_i c^*(c_i)c_i^* \) for any \( c^* \in (K\Gamma)^* \). Then

\[
\sum_i c^*(c_i)c_i^*(q) = (c^*v^*)(q) = \begin{cases} c^*(q), & \text{if } q \text{ ends at } v \\ 0, & \text{otherwise} \end{cases}
\]

If there exist infinitely many paths ending at \( v \), we can find one such path \( q \) which does not appear in the representation of any \( c_i \) as a linear combination of paths.. Then there exists \( c^* \in (K\Gamma)^* \) with \( c^*(q) \neq 0 \) and \( c^*(c_i) = 0 \) for any \( i \), in contradiction with (1). Thus only finitely many paths can end at \( v \). In particular \( \Gamma \) does not have cycles.

On the other hand, if we assume that there are infinitely many paths \( p_1, p_2, \ldots \) starting at \( v \), let \( c^* \in (K\Gamma)^* \) which is 1 on any \( p_i \), and 0 on any other path. Clearly \( c^* \notin \text{Im}(\psi) \). We show that \( c^* \in (K\Gamma)^\text{rat} \), and the obtained contradiction shows that only finitely many paths start at \( v \). Indeed, we have that

\[
(d^*c^*)(q) = \begin{cases} d^*(r), & \text{if } q = rp_i \text{ for some } i \geq 1 \text{ and some path } r \\ 0, & \text{otherwise} \end{cases}
\]

Let \( r_1, \ldots, r_m \) be all the paths ending at \( v \) (they are finitely many as we proved above). For each \( 1 \leq j \leq m \) we consider the element \( c_j^* \in (K\Gamma)^* \) which is 1 on any path of the form \( r_jp_i \), and 0 on any other path. Using (2) and the fact that \( r_jp_i \neq r_j'p_i \) for \((i, j) \neq (i', j')\) (this follows because \( r_j, r_j' \) end at \( v \) and \( p_i, p_i' \) begin at \( v \), and there are no cycles containing \( v \)), we see that \( d^*c^* = \sum_{1 \leq j \leq m} d^*(r_j)c_j^* \), and this will guarantee that \( c^* \) is a rational element.

(ii) ⇔ (iii) is similar to (i) ⇔ (iii).

(iii) ⇔ (iv) follows from [1, Corollary 6.3].

4. Coreflexivity for path subcoalgebras and subcoalgebras of incidence coalgebras

We recall from [9, 11] that a coalgebra \( C \) is called coreflexive if any finite dimensional left (or equivalently, any finite dimensional right) \( C^* \)-module is rational. This is also equivalent to asking that the natural embedding of \( C \) into the finite dual of \( C^* \), \( C \hookrightarrow (C^*)^0 \) is surjective (so an isomorphism), or that any left (equivalently, any right) cofinite ideal is closed in the finite topology. See [8, 9, 11, 12] for further equivalent characterizations.

Given the definition of coreflexivity and the characterizations of the previous section, it is natural to ask what is the connection between the situation when the path coalgebra can be recovered as the finite dual of the quiver algebra, and the coreflexivity of the path coalgebra. We note that these two are closely related. We have an embedding \( \iota : K\Gamma \hookrightarrow (K\Gamma)^* \); at the same time,
we note that the embedding of algebras (without identity) \( \psi : K[\Gamma] \hookrightarrow (K\Gamma)^* \) which is dense in the finite topology of \((K\Gamma)^*\), produces a comultiplicative morphism \( \varphi : (K\Gamma)^{0} \rightarrow K[\Gamma]^0 \). Note that \( \varphi \) is not necessarily a morphism of coalgebras, since it may not respect the counits. It is easy to see that these canonical morphisms are compatible with \( \theta \), i.e. they satisfy \( \theta = \varphi \circ \iota : K\Gamma \hookrightarrow K[\Gamma] \). It is then natural to ask what is the connection between the bijectivity of \( \theta \), and coreflexivity of \( K\Gamma \), i.e. bijectivity of \( \iota \). In fact, we notice that if \( C \) is coreflexive (equivalently, \( \iota \) is surjective), then \( \varphi \) is necessarily injective.

The following two examples will show that, in fact, \( C \) can be coreflexive and \( \theta \) not an isomorphism, and also that \( \theta \) can be an isomorphism without \( C \) being coreflexive.

**Example 4.1.** Consider the path coalgebra of the following quiver \( \Gamma \):

Here there are \( n \) arrows from vertex \( a \) to vertex \( b_n \) and \( n \) arrows from \( b_n \) to \( c \) for each natural number \( n \). We note that the 1 dimensional vector space spanned by \( a-c \) is a coideal, since \( a-c \) is an \((a,c)\)-skew-primitive element. It is not difficult to observe that the quotient coalgebra \( C/I \) is isomorphic to the coalgebra from [8, Example 3.4], and so \( C/I \) is not coreflexive as showed in [8]. By [5, 3.1.4], we know that if \( I \) is a finite dimensional coideal of a coalgebra \( C \) then \( C \) is coreflexive if and only if \( C/I \) is coreflexive. Therefore, \( C \) is not coreflexive. However, it is obvious that \( C \) satisfies the quiver conditions of Theorem 3.3, and therefore, \( K\Gamma = K[\Gamma]^0 \).

Hence, a path coalgebra of a quiver with no cycles and finitely many arrows between any two vertices is not necessarily coreflexive. Conversely, we note that in a coreflexive path coalgebra there are only finitely many arrows between any two vertices. This is true since a coreflexive coalgebra is locally finite by [5, 3.2.4], which means that the wedge \( X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y) \) of any two finite dimensional vector subspaces \( X, Y \) of \( C \) is finite dimensional (one applies this for \( X = Ka \) and \( Y = Kb \)). However, if a path coalgebra \( K\Gamma \) is coreflexive, \( \Gamma \) may contain cycles: consider the path coalgebra \( C \) of a loop (a graph with one vertex and one arrow); \( C \) is then the divided power coalgebra, \( C^* = K[[X]] \), the ring of formal power series, and its only ideals are \((X^n)\), which are closed in the finite topology of \( C^* \). Thus, every finite dimensional \( C^*\)-module is rational and \( C \) is coreflexive.

We will prove coreflexivity of an interesting class of path coalgebras, whose quiver satisfy a slightly stronger condition than that required by Theorem 3.3 (so in particular, they will satisfy \( K\Gamma = K[\Gamma]^0 \)). We first prove a general coreflexivity criterion.
Theorem 4.2. Let $C$ be a coalgebra with the property that for any finite dimensional subcoalgebra $V$ there exists a finite dimensional subcoalgebra $W$ such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. Then $C$ is coreflexive if and only if its coradical $C_0$ is coreflexive.

Proof. If $C$ is coreflexive, then so is $C_0$, since a subcoalgebra of a coreflexive coalgebra is coreflexive (see [5, Proposition 3.1.4]). Conversely, let $C_0$ be coreflexive. We prove that any finite dimensional left $C^\ast$-module $M$ is rational, by induction on the length $l(M)$ of $M$. If $l(M) = 1$, i.e. $M$ is simple, then $M$ is also a left $C^\ast/J(C^\ast)$-module. Since $C^\ast/J(C^\ast) \simeq C_0^\ast$ and $C_0$ is coreflexive, we have that $M$ is a rational $C_0^\ast$-module, so then it is a rational $C^\ast$-module, too.

Assume now that the statement is true for length $< n$, where $n > 1$, and let $M$ be a left $C^\ast$-module of length $n$. Let $M'$ be a simple submodule of $M$, and consider the associated exact sequence

$$0 \to M' \to M \to M'' \to 0$$

By the induction hypothesis $M'$ and $M''$ are rational. By [3, Theorem 2.2.14] we have that $ann_{C^\ast}(M')$ and $ann_{C^\ast}(M'')$ are co-finite dimensional closed two-sided ideals in $C^\ast$. Using [3, Corollary 1.2.8 and Proposition 1.5.23], we have that $ann_{C^\ast}(M') = U_1^\perp$ and $ann_{C^\ast}(M'') = U_2^\perp$ for some finite dimensional subcoalgebras of $C$. Using the hypothesis for $V = U_1 + U_2$, we have that there is a finite dimensional subcoalgebra $W$ of $C$ such that $U_1 \subseteq W$, $U_2 \subseteq W$ and $W^\perp W = W^\perp$. Then $W^\perp = W^\perp U_1^\perp U_2^\perp = ann_{C^\ast}(M') ann_{C^\ast}(M'') \subseteq ann_{C^\ast}(M)$, and $M$ is a rational $C^\ast$-module by using again [3, Theorem 2.2.14].

Proposition 4.3. Let $C$ be the path coalgebra $K\Gamma$, where $\Gamma$ is a quiver such that there are finitely many paths between any two vertices. Then for any finite dimensional subcoalgebra $V$ of $C$ there exists a finite dimensional subcoalgebra $W$ such that $V \subseteq W$ and $W^\perp W^\perp = W^\perp$. As a consequence, $C$ is coreflexive if and only if the coradical $C_0$ (which is the grouplike coalgebra over the set of vertices of $\Gamma$) is coreflexive.

Proof. Let $V$ be a finite dimensional subcoalgebra of $C = K\Gamma$. An element $c \in V$ is of the form $c = \sum_{i=1,n} a_i p_i$, with $a_i \neq 0$, a linear combination of paths $p_1, \ldots, p_n$. Consider the set of all vertices at least one of these paths passes through, and let $S_0$ be the union of all these sets of vertices when $c$ runs through the elements of $V$. Since $V$ is finite dimensional, we have that $S_0$ is finite (in fact, one can see that $S_0$ consists of all vertices in $\Gamma$ which belong to $V$, so that $KS_0$ is the socle of $V$). Let $P$ be the set of all paths $p$ such that $s(p), t(p) \in S_0$. We consider the set $S$ of all vertices at least one path of $P$ passes through. It is clear that $P$ is finite, and then so is $S$.

We note that if $v_1, v_2 \in S$ and $p$ is a path from $v_1$ to $v_2$, then any vertex on $p$ lies in $S$. Indeed, $v_1$ is on a path from $v_1$ to $u'_1$ (vertices in $S_0$), and let $p_1$ be its subpath from $v_1$ to $v_1$. Similarly, $v_2$ is on a path from $u_2$ to $u'_2$ (in $S_0$), and let $p_2$ be the subpath from $v_2$ to $u'_2$. Then $p_1 p_2 \in P$, so any vertex of $p$ is in $S$. Let $W$ be the subspace spanned by all paths starting and ending at vertices in $S$. It is clear that any subpath of a path in $W$ is also in $W$, so then $W$ is a finite dimensional subcoalgebra containing $V$ (since $S_0$ is contained in $S$).

We show that $W^\perp W^\perp = W^\perp$. For this, let $\eta \in W^\perp$, and we construct $f_1, f_2, g_1, g_2 \in W^\perp$ such that $\eta = f_1 g_1 + f_2 g_2$. We define $f_i(p)$ and $g_i(p)$, $i = 1, 2$, on all paths $p$ by induction on the length of $p$. For paths $p$ of length zero, i.e. $p$ is a vertex $v$, we define $f_i(v) = g_i(v) = 0$, $i = 1, 2$, for any $v \in S$, while for $v \notin S$, we set $f_1(v) = g_2(v) = 1$, and $f_2(v)$ and $g_2(v)$ are such that $g_1(v) + f_2(v) = \eta(v)$. Then clearly $\eta = f_1 g_1 + f_2 g_2$ on paths of length zero. For the induction step, assume that we have defined $f_i$ and $g_i$, $i = 1, 2$, on all paths of length $< l$, and that $\eta = f_1 g_1 + f_2 g_2$ on any such path. Let now $p$ be a path of length $l$, starting at $u$ and ending at $v$. If $u, v \in S$, then we define $f_1(p) = g_i(p) = 0$, $i = 1, 2$, and clearly $\eta(p) = \sum_{1 \leq i \leq 2} \sum_{v \neq p} f_i(q) g_i(r)$, since both sides are zero. If either $u \notin S$ or $v \notin S$, we need the following equality to hold.

$$f_1(u) g_1(p) + f_1(p) g_1(v) + f_2(u) g_2(p) + f_2(p) g_2(v) = \eta(p) - \sum_{i=1,2} \sum_{v \neq p, v \neq p} f_i(q) g_i(r)$$
We note that the terms of the right-hand side of the equality (3) have already been defined, because when \( p = qr \) and \( q \neq p, r \neq p \), the length of the paths \( q \) and \( r \) is strictly less than the length of \( p \). We define \( f_1(p) \) and \( g_2(p) \) to be any elements of \( K \), and then since either \( f_1(u) = 1 \) or \( g_2(v) = 1 \) (since either \( u \notin S \) or \( v \notin S \)), we can choose suitable \( g_1(p) \) and \( f_2(p) \) such that (3) holds true.

The fact that \( C \) is coreflexive if and only if so is \( C_0 \) follows now directly from Theorem 4.2. 

Moreover, we can extend the result in the previous proposition to subcoalgebras of path coalgebras.

**Proposition 4.4.** Let \( C \) be a subcoalgebra of a path coalgebra \( K\Gamma \), such that there are only finitely many paths between any two vertices in \( \Gamma \). Then \( C \) is coreflexive if and only if \( C_0 \) is coreflexive.

**Proof.** Let \( \Gamma' \) be the subquiver of \( \Gamma \) whose vertices are all the vertices \( v \) of \( \Gamma \) such that there is an element \( c = \sum \alpha_i p_i \in C \), where the \( \alpha_i \)'s are nonzero scalars and the \( p_i \)'s are distinct paths, and at least one \( p_i \) passes through \( v \). The arrows of \( \Gamma' \) are all the arrows of \( \Gamma \) between vertices of \( \Gamma' \). Clearly, there are only finitely many paths between any two vertices in \( \Gamma' \). Then we have that \( C \) is a subcoalgebra of \( K\Gamma' \) and \( C_0 = (K\Gamma')_0 \). Obviously, \( C_0 \subset (K\Gamma')_0 \); for the converse, let us consider a vertex \( u \) in \( \Gamma' \), so there is \( c \in C \) such that \( c = \sum \alpha_i p_i \), with \( \alpha_i \neq 0 \) and distinct \( p_i \)'s, and some \( p_k \) passes through \( u \). Let us write then \( p_k = qr \) such that \( q \) ends at \( u \) and \( r \) begins at \( u \). Since \( C \) is a subcoalgebra of \( K\Gamma' \) it is also a sub-bicomodule, so then \( r^*cq^* \in C \), where \( q^*, r^* \in (K\Gamma')^* \) are equal to 1 on \( q, r \) respectively and 0 on all other paths of \( K\Gamma' \). Now

\[
r^*pq^* = \sum_{p_i = stw} q^*(s)tr^*(w)
\]

and the only non-zero terms can occur if \( p_k = qt_ir \), so with \( t_i \) is a path starting and ending at \( u \) (loop at \( u \)). Let \( J \) be the set of these indices. In this situation \( r^*pq^* = t_i \). Note that since the \( p_i \)'s are different, the \( t_j \)'s, \( j \in J \) are different, too. Also, since \( p_k = qr \), there is at least such a \( j \). We have \( r^*cq^* = \sum_j \alpha_j p_j \), with all \( p_j \) beginning and ending at \( u \), and \( p_k = u \). Let \( l \in J \) be an index such that \( p_l \) has maximum length among the \( p_j \)'s. We note then that \( p_l^*p_j = 0 \) if \( j \neq l \), since for any decomposition \( p_j = st \), we have \( t \neq p_l \) because of the maximality of \( p_l \) and of the fact that \( p_j \neq p_k \). However, \( p_l^*p_l = u \). Therefore, \( p_l^*c = \alpha_l u \in C \), so \( u \in C \) since \( \alpha_l \neq 0 \).

Thus if \( C_0 \) is coreflexive, we have that \( (K\Gamma')_0 \) is coreflexive, and then by Lemma 4.3, we have that \( K\Gamma' \) is coreflexive. Then \( C \) is coreflexive, as a subcoalgebra of \( K\Gamma' \). Conversely, if \( C \) is coreflexive, then clearly \( C_0 \) is coreflexive.

We now give an example to show that it is possible to have a coalgebra which is both coreflexive, and satisfies the path coalgebra “recovery” conditions of Theorem 3.3; however, in its quiver, some vertices are joined by infinitely many paths. Thus, in general, the coreflexivity question for path coalgebras is more complicated.
Example 4.5. Consider the path coalgebra $C$ of the following quiver $\Gamma$:

Here there are infinitely many vertices $b_n$, one for each positive integer $n$. Let $W_n$ be the finite dimensional subcoalgebra of $C$ with basis

$$B = \{a, c, b_1, \ldots, b_n, x_1, \ldots, x_n, y_1, \ldots, y_n, x_1 y_1, \ldots, x_n y_n\}.$$  

We show that $W_n^\perp = W_n^\perp \cdot W_n^\perp$. Let $f \in W_n^\perp$. We show that we can find $g_1, g_2, h_2, h_2 \in W_n^\perp$ such that $f = g_1 h_1 + g_2 h_2$. This condition is already true on elements of $B$. We need to verify that for $k > n$:

$$f(x_n y_n) = \sum_{i=1,2} g_i(a) h_i(x_n y_n) + g_i(x_n) h_i(y_n) + g_i(x_n y_n) h_i(c)$$

$$f(x_n) = \sum_{i=1,2} g_i(a) h_i(x_n) + g_i(x_n) h_i(b_n)$$

$$f(y_n) = \sum_{i=1,2} g_i(b_n) h_i(y_n) + g_i(y_n) h_i(c)$$

$$f(b_n) = \sum_{i=1,2} g_i(b_n) h_i(b_n)$$

and since $g_i(a) = h_i(a) = g_i(c) = h_i(c) = 0$ this is equivalent to the matrix equality

$$\begin{pmatrix} f(b_n) & f(y_n) \\ f(x_n) & f(x_n y_n) \end{pmatrix} = \begin{pmatrix} g_1(b_n) & g_1(b_n) \\ g_1(x_n) & g_1(x_n) \end{pmatrix} \cdot \begin{pmatrix} h_1(b_n) & h_1(y_n) \\ h_1(b_n) & h_1(y_n) \end{pmatrix} + \begin{pmatrix} g_2(b_n) & g_2(b_n) \\ g_2(x_n) & g_2(x_n) \end{pmatrix} \cdot \begin{pmatrix} h_2(b_n) & h_2(y_n) \\ h_2(b_n) & h_2(y_n) \end{pmatrix}$$

But it is standard linear algebra fact that any arbitrary $2 \times 2$ matrix can be written this way as a sum of two matrices of rank 1, and thus the claim is proved. Since every finite dimensional subcoalgebra $V$ of $C$ is contained in some $W_n$ with $W_n^\perp = W_n^\perp \cdot W_n^\perp$. $W_n^\perp$ and $C_0 \cong K^N$ is coreflexive, by Theorem 4.2 we obtain that $C$ is coreflexive.

Corollary 4.6. Let $C$ be a subcoalgebra of an incidence coalgebra $KX$. Then $C$ is coreflexive if and only if $C_0$ is coreflexive.

Proof. As explained in the Introduction, $KX$ can be embedded in a path coalgebra $K\Gamma$ for which there are finitely many paths between any two vertices. Then $C$ is isomorphic to a subcoalgebra of a $K\Gamma$ and we apply Proposition 4.4. \qed

We give now an application of our considerations on coreflexive coalgebras. If $\Gamma, \Gamma'$ are quivers, then we consider the quiver $\Gamma \times \Gamma'$ defined as follows. The vertices are all pairs $(a, a')$ for $a, a'$ vertices in $\Gamma$ and $\Gamma'$ respectively. The arrows are the pairs $(a, x')$, which is an arrow from $(a, a'_1)$ to $(a, a'_2)$, where $a$ is a vertex in $\Gamma$ and $x'$ is an arrow from $a'_1$ to $a'_2$ in $\Gamma'$, and the pairs $(x, a')$, which is an arrow from $(a_1, a')$ to $(a_2, a')$, where $x$ is an arrow from $a_1$ to $a_2$ in $\Gamma$, and $a'$ is a vertex.
in $\Gamma'$. Let $p = x_1x_2 \ldots x_n$ be a path in $\Gamma$ going (in order) through the vertices $a_0, a_1, \ldots, a_n$ and $q = y_1y_2 \ldots y_k$ be a path in $\Gamma'$ going through vertices $b_0, b_1, \ldots, b_k$ (some vertices may repeat). We consider the 2 dimensional lattice $L = \{0, \ldots, n\} \times \{0, \ldots, k\}$. A lattice walk is a sequence of elements of $L$ starting with $(0, 0)$ and ending with $(n, k)$, and always going either one step to the right or one step upwards in $L$, i.e. $(i, j)$ is followed either by $(i + 1, j)$ or by $(i, j + 1)$. There are $\binom{n+k}{k}$ such walks.

To $p, q$ and a lattice walk $(0, 0) = (i_0, j_0), (i_1, j_1), \ldots, (i_{n+k}, j_{n+k}) = (n, k)$ in $L$ we associate a path of length $n + k$ in $\Gamma \times \Gamma'$, starting at $(a_0, b_0)$ and ending at $(a_n, b_k)$ such that the $r$-th arrow of the path, from $(a_{i_r-1}, b_{j_r-1})$ to $(a_{i_r}, b_{j_r})$ is $(x_{r-1}, b_{j_r-1})$ if $i_r = i_{r-1} + 1$, and $(a_{i_r-1}, y_{r-1})$ if $j_r = j_{r-1} + 1$.

Conversely, if $\gamma$ is a path in $\Gamma \times \Gamma'$, there are (uniquely determined) paths $p$ in $\Gamma$ and $q$ in $\Gamma'$, and a lattice walk such that $\gamma$ is associated to $p, q$ and that lattice walk as above. Indeed, we take $p$ to be the path in $\Gamma$ formed by considering the arrows $x$ such that there are arrows of the form $(x, a')$ in $\gamma$, taken in the order they appear in $\gamma$. Similarly, $q$ is formed by considering the arrows of the form $(a, y)$ in $\gamma$. The lattice walk is defined according to the succession of arrows in $\gamma$.

For two such paths $p, q$ let us denote $W(p, q)$ the set of all paths in $\Gamma \times \Gamma'$ associate to $p$ and $q$ via lattice walks.

**Lemma 4.7.** The linear map $\alpha : K\Gamma \otimes K\Gamma' \hookrightarrow K(\Gamma \times \Gamma')$ defined as $\alpha(p \otimes q) = \sum_{w \in W(p, q)} w$ for $p \in \Gamma, q \in \Gamma'$ paths, is an injective morphism of $K$-coalgebras.

**Proof.** We keep the notations above. Denote $\delta$ and $\Delta$ the comultiplications of $K\Gamma \otimes K\Gamma'$ and $K(\Gamma \times \Gamma')$. We have that

$$\delta \alpha(p \otimes q) = \sum_{w \in W(p, q)} \sum_{w'w'' = w} w' \otimes w''$$

$$(\alpha \otimes \alpha) \Delta(p \otimes q) = \sum_{p'p'' = p} \sum_{q'q'' = q} \sum_{w \in W(p', q', p'', q'')} u \otimes v$$

On one hand, if $p = p'p'', q = q'q''$, $u \in W(p', q')$ and $v \in W(p'', q'')$, we have that $uv \in W(p, q)$. On the other hand, if $w \in W(p, q)$ and $w = w'w''$, then there exist $p'p''$ in $\Gamma$ and $q'q''$ in $\Gamma'$ such that $p = p'p''$, $q = q'q''$, $w' \in W(p', q')$ and $w'' \in W(p'', q'')$. These show that $\delta \alpha(p \otimes q) = (\alpha \otimes \alpha) \Delta(p \otimes q)$, i.e. $\alpha$ is a morphism of coalgebras (the compatibility with counits is easily verified).

To prove injectivity, if $p = x_1x_2 \ldots x_n$ is a path in $\Gamma$ starting at $a_0$ and ending at $a_n$, and $q = y_1y_2 \ldots y_k$ is a path in $\Gamma'$ starting at $b_0$ and ending at $b_k$, we denote by $(p^*, q^*)$ the linear map on $K(\Gamma \times \Gamma')$ which equals 1 on the path $(x_1, b_0), \ldots, (x_n, b_0), (a_n, y_1), \ldots, (a_n, y_k)$ (for simplicity we also denote this path by $(p, b_0); (a_n, q)$) and 0 on the rest of the paths. Let $\sum_i \lambda_ip_i \otimes q_i \in \text{Ker}(\alpha)$. Then we have that

$$\sum_i \sum_{w \in W(p_i, q_i)} \lambda_i w = 0$$

Fix some $j$. Say that $p_j$ ends at $a_n$ and $q_j$ starts at $b_0$. We have that

$$(p_j^*, q_j^*) \in W(p, q) = \begin{cases} 0, & \text{if } w \in W(p_i, q_i), i \neq j \\ 0, & \text{if } w \in W(p_j, q_j) \text{ and } w \neq (p_j, b_0), (a_n, q_j) \\ 1, & \text{if } w = (p_j, b_0), (a_n, q_j) \end{cases}$$

Note that we used the fact that $W(p, q) \cap W(p', q') = \emptyset$ for $(p, q) \neq (p', q')$. Now applying $(p_j^*, q_j^*)$ to (4) we see that $\lambda_j = 0$. We conclude that $\alpha$ is injective. 

Combining the above, we derive a result about tensor products of certain coreflexive coalgebras. It is known that a tensor product of a coreflexive and a strongly coreflexive coalgebra is coreflexive.
Proposition 4.8. Let $C, D$ be co-reflexive subcoalgebras of path coalgebras $K\Gamma$ and $K\Gamma'$ respectively such that between any two vertices in $\Gamma$ and $\Gamma'$ respectively there are only finitely many paths. Then $C \otimes D$ is co-reflexive.

Proof. Without any loss of generality we may assume that $C_0 = (K\Gamma)_0 = K(\Gamma_0)$ and $D_0 = (K\Gamma')_0 = K(\Gamma'_0)$ (otherwise we replace $\Gamma$ and $\Gamma'$ by appropriate subquivers), where $K(\Gamma_0)$ denotes the grouplike coalgebra with basis the set $\Gamma_0$ of vertices of $\Gamma$. Now $C \otimes D$ is a subcoalgebra of $K\Gamma \otimes K\Gamma'$, so by Lemma 4.7, it also embeds in $K(\Gamma \times \Gamma')$. Since the coradical of $K(\Gamma \times \Gamma')$ is $K((\Gamma_0 \times \Gamma'_0))$, and $K((\Gamma_0 \times \Gamma'_0)) \simeq K(\Gamma_0) \otimes K(\Gamma'_0) = C_0 \otimes D_0 \subseteq C \otimes D$, we must have that $(C \otimes D)_0 = K((\Gamma_0 \times \Gamma'_0))$. We claim that $K((\Gamma_0 \times \Gamma'_0))$ is coreflexive. Indeed, this is obvious if $\Gamma_0$ and $\Gamma'_0$ are both finite. Otherwise, $\text{card}(\Gamma_0 \times \Gamma'_0) = \max\{\text{card}(\Gamma_0), \text{card}(\Gamma'_0)\}$, hence $K((\Gamma_0 \times \Gamma'_0))$ is isomorphic either to $K(\Gamma_0)$ or to $K(\Gamma'_0)$, so it is coreflexive. Since it is clear that in $\Gamma \times \Gamma'$ there are also finitely many paths between any two vertices, we can use Proposition 4.4 and see that $C \otimes D$ is coreflexive. □

Corollary 4.9. If $C, D$ are coreflexive subcoalgebras of incidence coalgebras, then $C \otimes D$ is coreflexive.

Proof. It follows immediately from the embedding of $C$ and $D$ in path coalgebras verifying the hypothesis or Proposition 4.8. □

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