Student
Fill in your S-#

The exam is closed book. Use only the paper provided and make sure that each page is signed with your number. Do not write answers to different problems on the same page. Staple separately your answers to each problem.

Please solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed 2 hrs.

Please, indicate problems you are turning in

□ II-1 □ II-2 □ II-3 □ II-4
II-1. (Classical Mechanics)

A particle of mass \( m \) is moving in the field of a force

\[
\mathbf{F} = 2k \frac{r}{r^4}, \quad k \neq 0,
\]

where \( r \) is the position of the particle, \( r = |\mathbf{r}| \), and \( k \) is a constant.

(i) Argue that the motion of the particle is planar.

(ii) Using polar coordinates \( r \) and \( \phi \) on the plane, write down the Lagrangian and the Hamiltonian of the particle.

(iii) Show that trajectories of the particle on the plane satisfy

\[
\frac{d^2u}{d\phi^2} + a u = 0, \quad u(\phi) = \frac{1}{r(\phi)},
\]

where \( a \) is constant along the trajectory. Determine \( a \) as a function of the usual constants of motion, the energy \( E \) and the magnitude of the angular momentum \( \ell \).

(iv) Find conditions that \( E, \ell \) and \( k \) must satisfy in order that a trajectory is bounded, i.e., \( r(\phi) < r_0 < \infty \).

(v) For the case \( a = 0 \), sketch a trajectory for which the particle approaches the center of force from infinity with some nonzero angular momentum.

(vi) Argue that when the force is repulsive, \( k > 0 \), all trajectories are of a scattering type as in the figure below and show that the differential cross section \( \sigma(\vartheta) \) for this scattering is given by

\[
\sigma(\vartheta) = \frac{k}{E} \frac{\pi^2(\pi - \vartheta)}{\vartheta^2(2\pi - \vartheta)^2 \sin \vartheta},
\]
Consider an infinite, grounded, conducting sheet which is a plane surface except for a hemispherical bump of radius $a$. A point charge $q$ is placed directly above the hemispherical bump, at a distance $d$ from its center.

(i) There are three image charges. What are their magnitudes and locations? Use a coordinate system whose origin is at the center of the hemispherical bump, with the $z$-axis perpendicular to the plane, so the charge $q$ resides at $z = d$.

(ii) Using spherical coordinates $(r, \theta, \phi)$, write down the potential at any point in the region $r \geq a$ and $0 \leq \theta \leq \pi/2$.

(iii) What is the electric field on the surface of the hemispherical bump?

(iv) Show that the total charge on the hemispherical bump is

$$-q \left(1 - \frac{d^2 - a^2}{d \sqrt{d^2 + a^2}}\right).$$
II-3. (Quantum Mechanics)

An effective interaction between two heavy quarks in QCD is described by a Hamiltonian

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_1, r_2), \]

with a linear potential

\[ V(r_1, r_2) = \gamma |r_1 - r_2|, \]

proportional to the distance between the quarks as in the figure. In this problem you will examine the energy spectrum of this Hamiltonian.

(i) **Outline** the argument that the radial Schrödinger equation for the radial component of the wave function \( R(r) \), \( r = |r_1 - r_2| \), of the system is

\[ \left[ \frac{\hbar^2}{2\mu} \left( -\frac{1}{r} \frac{\partial}{\partial r} r^2 + \frac{\ell(\ell + 1)}{r^2} \right) + \gamma r \right] R(r) = E R(r). \]

What is \( \mu \)? What is \( \ell \)? What are the allowed values of \( \ell \)?

(ii) Estimate the ground state energy of the Hamiltonian using the dimensional analysis.

(iii) Using the uncertainty principle, or any other rough approximation scheme, **estimate** the energy \( E \) of the lowest energy state for each fixed value of \( \ell \). Express your answer as a function of the parameters \( m, \gamma, \hbar, \) and \( \ell \).

To further simplify the radial equation (i) use the substitution \( f(r) = r R(r) \) and introduce the dimensionless coordinate, \( u \), and energy, \( \epsilon \),

\[ r = r_0 u, \quad E = E_0 \epsilon, \]

where \( r_0 \) and \( E_0 \) are the “Bohr radius” and the energy of the ground state, respectively.

(iv) Show that for \( \ell = 0 \), the spectrum of energies is determined by the zeros of the Airy function,

\[ \text{Ai}(x) = C \int_0^\infty dt \cos(t^3/3 + tx), \]

where \( C \) is a constant.

**Hint:** Use the Fourier transform and implement carefully the boundary conditions for the radial wave function \( f(u) \).
(Mathematical Methods)

Prove the identities (i)-(iii) below and then use them to prove the identity (iv) for the principal value generalized functions.

(i) Prove the equivalence of the two definitions of the principal value generalized function:

\[ P \left( \frac{1}{x} \right) = \lim_{\delta \to 0} \begin{cases} \frac{1}{x} & \text{for } |x| > \delta \\ 0 & \text{for } |x| \leq \delta \end{cases} \]

\[ P \left( \frac{1}{x} \right) = \lim_{\epsilon \to 0} \Re \left( \frac{1}{x - i\epsilon} \right) \]

*Hint:* Consider an integral (over the real axis) of a product of \( P \left( \frac{1}{x} \right) \) with a regular at zero test function \( \phi(x) \).

(ii) Prove that

\[ \lim_{\epsilon \to 0} \Im \left( \frac{1}{x - i\epsilon} \right) = i \pi \text{sign}(\epsilon) \delta(x). \]

*Hint:* As in part (i).

(iii) Show that

\[ \delta(x-y)\delta(x-z) = \delta(x-y)\delta(y-z) = \delta(x-z)\delta(y-z). \]

*Hint:* Consider two-dimensional integrals of products \( \delta(\cdot)\delta(\cdot)\phi(x, y) \), where \( \phi(x, y) \) is a test function regular at \( x = y, x = z \) and \( y = z \).

(iv) Using (i)-(iii) prove that

\[ P \left( \frac{1}{x-z} \right) P \left( \frac{1}{y-z} \right) = \pi^2 \delta(x-z)\delta(y-z) + P \left( \frac{1}{y-x} \right) P \left( \frac{1}{x-z} \right) + P \left( \frac{1}{x-y} \right) P \left( \frac{1}{y-z} \right). \]